ISSN 0005-1179 (print), ISSN 1608-3032 (online), Automation and Remote Control, 2026, Vol. 87, No. 9, pp. 809–830.
© The Author(s), 2026 published by Trapeznikov Institute of Control Sciences, Russian Academy of Sciences, 2026.
Russian Text © The Author(s), 2026, published in Avtomatika i Telemekhanika, 2026, No. 9, pp. 63–90.

LINEAR SYSTEMS

Closed-Loop State Predictor: Robustness and Safe Implementation

V. O. Nikiforov[*,a] **and D. N. Gerasimov**[*,b]

[*] *ITMO University, Saint-Petersburg, Russia*
e-mail: [a] *nikiforov_vo@itmo.ru,* [b] *dngerasimov@itmo.ru*

Received August 31, 2025
Revised January 29, 2026
Accepted February 29, 2026

Abstract—The paper addresses the problem of robust and safe implementation of a state predictor used for stabilization of the linear time-invariant system with known arbitrarily large yet bounded input delay. To this end, we design a new simple observer with resetting of initial conditions generating an estimate of the state prediction error. This estimate is used as the feedback for closed-loop correction of the state prediction. The robustness of the proposed closed-loop predictor with respect to parametric perturbations as well as to its numerical realization is proved and illustrated by simulation.

Keywords: linear systems, control delay, robustness

DOI: 10.7868/S1608303226090029

1. INTRODUCTION AND PROBLEM STATEMENT

It is considered a problem of practical implementation of the state-feedback control stabilizing linear time-invariant plant with the input delay[1]:

$$\dot{x}(t) = Ax(t) + bu(t-h), \tag{1}$$

where $x \in \mathbb{R}^n$ is the state with the initial condition x_0, $u \in \mathbb{R}^1$ is the control ($u(t) \equiv 0$ for $t < h$), $A \in \mathbb{R}^{n\times n}$ is not Hurwitz, $b \in \mathbb{R}^n$, the pair (A, b) is controllable, and h is the known arbitrarily large yet bounded time delay. We also assume that A and b are uncertain and their constant estimates $\hat{A}$ and $\hat{b}$ are used in the control design. In general, $\hat{A} \neq A$ and $\hat{b} \neq b$.

Remark 1. We will assume that the states of all considered dynamical systems are zero for $t < 0$, and under the initial condition x_0 we will assume that the state at the instance $t = 0$, i.e., $x_0 = x(0)$. For a dynamical subsystem with the state χ, by χ_ξ^+ we will denote the value of the state χ reset at the time instant ξ in accordance with some resetting rule.

If A and b are known exactly (i.e., $\hat{A} = A$ and $\hat{b} = b$), one can use the following *prediction-based* control [1–3]:

$$u(t) = -k^\top \left(e^{Ah}x(t) + \int\limits_{t-h}^{t} e^{A(t-\tau)}bu(\tau)d\tau \right), \tag{2}$$

where the vector of feedback gains $k \in \mathbb{R}^n$ is such that the matrix $A_k = A - bk^\top$ is Hurwitz. It is worth noting that the expression in the brackets is exactly the predicted value of the state, i.e.,

$$e^{Ah}x(t) + \int\limits_{t-h}^{t} e^{A(t-\tau)}bu(\tau)d\tau = x(t+h), \tag{3}$$

while the control (2) yields the closed-loop characteristic equation $\det\left(sI - A + bk^\top\right) = 0$.

[1] Sometimes the argument t will be omitted if it does not cause misunderstanding.

The control law (2) is widely used in such control problems as finite spectrum assignment [3, 4], adaptive regulation in systems with input delay [5, Chapter 6; 6–9], stabilization of multi-input multi-output systems with multiple delays [10] and so on (see also specialized books [2, 4, 11, 12]). However, this elegant theoretical result has essential drawback related to practical implementation of the integral term for unstable plants. In [3], it was proposed to approximate the integral with some numerical quadrature rule when the integral is replaced by a discrete sum with a finite number of terms. It was suggested that the closed-loop system should be stable for sufficiently high accuracy (i.e., for sufficiently large number of the terms).

However, in [13] it was demonstrated that for some conventional numerical quadrature rules the control law (2) may not stabilize the closed-loop system even for an arbitrary large number of the discrete terms in the sum. In other words, the predicted-based control is not robust with respect to numerical implementation of the predictor[2]. At the same time, as it was demonstrated in [3], implementation of (3) as a solution of the differential equation leads to cancellation of the unstable poles and zeros in the closed-loop system and results in instability for arbitrarily small parametric perturbations. In other words, in this case the predicted-based control is not robust with respect to parametric perturbations of the plant. It is worth noting that the control of asymptotically stable plants with input delays does not meet similar problems that gives opportunity to design a wide class of stabilizing control laws (for example, see [14, Section 2.2; 7, 15–17] and references therein).

To overcome this obstacle, in [18] (and in [19] for systems with both input and state delays) it was proposed to include a low-pass filter in the control loop and use *dynamic prediction-based* control of the form:

$$\dot{u} = -(\gamma + k^\top b)u - (k^\top A + \gamma k^\top)\left[e^{Ah}x(t) + \int\limits_{t-h}^{t} e^{A(t-\tau)}bu(\tau)d\tau\right], \tag{4}$$

where $\gamma > 0$. In this case, the closed-loop characteristic equation is $(s+\gamma)\det\left(sI - A + bk^\top\right) = 0$. It was demonstrated that the closed-loop system (1), (4) preserves the stability if the integral is approximated by a numerical quadrature rule with sufficiently high accuracy. Later, in [21–23] robustness of (4) with respect to small (in a certain sense) parametric (as well as delay) perturbations was proved.

Thus, the dynamic prediction-based controller (4) preserves the closed-loop stability even if the integral is approximated by a finite number of discrete terms and the real parameters A and b are slightly different form the nominal ones $\hat{A}$ and $\hat{b}$ used in the control law design. However, practical implementation of (4) requires the use of a special numerical quadrature rules (such as rectangular, trapezoidal, or Simpson's rules) of high precision (i.e., with a large number of the discrete terms).

There are two conceptually different approaches that completely avoid the problem of practical implementation of the infinite dimensional integral term. One of them assumes the use of different types of static finite dimension (SFD) feedbacks, while another one is based on the use of Luenberger-like observers for estimation of $x(t+\tau)$.

A simple SFD control law is represented by the *truncated predictor feedback* (TPF) [24–27]

$$u(t) = -k^\top(\sigma)e^{Ah}x(t), \tag{5}$$

where the feedback low gain $\hat{k} = \hat{k}(\sigma)$ is parameterized by a small parameter $\sigma \in (0, 1]$. As demonstrated in [26, 27], under a special choice of the feedback low gain $\hat{k}(\sigma)$ with sufficiently small σ, the

[2] In [20] with the use of linear matrix inequalities it was derives an estimate on admissible input delay allowing one to preserve stability of the closed-loop system under numerical implementation of the predictor for a given plant.

TPF provides global stabilization of a linear system with arbitrarily large input delay if the open-loop system is not exponentially unstable (i.e., eigenvalues of A are on the closed left-half plane). Further modification of a static finite dimension feedback known as *pseudo-predictor feedback* is applicable to a class of linear time-varying systems which are only polynomially unstable [28]. As mentioned in [28], since the static feedbacks are linear and finite dimensional, they are robust with respect to small parametric perturbations.

Thus, the static feedback laws provide robust stabilization of linear systems with arbitrarily large input delay and simultaneously completely remove the problem of practical implementation. However, the class of systems they can cope with is very restrictive.

