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NOTES, CHRONICLES, INFORMATION

The Problem of Optimal Iterative Learning

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Abstract—It is shown that three variants of the optimal iterative learning problem — maximizing the expected result for a fixed total number of trials and time; minimizing the time to achieve a given expected result; minimizing the number of trials required to achieve a given expected result — have either a degenerate (independent of the learning intensity dynamics) or a trivial (maximum permissible learning intensity) solution.

Keywords: iterative learning, learning curve, optimal learning intensity

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1. INTRODUCTION

Learning in general is "the process and result of acquiring individual experience" [1–5]. We consider one of the types of learning, namely iterative learning (from the Latin *iterativus* — repetitive) — the repeated repetition by a learning system (living or non-living) of actions, trials, attempts, etc. to achieve a fixed goal under constant external conditions.

Iterative learning as learning in strictly repetitive conditions is one of the simplest types of learning. It occurs in a wide class of phenomena: the formation of various skills, the assimilation of information by humans, animal learning (conditioned reflex formation), and the training of technical and cybernetic systems. Various aspects of iterative learning are studied in pedagogy, psychology and physiology of humans and animals, control theory, and other sciences (see reviews in [4, 6]). Modern mathematical models of iterative learning and a corresponding review can be found in [2].

The constancy of external conditions allows for a quantitative description of iterative learning in the form of graphs — learning curves, which represent the dependence of the learning level criterion on time or on the number of repetitions (iterations) [4, 7]. As a rule, the learning level is interpreted as the probability of correctly performing an action or operation and accordingly takes values from zero to one.

2. PROBLEM STATEMENT

We consider a learning process in continuous time. The following assumptions are introduced.

A.1. The learning intensity $u(t) \geq 0$ is a continuous, non-negative, and bounded above function of time:

$$0 \leq u(t) \leq u_{\max}(t).$$

A.2. Learning is described by a continuous learning curve $L(U(t))$ — a mapping of the total number of attempts, trials, or repetitions made by time t into the interval $[0; 1]$.

A.3. The total number of attempts $U(T) > 0$ that the learning system can make over time $T > 0$ is fixed. Moreover,

$$U(t) = \int_0^t u(\tau)\, d\tau, \quad U(0) = 0.$$

Obviously, $u(t) = U'(t)$.

A.4. The expected number of successful attempts at time t is equal to the product of the learning intensity and the current learning level: $u(t)L(U(t))$. Then the expected number of successful attempts $Z(T)$ over time T is equal to

$$Z(T) = \int_0^T U'(t)\, L(U(t))\, dt. \tag{1}$$

The optimal iterative learning problem consists of maximizing the expected number of successful attempts by choosing the learning intensity dynamics (in [5] this problem was formulated in discrete time):

$$\int_0^T U'(t)\, L(U(t))\, dt \to \max_{U(\cdot)}. \tag{2}$$

3. MAIN RESULT

Let us make a change of variables in the integral (1):

$$Z(T) = \int_0^T U'(t)\, L(U(t))\, dt = \int_{U(0)}^{U(T)} L(U)\, dU.$$

Denote by $W(\cdot)$ the anti-derivative of the learning curve $L(\cdot)$. Then

$$Z(T) = W(U(T)) - W(0). \tag{3}$$

From (3) it is clear that the expected number of successful attempts *does not depend* on the learning intensity dynamics, but is determined only by the shape of the learning curve and the total number of attempts $U(T)$.

Interpretation: in iterative learning (fixed goal, constant external conditions), the learning level depends only on the total number of attempts made and does not depend on their distribution over time. The effects of fatigue, forgetting, warm-up, etc. [3] are not taken into account in this model.

Statement. *If assumptions A.1–A.4 hold, then the value of the efficiency criterion in the optimal iterative learning problem* (2) *does not depend on the learning intensity dynamics.*

4. EXAMPLE: EXPONENTIAL LEARNING CURVE

The most common example of a learning curve is the exponential curve:

$$L(U) = 1 - \exp[-\gamma U(t)], \quad \gamma > 0,$$

where γ is the learning rate. If the intensity u is constant over time, the classical curve $1 - e^{-gt}$ is obtained, with the reduced learning rate $g = \gamma u$.

For the exponential learning curve, expression (1) takes the form

$$Z(T) = \int_0^T U'(t)\big[1 - e^{-\gamma U(t)}\big]dt = U(T) - \int_0^T U'(t)e^{-\gamma U(t)}dt. \tag{4}$$

Problem (2) taking into account (4) reduces to minimizing the number of unsuccessful attempts:

$$\int_0^T U'(t)e^{-\gamma U(t)}dt \to \min_{U(t)}. \tag{5}$$

For the exponential learning curve, we can make the substitution $v(t) = e^{-\gamma U(t)}$, $U(t) = -\frac{1}{\gamma}\ln v(t)$. Then (5) transforms to

$$-\frac{1}{\gamma}\int_0^T v'(t)\,dt \to \min,$$

which is equivalent to $\frac{1}{\gamma}\big[1 - e^{-\gamma U(T)}\big] \to \min$, which also does not depend on the form of $u(\cdot)$.

5. TWO ADDITIONAL PROBLEM FORMULATIONS

By analogy, the following two variants of optimization problems are possible.

(1) **Minimizing time** T to achieve a given expected result Z. By virtue of (3) and the monotonicity of the learning curve, the solution is trivial: the learning time is minimal provided that at each instant the learning intensity is maximal:

$$u(t) \equiv u_{\max}(t).$$

(2) **Minimizing the number of attempts** U required to achieve a given expected result Z. By virtue of (3), this problem has a degenerate solution (independent of the learning intensity dynamics). In substantive terms: the learning system must inevitably make a certain number of errors; otherwise, learning is impossible.

6. OPTIMAL CONTROL PROBLEM (OCP)

6.1. Reduction to an OCP

We introduce state variables $x_1 = U$ and $x_2 = u$. Then the two-point boundary value problem has the form

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = w, \quad w \in [0, 1],$$

$c = (0, ?)^T$, $d = (?, ?)^T$, $-\int_0^T x_2(t)L(x_1(t))\,dt \to \min$.

6.2. Application of Pontryagin's Maximum Principle (PMP)

The problem is autonomous, T is fixed, and the system is linear. Let us apply the local maximum principle. The Pontryagin function with $\lambda_0 = 1$ has the form

$$H = \psi_1 x_2 + \psi_2 w + x_2(t)L(x_1(t)).$$

The adjoint system:

$$\dot{\psi}_1(t) = -x_2(t)L_{x_1}(x_1(t)), \qquad \dot{\psi}_2(t) = -\psi_1(t) - L(x_1(t)).$$

It follows that $\psi_1(t) = L(x_1(0)) - L(x_1(t))$, and $\dot{\psi}_2(t) = -L(x_1(0))$, which is a constant. Then $\psi_2(t) = -L(x_1(0))t + a$ is a linear function.

The optimal control is $u^* = \operatorname{sgn}(\psi_2)$. Since ψ_2 is linear, there is at most one switching of the control. However, since the right endpoint with respect to x_2 is not fixed, the transversality condition implies that $\psi_2(T) = 0$. Hence, there are no control switchings, and only trivial solutions of the OCP exist.

7. CONCLUSION

All three variants of the optimal iterative learning problem have either a degenerate (independent of the learning intensity dynamics) or a trivial (maximum permissible learning intensity) solution. To obtain non-trivial solutions, it is necessary to incorporate into the model the effects of fatigue, forgetting, as well as possibly the limited cognitive and other capabilities of the learning system.

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