Alternative method allowing one to completely remove the problem of the integral term implementation is based on the use of Luenberger-like observer [29]

$$\dot{\bar{x}}(t) = A\bar{x}(t) + bu(t) + L\left(\bar{x}(t-h) - x(t)\right), \tag{6}$$

where the feedback matrix L is chosen so that the model

$$\dot{e} = Ae + Le(t-h) \tag{7}$$

of the prediction error $e(t) = \bar{x}(t-h) - x(t)$ is asymptotically stable. If $e(t) \to 0$, then $\bar{x}(t) \to x(t+h)$ and one can use the simple state feedback $u(t) = -k^\top \bar{x}(t)$.

As well known [30], if $(A+L)$ is Hurwitz, then there exists sufficiently small $h_m > 0$ such that (7) is asymptotically stable for all $h \in [0, h_m]$. To cope with arbitrarily large input delays in [29], it was proposed to use *sequential sub-predictors* (SSP)—a sequence of coupled sub-predictors for small segments of the large delay (see also [31, 34, 36, 37], where SSP are used for different classes of linear and nonlinear systems with input, state, and output delays). In [29], it was demonstrated that the controller using the SSP is robust with respect to parametric perturbations, while in [37] it was proposed to couple infinite dimension prediction $x(t+r)$ with $r < h$ like (3) with the observer-based prediction $\bar{x}(t+h-r)$ like (6) to provide low-pass filtering needed for safe implementation of the integral term.

Thus, the observer-based predictors are robust with respect to parametric perturbations. However, for large input delays they require using a sequence of coupled sub-predictors or single observer-based predictor coupled with numerical implementation of infinite dimensional integral term.

In this paper, we propose a new closed-loop state predictor that is robust with respect to parametric perturbations and as well as numerical implementation. The proposed predictor consists of a single observer of simple structure and a resetting mechanism and has the following advantages over the known solutions:

—it is simple and does not require employment of special techniques of delayed systems stabilization;

—it is applicable for systems with arbitrarily large input delay without increasing complexity in the form of sequential sub-predictors;

—it is closed by actual value of the prediction error;

—it is robust with respect to parametric perturbations and numerical implementation;

—the predictor dynamic blocks can be practically implemented with the use of standard numerical solvers like `ode1 (Euler)`, `ode2 (Heun)`, `ode3 (Bogacki-Shampine)`, `ode4 (Runge-Kutta)`, `ode5`, or `ode8 (Dormand-Prince)` proposed in MatLab.

The present paper extends the results of the conference paper [38], where the closed-loop predictor was proposed but its robustness with respect to numerical realization was just conjected. In this paper, the properties of robustness with respect to both parametric perturbations and numerical

implementation are rigorously proved (Sections 3 and 4 respectively). The problem of robustness with respect to time-delay variations are not considered in this paper and perspective readers are referred to the corresponding works [2, 11].

The paper is organized as follows. In Section 2, the key idea of the proposed closed-loop predictor is introduced and its stability properties are analyzed for the ideal case, i.e., under assumption that the parameters A and b of the plant are known exactly, and there are no approximation errors in the integral implementation. Section 3 demonstrates the robustness of the proposed controller with respect to parametric perturbations, while Section 4 proves robustness with respect to numerical realization. Simulation results are presented in Section 5, while Appendices present the derivations of some models, proof of auxiliary lemmas.

2. CLOSED-LOOP PREDICTOR: KEY IDEA

Let us consider the integral term in (2) as the solution of the ordinary differential equation

$$\dot{\psi}(t) = A\psi(t) + bu(t) - e^{Ah}bu(t-h), \quad \psi(0) = 0. \tag{8}$$

In this case

$$x(t+h) = e^{Ah}x(t) + \psi(t), \tag{9}$$

and the control (2) takes the form

$$u = -k^\top \left(e^{Ah}x + \psi\right). \tag{10}$$

Hereafter, we will consider (8) and (9) as a virtual mathematical model of the *true state prediction* $x(t+h)$, while a solution $\psi(t)$ of (8) will be considered as the unmeasured *true integral term*.

We will assume that for implementation of a control law, the *integral term estimate* $\hat{\psi}(t)$ is generated as a solution (may be approximated by a numerical solver) of the equation

$$\dot{\hat{\psi}}(t) = \hat{A}\hat{\psi}(t) + \hat{b}u(t) - e^{\hat{A}h}\hat{b}u(t-h), \quad \hat{\psi}(0) = \hat{\psi}_0, \tag{11}$$

where $\hat{\psi} \in \mathbb{R}^n$. Then, the control law is defined as

$$u(t) = -\hat{k}^\top \left(e^{\hat{A}h}x(t) + \hat{\psi}(t)\right) \tag{12}$$

with the vector of feedback gains $\hat{k}$ selected so that the matrix $\hat{A}_k = \hat{A} - \hat{b}\hat{k}^\top$ is Hurwitz. In this case, *the state prediction estimate* is

$$\hat{x}(t+h) = e^{\hat{A}h}x(t) + \hat{\psi}(t). \tag{13}$$

As was mentioned above, the control (11), (12) is not robust due to cancellation of unstable (if A is not Hurwitz) poles and zeros in the closed-loop system [3]. It means that the closed-loop system (1), (11), (12) may lose the stability under *arbitrarily small* perturbations. Such perturbations can be caused either by parametric uncertainty of the plant (when some estimates $\hat{A}$ and $\hat{b}$ are used in the control law (11), (12) and $\hat{A} \neq A$, $\hat{b} \neq b$) or by the approximation errors of a numerical solver. To cope with this problem, we introduce a prediction error observer with resetting initial conditions.

2.1. Control Law Design: Ideal Case

To clarify the key idea of the design, we start with the ideal case when the control involves the true values of A and b, and the equation (11) is solved absolutely precisely. In this case, instead of the equations (11)–(13) we have

$$\dot{\hat{\psi}}(t) = A\hat{\psi}(t) + bu(t) - e^{Ah}bu(t-h),\ \hat{\psi}(0) = \hat{\psi}_0, \tag{14}$$

$$u(t) = -k^\top \left(e^{Ah}x(t) + \hat{\psi}(t)\right), \tag{15}$$

$$\hat{x}(t+h) = e^{Ah}x(t) + \hat{\psi}(t). \tag{16}$$

Here, we set $\hat{\psi}_0 \neq 0$ just to stress the fact that the designed closed-loop predictor is asymptotically stable.

Let us introduce *the state prediction error*

$$\varepsilon(t) := x(t+h) - \hat{x}(t+h). \tag{17}$$

In view of (9) and (16), we have

$$\varepsilon(t) = \psi(t) - \hat{\psi}(t), \tag{18}$$

and in view of (8) and (14) we obtain

$$\dot{\varepsilon}(t) = A\varepsilon(t), \quad \varepsilon_0 = -\hat{\psi}_0. \tag{19}$$

The state prediction error $\varepsilon(t)$ is not measured. However, the value[3]

$$y(t) := \varepsilon(t-h) = x(t) - \hat{x}(t) \tag{20}$$

or (see (16))

$$y(t) = x(t) - e^{Ah}x(t-h) - \hat{\psi}(t-h) \tag{21}$$

is measured and can be considered as the delayed output of the model (19). Then, in view of (19) and (20) we can write:

$$y(t) = e^{-Ah}\varepsilon(t). \tag{22}$$

The key idea of the closed-loop predictor design is to estimate the difference between the inaccessible true state prediction $x(t+h)$ and the state prediction estimate $\hat{x}(t+h)$ described by the model

$$\dot{\varepsilon}(t) = A\varepsilon(t), \quad y(t) = \varepsilon(t-h) \tag{23}$$

with the measured delayed output y, and then to use this estimate for online correction of $\hat{x}(t+h)$. To this end, we use the following observer (which is a particular case of the observer from Section 3.3 of [2]):

$$\dot{\hat{\varepsilon}}(t) = A\hat{\varepsilon}(t) + Le^{Ah}\left(y(t) - e^{-Ah}\hat{\varepsilon}(t)\right), \quad \hat{\varepsilon}_0 = 0, \tag{24}$$

where $\hat{\varepsilon} \in \mathbb{R}^n$ is *the estimate of the state prediction error* and the designer-chosen matrix $L \in \mathbb{R}^{n\times n}$ is such that $A_L = A - L$ is Hurwitz.

[3] Similar approach was employed in [39], where the delayed value of the state prediction error $\varepsilon(t-h) = x(t) - \hat{x}(t)$ was used to cope with constant external disturbances.

Introducing *the estimate error* $\tilde{\varepsilon} := \varepsilon - \hat{\varepsilon}$ and calculating its time derivative in view of (19), (22), and (24), we obtain

$$\dot{\tilde{\varepsilon}} = A_L \tilde{\varepsilon} \tag{25}$$

and, therefore, $\hat{\varepsilon}(t) \to \varepsilon(t) = x(t+h) - \hat{x}(t+h)$ as $t \to \infty$.

The expression (18) and the model (25) motivate us to correct the state prediction estimate according to the expression

$$\hat{x}^*(t+h) := e^{Ah}x(t) + \hat{\psi}(t) + \hat{\varepsilon}(t). \tag{26}$$

It is easy to see that the following equality holds:

$$\hat{x}^*(t+h) = x(t+h) - \tilde{\varepsilon}(t), \tag{27}$$

where $\tilde{\varepsilon}(t) \to 0$ as $t \to \infty$. Therefore, instead of (15) we use the control

$$u(t) = -k^\top \left(e^{Ah}x(t) + \hat{\psi}(t) + \hat{\varepsilon}(t) \right). \tag{28}$$

However, despite the fact that $\tilde{\varepsilon}(t) \to 0$ as $t \to \infty$, the estimates $\hat{\varepsilon}$ and $\hat{\psi}$ may grow unboundedly since the model (23) is not stable. To prevent unbounded growth of $\hat{\psi}$ and $\hat{\varepsilon}$, we propose to reset the states of the models (14) and (24) at each instance iT, where $i = 1, 2, 3, \ldots$ and T is a sufficiently large time interval to be defined later, according to the rule[4]

$$\hat{\psi}^+_{iT} = \hat{\psi}(iT) + \hat{\varepsilon}(iT), \qquad \hat{\varepsilon}^+_{iT} = 0. \tag{29}$$

Taking into account that $\hat{\varepsilon} = \varepsilon - \tilde{\varepsilon} = \psi - \hat{\psi} - \tilde{\varepsilon}$, we obtain

$$\hat{\psi}_{iT} = \psi(iT) - \tilde{\varepsilon}(iT). \tag{30}$$

Therefore, if the sequence $\tilde{\varepsilon}(iT)$ is asymptotically decaying, i.e., $\tilde{\varepsilon}(iT) \to 0$ for $i \to \infty$, then $\hat{\psi}_{iT} \to \psi(iT)$ for $i \to \infty$.

Remark 2. Variables $\hat{\varepsilon}(t)$ and $\hat{\psi}(t)$ are inner state variables of the controller and do not affect asymptotic behavior of the physical signals—state $x(t)$ and control $u(t)$. However, a rigorous poof of the system closed-loop stability as well as practical issue concerning large signal manipulations require to provide the boundedness of all the closed loop signal that is achieved by periodic resetting defined by (29).

It is worth noting that at the time instants $t = iT$ the signal $e^{-Ah}\hat{\varepsilon}(t)$ being the estimate of $y(t) = \varepsilon(t-h)$ is zeroed. Therefore, to provide correct work of the feedback $\left(y(t) - e^{-Ah}\hat{\varepsilon}(t)\right)$ in the observer (24) it is proposed to switch off this feedback on intervals $t \in [iT, iT+h]$ and instead of the observer (24) to use the observer with switching feedback

$$\dot{\hat{\varepsilon}}(t) = A\hat{\varepsilon}(t) + \gamma(t)Le^{Ah}\left(y(t) - e^{-Ah}\hat{\varepsilon}(t)\right), \quad \hat{\varepsilon}_0 = 0, \tag{31}$$

where

$$\gamma(t) = \begin{cases} 0 & \text{for} \quad iT \leqslant t < iT + h, \\ 1 & \text{for} \quad iT + h \leqslant t < (i+1)T. \end{cases} \tag{32}$$

[4] An alternative approach to the resetting of the model (14) was proposed in [40]. However, in this paper the predictor remains open loop and does not provide *asymptotic* behavior of the prediction.

Then, instead of the model (25) we have

$$\dot{\tilde{\varepsilon}} = \bar{A}_L \tilde{\varepsilon}, \tag{33}$$

where

$$\bar{A}_L = \begin{cases} A & \text{for} \quad iT \leqslant t < iT + h, \\ A_L & \text{for} \quad iT + h \leqslant t < (i+1)T. \end{cases}$$

Thus, the proposed control law consists of: the closed-loop state predictor (14), (26); the prediction error observer (31), (21); the controller (28); and the resetting rule (29).

The resetting interval T is chosen to provide the asymptotic stability of the model (33). To this end, we define a positive defined matrix P with minimum λ_1 and maximum λ_2 eigenvalues as the solution to the matrix equation $A_L^\top P + P A_L = -I$. Then, in view of (33) for the time interval $t \in [iT, (i+1)T]$, we can write

$$|\tilde{\varepsilon}((i+1)T)|^2 \leqslant \frac{\lambda_2}{\lambda_1} e^{-\frac{T-h}{\lambda_2}} |\tilde{\varepsilon}(iT+h)|^2 \leqslant \frac{\lambda_2}{\lambda_1} e^{-\frac{T-h}{\lambda_2}} \|e^{Ah}\|^2 |\tilde{\varepsilon}(iT)|^2.$$

To provide inequality $|\tilde{\varepsilon}((i+1)T)| < |\tilde{\varepsilon}(iT)|$, T is chosen from the condition[5]

$$T \geqslant h - \lambda_2 \ln\left(\frac{\lambda_1}{\lambda_2 \|e^{Ah}\|^2}\right). \tag{34}$$

Thus, we have proved the following lemma.

Lemma 1. *If the resetting time interval T obeys the inequality (34), then $\tilde{\varepsilon}(iT) \to 0$ for $i \to \infty$.*

Corollary 1. *Under the validity of (34), we have $\tilde{\varepsilon}(iT + h) \to 0$ for $i \to \infty$.*

Remark 3. The validity of (34) guarantees asymptotic convergence $\tilde{\varepsilon}(iT) \to 0$ for $i \to \infty$. At the same time, numerous simulations demonstrate that smooth transients without splashes after resetting in the closed-loop system are provided if T is chosen to be one and a half to two times greater than the value defined by inequality (34).

2.2. Stability under Ideal Conditions

To analyze the stability properties, we derive a model of the closed-loop system. It is convenient to derive separately two dynamic subsystems. The first one will be represented in the components $(\tilde{\varepsilon}, x, \psi)$, while the second one will be represented in $(\varepsilon, \hat{\varepsilon}, \hat{\psi})$.

Since the states $\hat{\psi}$ and $\hat{\varepsilon}$ are reset at the instances iT, and the feedback loop in the observer (31) is switched on with delay h, the derived dynamic models will be valid for the time intervals $iT + h \leq t < (i+1)T$. However, these models are used for analysis of the closed-loop stability for all time $t \in [0, \infty)$ taking into account asymptotic behavior of the discrete sequences $\tilde{\varepsilon}(iT)$ and $\tilde{\varepsilon}(iT + h)$.

Substituting (27) and (28) into (1), we obtain

$$\dot{x} = Ax - bk^\top x + bk^\top \tilde{\varepsilon}(t-h) = A_k x + bk^\top e^{-A_L h} \tilde{\varepsilon}.$$

Further, substituting (28) into (8), we have

$$\dot{\psi} = A\psi - bk^\top \left(e^{Ah} x + \hat{\psi} + \hat{\varepsilon} \pm \psi\right) + e^{Ah} bk^\top x - e^{Ah} bk^\top \tilde{\varepsilon}(t-h) = A_k \psi + Dx + C\tilde{\varepsilon},$$

where $C = bk^\top - e^{Ah} bk^\top e^{-A_L h}$ and $D = e^{Ah} bk^\top - bk^\top e^{Ah}$.

[5] Here we assume that A is not Hurwitz, hence $\|e^{Ah}\| \geqslant 1$.

Therefore, on the time intervals $iT + h \le t < (i+1)T$ we have

$$\begin{bmatrix} \dot{\tilde{\varepsilon}} \\ \dot{x} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} A_L & 0 & 0 \\ bk^\top e^{-A_L h} & A_k & 0 \\ C & D & A_k \end{bmatrix} \begin{bmatrix} \tilde{\varepsilon} \\ x \\ \psi \end{bmatrix},$$

where A_k and A_L are Hurwitz. Therefore, this subsystem is asymptotically stable due to its diagonal structure. If T is defined by (34), the resetting does not destroy asymptotic convergence $\tilde{\varepsilon}(iT) \to 0$ as $i \to \infty$, and $\tilde{\varepsilon}(t) \to 0$ as $t \to \infty$. Therefore, we conclude that $\tilde{\varepsilon}(t), x(t), \psi(t) \to 0$ as $t \to \infty$.

Now, we will derive a subsystem in the components $(\varepsilon, \hat{\varepsilon}, \hat{\psi})$. For $\hat{\varepsilon}$, we have

$$\dot{\hat{\varepsilon}} = A\hat{\varepsilon} + Le^{Ah}\left(e^{-Ah}\varepsilon - e^{-Ah}\hat{\varepsilon}\right) = A_L\hat{\varepsilon} + L\varepsilon.$$

Substituting (28) into (14), we have

$$\dot{\hat{\psi}} = A\hat{\psi} - bk^\top\left(e^{Ah}x + \hat{\psi} + \hat{\varepsilon}\right) + e^{Ah}bk^\top x - e^{Ah}bk^\top\left(\varepsilon(t-h) - \hat{\varepsilon}(t-h)\right) = A_k\hat{\psi} + Dx - K\varepsilon + \hat{K}\hat{\varepsilon},$$

where $K = e^{Ah}bk^\top e^{-Ah}$ and $\hat{K} = K - bk^\top$.

Therefore, on the time intervals $iT + h \le t < (i+1)T$ we have

$$\begin{bmatrix} \dot{\varepsilon} \\ \dot{\hat{\varepsilon}} \\ \dot{\hat{\psi}} \end{bmatrix} = \begin{bmatrix} A & 0 & 0 \\ L & A_L & 0 \\ -K & \hat{K} & A_k \end{bmatrix} \begin{bmatrix} \varepsilon \\ \hat{\varepsilon} \\ \hat{\psi} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ D \end{bmatrix} x. \tag{35}$$

This subsystem is unstable because of the matrix A. However, due to the states resetting (29), we have $\varepsilon(iT) = \tilde{\varepsilon}(iT)$, where $\tilde{\varepsilon}(iT) \to 0$ for $i \to \infty$. Therefore, $\varepsilon(iT)$ and $\varepsilon(iT + h)$ are asymptotically decreasing sequences. Now, taking into account that $x(t), \varepsilon(t) \to 0$ for $t \to \infty$, from (35) we conclude that $\hat{\varepsilon}(t), \hat{\psi}(t) \to 0$ for $t \to \infty$. Thus, we have proved the following statement.

Proposition 1. *For the ideal case, in the closed-loop system consisting of the plant* (1)*, the closed-loop state predictor* (14) *and* (26)*, the prediction error observer* (31) *and* (21)*, the controller* (28)*, and the resetting rule* (29) *all the signals are bounded and* $\tilde{\varepsilon}(t)$*,* $\hat{\varepsilon}(t)$*,* $\varepsilon(t)$*,* $x(t)$*,* $\psi(t)$*,* $\hat{\psi}(t) \to 0$ *as* $t \to \infty$ *for any initial conditions* x_0 *and* $\hat{\psi}_0$*.*

Thus, the closed-loop predictor provides the boundedness and asymptotic convergence of the signals for any initial conditions x_0 and $\hat{\psi}_0$, but not only for $\hat{\psi}_0 = 0$ as classical open-loop predictor does. In the next sections, we rigorously prove the robustness of the proposed controller with respect to parametric perturbations as well as to numerical solution of (14).

3. CLOSED-LOOP PREDICTOR: PARAMETRIC ROBUSTNESS

Now, let us assume that in the control law we use the estimate $\hat{A}$ instead of the true value A, and $\hat{A} \ne A$ and investigate the stability of the closed-loop system for small (in the sense of the norm $\|A - \hat{A}\|$) deviations of $\hat{A}$ from A.

Remark 4. All the results can be extended to the estimation of $\hat{b}$. However, we consider the case of $\hat{A}$ only to avoid tedious calculations.

Therefore, the control law consists of:

—the closed-loop state predictor

$$\dot{\hat{\psi}}(t) = \hat{A}\hat{\psi}(t) + bu(t) - e^{\hat{A}h}bu(t-h), \tag{36}$$

$$\hat{x}^*(t+h) := e^{\hat{A}h}x(t) + \hat{\psi}(t) + \hat{\varepsilon}(t); \tag{37}$$

—*the prediction error observer*

$$\dot{\hat{\varepsilon}}(t) = \hat{A}\hat{\varepsilon}(t) + \gamma(t)\hat{L}e^{\hat{A}h}\left(y(t) - e^{-\hat{A}h}\hat{\varepsilon}(t)\right), \tag{38}$$

$$y(t) = x(t) - e^{\hat{A}h}x(t-h) - \hat{\psi}(t-h), \tag{39}$$

where $\hat{L}$ is a designer-chosen matrix such that $\hat{A}_L = \hat{A} - \hat{L}$ is Hurwitz and $\gamma(t)$ is defined by (32);

—*the controller*

$$u(t) \ = -\hat{k}^\top \hat{x}^*(t+h), \tag{40}$$

where a vector $\hat{k}$ is such that the matrix $\hat{A}_k = \hat{A} - b\hat{k}^\top$ is Hurwitz;

—*the scheme of the states resetting* (29), where T is chosen as

$$T \geqslant h - \hat{\lambda}_2 \ln\left(\frac{\hat{\lambda}_1}{\hat{\lambda}_2\|e^{\hat{A}h}\|^2}\right), \tag{41}$$

$\hat{\lambda}_1$ and $\hat{\lambda}_2$ are the minimum and maximum eigenvalues of the matrix $\hat{P}$, respectively, while $\hat{P}$ is the solution of the matrix equation $\hat{A}_L^\top\hat{P} + \hat{P}\hat{A}_L = -I$.

As in Section 2.2, we first derive a model of the closed-loop system in the components $(\tilde{\varepsilon}, x, \psi)$.

In view of (13), for the prediction state error we have

$$\varepsilon(t) = x(t+h) - \hat{x}(t+h) = \tilde{E}x(t) + \psi(t) - \hat{\psi}(t), \tag{42}$$

where $\tilde{E} = e^{Ah} - e^{\hat{A}h}$. Therefore,

$$\hat{\varepsilon} = \tilde{E}x + \psi - \hat{\psi} - \tilde{\varepsilon}, \tag{43}$$

and (see (37))

$$\begin{aligned}\hat{x}^*(t+h) &= e^{\hat{A}h}x + \hat{\psi} + e^{Ah}x - e^{\hat{A}h}x + \psi \\ &= -\hat{\psi} - \tilde{\varepsilon}e^{Ah}x + \psi - \tilde{\varepsilon} = x(t+h) - \tilde{\varepsilon}.\end{aligned} \tag{44}$$

As demonstrated in Appendix A, on the time intervals $iT + h \leq t < (i+1)T$ we have

$$\begin{bmatrix}\dot{\tilde{\varepsilon}}\\ \dot{x}\\ \dot{\psi}\end{bmatrix} = \begin{bmatrix}\hat{A}_L & \tilde{E} & \tilde{A}\\ 0 & \bar{A}_k & 0\\ b\hat{k}^\top & \hat{D} & \bar{A}_k\end{bmatrix}\begin{bmatrix}\tilde{\varepsilon}\\ x\\ \psi\end{bmatrix} + \begin{bmatrix}0\\ b\hat{k}^\top\\ -e^{Ah}b\hat{k}^\top\end{bmatrix}\tilde{\varepsilon}(t-h) + \begin{bmatrix}\hat{L}e^{\hat{A}h}\\ 0\\ 0\end{bmatrix}\tilde{\eta}(x,\psi), \tag{45}$$

where $\tilde{\eta}(x,\psi)$ is defined in (A.3), and $\hat{D} = e^{Ah}b\hat{k}^\top - b\hat{k}^\top e^{Ah}$.

In view of Corollary 2, from Appendix B we conclude that the matrix $\bar{A}_k$ is Hurwitz for sufficiently small $\|\tilde{A}\|$. Since the matrices $\hat{A}_L$, $\bar{A}_k$ are Hurwitz, then the matrix

$$\begin{bmatrix}\hat{A}_L & 0 & 0\\ 0 & \bar{A}_k & 0\\ e^{Ah}b\hat{k}^\top & \hat{D} & \bar{A}_k\end{bmatrix}$$

is Hurwitz due to its diagonal structure. By virtue of Lemma 3 from Appendix B the matrix

$$\begin{bmatrix}\hat{A}_L & \tilde{E} & \tilde{A}\\ 0 & \bar{A}_k & 0\\ b\hat{k}^\top & \hat{D} & \bar{A}_k\end{bmatrix}$$

is Hurwitz for sufficiently small $\|\tilde{A}\|$ (and, as a result, for sufficiently small $\|\tilde{E}\|$). By Lemmas 4 and 5, from Appendix B we conclude that the origin of the system (45) is asymptotically stable on time intervals $iT \leqslant t < (i+1)T$. Further, we can prove that the sequence $\tilde{\varepsilon}(iT)$ is decreasing for sufficiently small $\|\tilde{A}\|$ if T is defined by (41). As a result, we have that $\tilde{\varepsilon}(t)$, $x(t)$, $\psi(t)$, $\eta(t) \to 0$ as $t \to \infty$ for sufficiently small $\|\tilde{A}\|$.

Now, we derive the subsystem in the coordinates $(\varepsilon, \hat{\varepsilon}, \hat{\psi})$. The model for ε is given by (A.7). As demonstrated in Appendix A (see (A.7) and (A.8)), on the time intervals $iT \leq t < (i+1)T$ we have

$$\begin{bmatrix} \dot{\varepsilon} \\ \dot{\hat{\varepsilon}} \\ \dot{\hat{\psi}} \end{bmatrix} = \begin{bmatrix} \hat{A} & 0 & 0 \\ \hat{L} & \hat{A}_L & 0 \\ 0 & -b\hat{k}^\top & \hat{A}_k \end{bmatrix} \begin{bmatrix} \varepsilon \\ \hat{\varepsilon} \\ \hat{\psi} \end{bmatrix} + \begin{bmatrix} \tilde{E}_A \\ 0 \\ \bar{D} \end{bmatrix} x + \begin{bmatrix} \tilde{A} \\ 0 \\ 0 \end{bmatrix} \psi + \begin{bmatrix} 0 \\ \hat{L}e^{\hat{A}h}\tilde{A} \\ 0 \end{bmatrix} \eta - \begin{bmatrix} 0 \\ 0 \\ e^{\hat{A}h}b\hat{k}^\top \end{bmatrix} \tilde{\varepsilon}(t-h).$$

As it was proved above, $x(t)$, $\tilde{\varepsilon}(t)$, $\psi(t)$, $\eta(t) \to 0$ as $t \to \infty$. However, this subsystem is unstable because of the matrix $\hat{A}$. At the same time, due to the states resetting (29) at the instances (41) we have $\varepsilon(iT) = \tilde{\varepsilon}(iT)$, where $\tilde{\varepsilon}(iT) \to 0$. As a result, $\varepsilon(iT)$ is the decreasing sequence. Taking into account that $\hat{A}_L$ and $\hat{A}_k$ are Hurwitz, we can conclude that $\varepsilon(t)$, $\hat{\varepsilon}(t)$, $\hat{\psi}(t) \to 0$ as $t \to \infty$. Thus, we have proved the following statement.

Proposition 2. *The control law* (36)–(40) *and* (29) *is robust with respect to parametric perturbations in the sense that the asymptotic stability declared by the Proposition 1 is preserved for small parametric variations* $\|\tilde{A}\|$*, i.e., all the closed-loop signals are bounded, and* $\tilde{\varepsilon}(t)$*,* $\hat{\varepsilon}(t)$*,* $\varepsilon(t)$*,* $x(t)$*,* $\psi(t)$*,* $\hat{\psi}(t) \to 0$ *for* $t \to \infty$ *and any initial conditions* x_0*,* $\hat{\psi}_0$*.*

4. CLOSED-LOOP PREDICTOR: ROBUSTNESS WITH RESPECT TO NUMERICAL REALIZATION

In this section, we consider the case when the parameters of the plant are known exactly (i.e., $\hat{A} = A$ and $\hat{b} = b$), while to implement the integral term ψ we solve the differential equation (14) with the use of a numerical solver like `ode1 (Euler)`, `ode2 (Heun)`, `ode3 (Bogacki-Shampine)`, `ode4 (Runge-Kutta)`, `ode5`, or `ode8 (Dormand-Prince)` proposed by MatLab. In other words, we investigate robustness of the open-loop predictor with respect to numerical solution of the differential equation (14).

To this end, let us assume that we use a discrete controller

$$u(m) = -k^\top \left(e^{Ah}x(m) + \bar{\psi}(m) \right) \tag{46}$$

with the sampling time δ, where $u(m)$ and $x(m)$ are the values of the signals u and x at the time instants $t = m\delta$, $m = 1, 2, 3 \ldots$, while $\bar{\psi}(m)$ is the *numerical solution* of the differential equation (14) obtained with the use of a numerical solver and the same sampling time δ. To interpolate the control signal within the sampling time δ, we use the zero-order hold.

It is reasonable to make the following assumption.

Assumption. The numerical solution $\bar{\psi}(m)$ is *convergent*, i.e.,

$$\lim_{\delta \to 0} \bar{\psi}(m) = \hat{\psi}(t)$$

for any fixed $t = m\delta$.

Since the equation (14) is linear in the state $\hat{\psi}$, we can construct such continuous-time model referred as *the model of the continuous majorant*

$$\dot{\bar{\psi}}^*(t) = \bar{A}\bar{\psi}^*(t) + \bar{b}u(t) - e^{\bar{A}h}\bar{b}u(t-h) \tag{47}$$

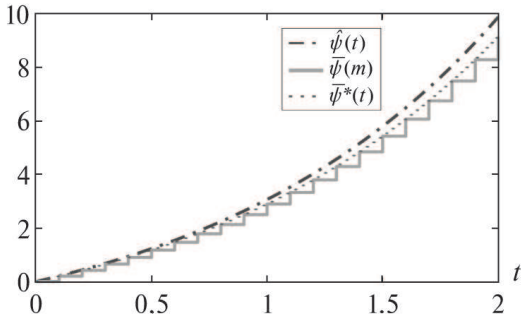


Fig. 1. The solution $\hat{\psi}(t)$ to the equation (14), numerical solution $\bar{\psi}(m)$, and the continuous-time majorant $\bar{\psi}^*(t)$.

with some $\bar{A} = \bar{A}(\delta)$ and $\bar{b} = \bar{b}(\delta)$ depending on δ that (see Fig.1)

$$\bar{\psi}^*(t) = \bar{\psi}(m) \quad \text{for any} \quad t = m\delta.$$

The properties of the model (47) are defined by the following statement proved in Appendix C.

Lemma 2. *The parameters $\bar{A}$ and $\bar{b}$ of the model* (47) *are such that*

$$\lim_{\delta\to 0} \bar{A}(\delta) = A, \quad \text{and} \quad \lim_{\delta\to 0} \bar{b}(\delta) = b.$$

Thus, at the time instants $t = m\delta$ the numerical solution $\bar{\psi}(m)$ can be replaced by the continuous majorant $\bar{\psi}^*(t)$ generated by the parametrically perturbed (since $\bar{A} \neq A$) model (47). In other words, we reduce the robustness analysis with respect to numerical solution to the robustness analysis with respect to parametric perturbations. Thus, in view of Proposition 2 we immediately obtain the following result.

Proposition 3. *The control law* (14), (26), (24), (21), (28), *and* (29) *is robust with respect to numerical solution of* (14) *in the sense that the property of the asymptotic convergence defined by Proposition 1 is preserved for a convergent numerical solution $\bar{\psi}(m)$ with sufficiently small sampling time δ, i.e., all the closed-loop signals are bounded and $\tilde{\varepsilon}(t)$, $\hat{\varepsilon}(t)$, $\varepsilon(t)$, $x(t)$, $\psi(t)$, $\hat{\psi}(t) \to 0$ as $t \to \infty$ for any initial conditions x_0, $\hat{\psi}_0$.*

Remark 5. It is obvious that Propositions 2 and 3 can be combined, and we can prove the robustness of the proposed closed-loop predictor simultaneously with respect to the parametric perturbations as well as to the numerical solution.

5. SIMULATION RESULTS

Example 1. Convergence under nonzero initial condition $\hat{\psi}_0 \neq 0$. For the methodological purposes we consider behavior of the open-loop predictor, closed-loop predictor without state resetting and closed-loop predictor with state resetting under nonzero initial condition $\hat{\psi}_0$.

Consider the second-order unstable plant (1) with $h = 1$ *s*,

$$\hat{A} = A = \begin{bmatrix} 0 & 1 \\ 0 & 0.1 \end{bmatrix}, \quad \hat{b} = b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad x_0 = \begin{bmatrix} 0 \\ 0.5 \end{bmatrix} \tag{48}$$

and set $k = [2 \;\; 3.1]^\top$. In this example, the simulations were carried out in MatLab/Simulink environment with the use of `ode8 (Dormand-Prince)` solver and fixed sampling time 0.00001. It is also assumed that A and b are known and used in the control law, i.e., $\hat{A} = A$ and $\hat{b} = b$.

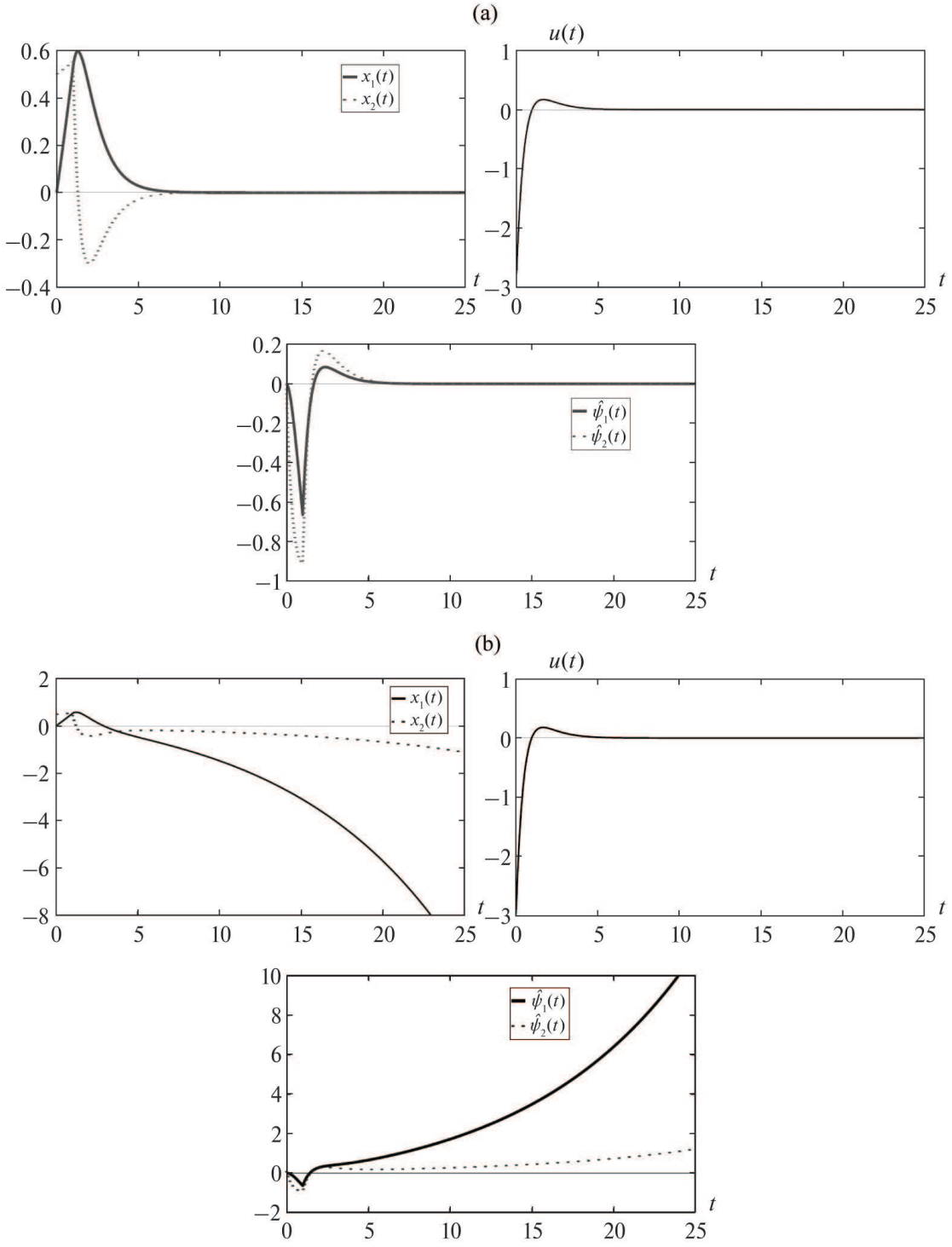


Fig. 2. (Example 1) Transients in the system obtained using the classical open-loop predictor with the parameters (48) and $h = 1$ s: (a) $\hat{\psi}_0 = 0$; (b) $\hat{\psi}_0 \neq 0$.

First, we consider controller with the open-loop state prediction estimate (14) and (15). Simulations results for $\hat{\psi}_0 = 0$ are presented in Fig. 2a and demonstrate asymptotic convergence. However, for $\hat{\psi}_0 = [0, 0.1]^\top$ we obtain unbounded growth of x and ψ (see Fig. 2b).

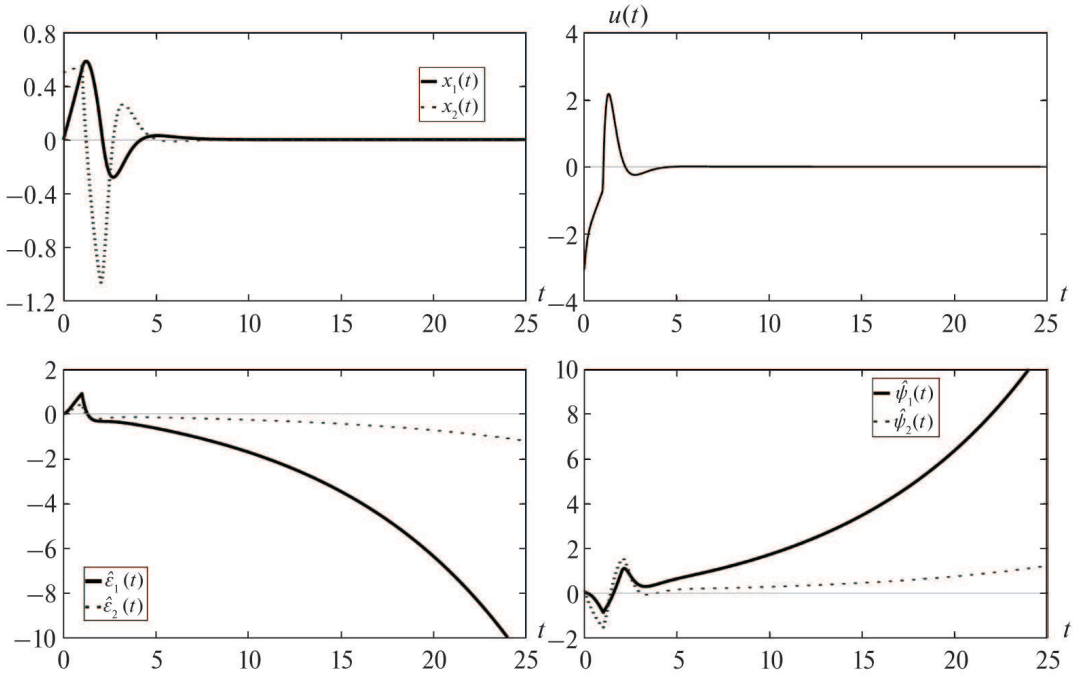


Fig. 3. (Example 1) Transients in the system obtained for $\hat{\psi}_0 \neq 0$ using the closed-loop predictor without state resetting with the parameters (48) and $h = 1$ s.

Now, consider controller with the closed-loop predictor but without state resetting. In this case the controller is described by the following equations: (14), (20), (24), and (28). In simulation, we set:

$$L = \begin{bmatrix} 5 & 0 \\ 6 & 0.1 \end{bmatrix}, \quad T = 10 \text{ s}. \tag{49}$$

Simulation results for $\hat{\psi}_0 = [0, 0.1]^\top$ are presented in Fig. 3 and demonstrate asymptotic convergence on x (in contrast to the open-loop predictor), but unbounded growth on $\hat{\psi}$ and $\hat{\varepsilon}$.

Finally, Fig. 4 presents simulation results obtained for $\hat{\psi}_0 = [0, 0.1]^\top$ with the use of the proposed closed-loop predictor with the state resetting. In this case, the controller is described by the following equations: (14), (26), (31), (21), (28), and (29). As seen from the presented plots, the proposed controller provides asymptotic convergence of x and boundedness and zeroing of $\hat{\psi}$ and $\hat{\varepsilon}$.

Example 2. Robustness with respect to parametric disturbances. Consider the plant (1) with $h = 1$ s,

$$A = \begin{bmatrix} 0 & 1 \\ 0.45 & 0.2 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad x_0 = \begin{bmatrix} 0 \\ 0.5 \end{bmatrix}, \tag{50}$$

and

$$\hat{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0.1 \end{bmatrix}, \quad \hat{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{51}$$

We set $\hat{k} = [2 \;\; 3.1]^\top$. Simulations are carried out in MatLab/ Simulink with the use of `ode8 (Dormand -Prince)` solver and fixed sampling time 0.00001.

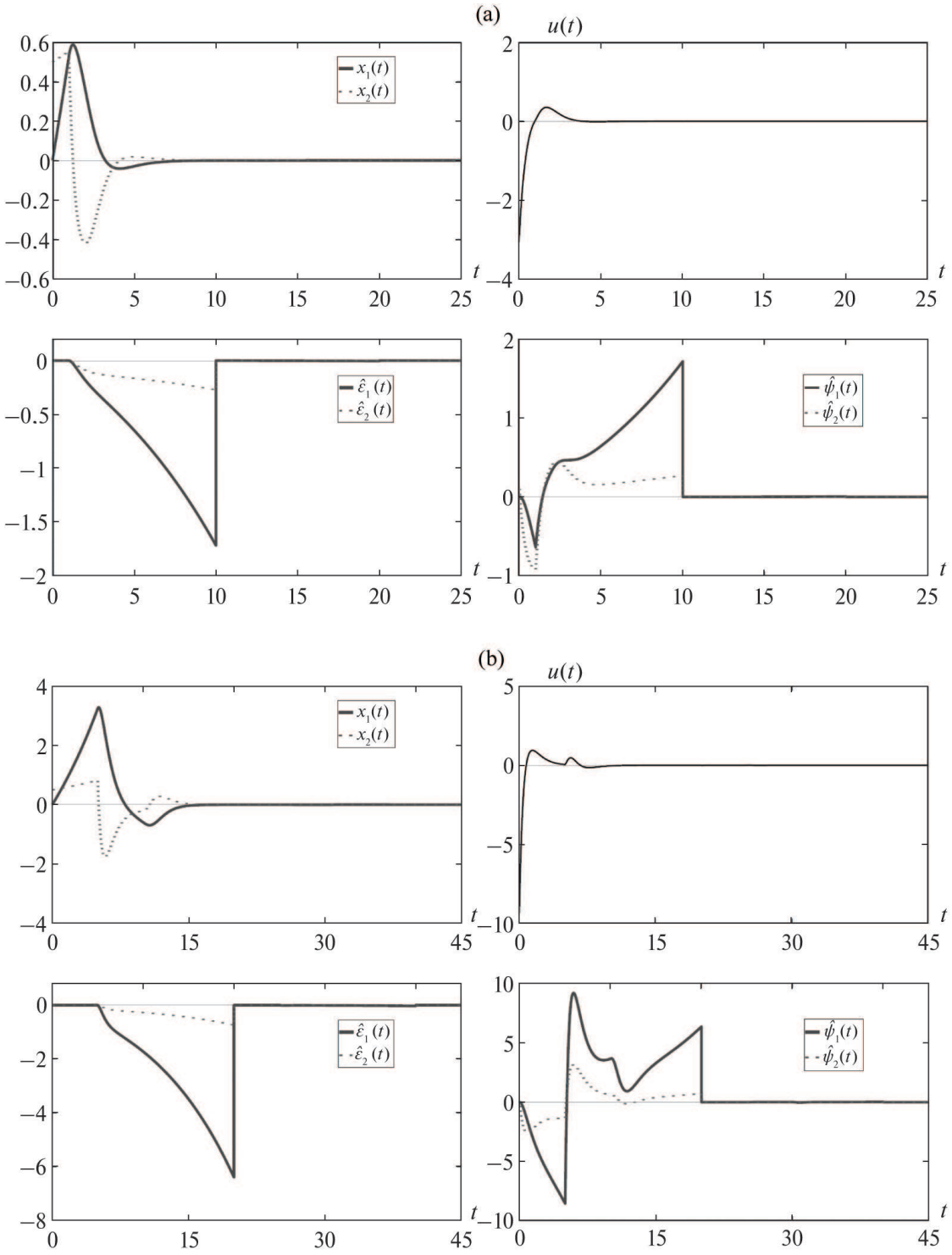


Fig. 4. (Example 1) Transients in the system obtained for $\hat{\psi}_0 \neq 0$ using the closed-loop predictor with state resetting with the parameters (48): (a) $h = 1$ s and $T = 10$ s; (b) $h = 5$ s and $T = 20$ s.

Simulation results obtained for the controller with the open-loop state prediction estimate with $\hat{\psi}_0 = 0$ are presented in Fig. 5 and demonstrate unbounded growth of $x(t)$. Figure 6 presents simulation results obtained for the proposed controller with the closed-loop state prediction and the states resetting. In this simulation, we set L and T as in (49). As seen from the plots, in spite

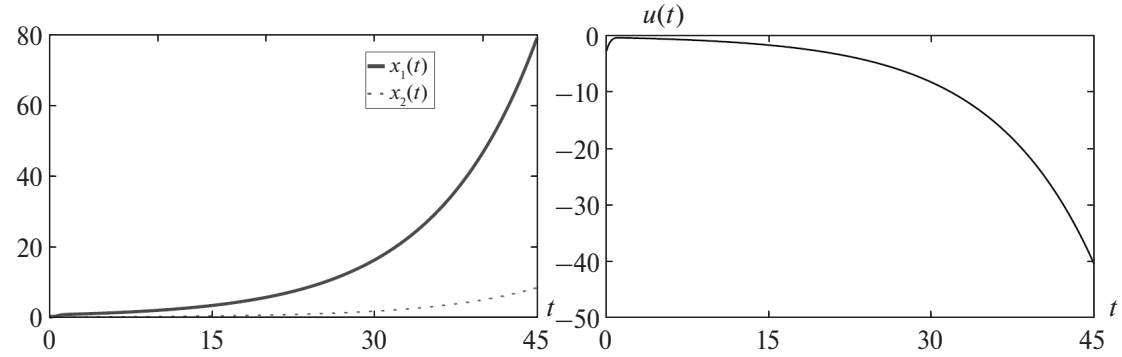


Fig. 5. (Example 2) Transients in the system obtained using the classical open-loop predictor with the parameters (50), (51), and $h = 1$ s.

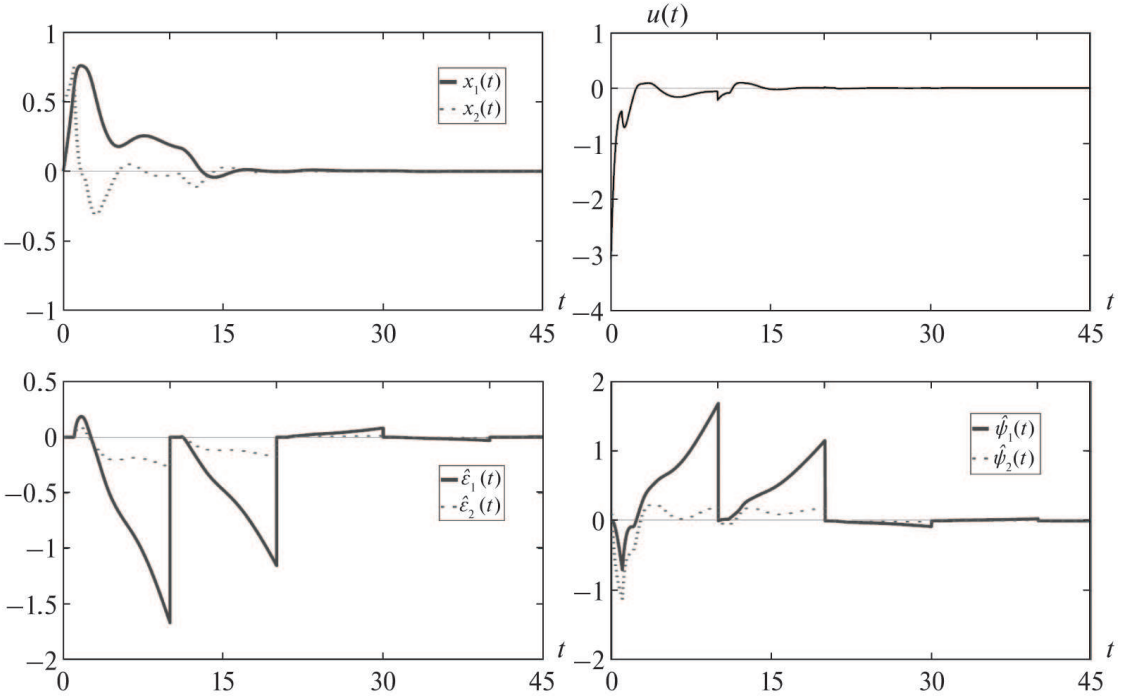


Fig. 6. (Example 2) Transients in the system obtained using the closed-loop predictor with state resetting with the parameters (50), (51), and $h = 1$ s.

of the parametric perturbations the proposed controller provides asymptotic zeroing of $x(t)$, $\hat{\psi}(t)$, and $\hat{\varepsilon}(t)$ with all the closed-loop signals bounded.

Example 3. Robustness with respect to numerical solution. Consider again the plant (1) with the parameters (48), $h = 1$ s, and $k = [2\ \ 3.1]^\top$. Simulation results for the controller with the open-loop state prediction obtained with the use of `ode8 (Dormand-Prince)` solver and fixed sampling time 0.00001 are presented in Fig. 2a and demonstrate zeroing of the state $x(t)$ visible at least for the given zoom and $t \leqslant 25$ s. However, for the less accurate `ode1 (Euler)` solver with fixed sampling time 0.0001 we obtain fast diverging of $x(t)$ (see Fig. 7). The proposed controller with the closed-loop state predictor and L, T defined by (49) provides for `ode1 (Euler)` solver with fixed sampling time 0.0001 the boundedness of all the closed-loop signals and asymptotic zeroing of $x(t)$, $\hat{\psi}(t)$, and $\hat{\varepsilon}(t)$ with small transformations of transient processes presented in Fig. 4a.

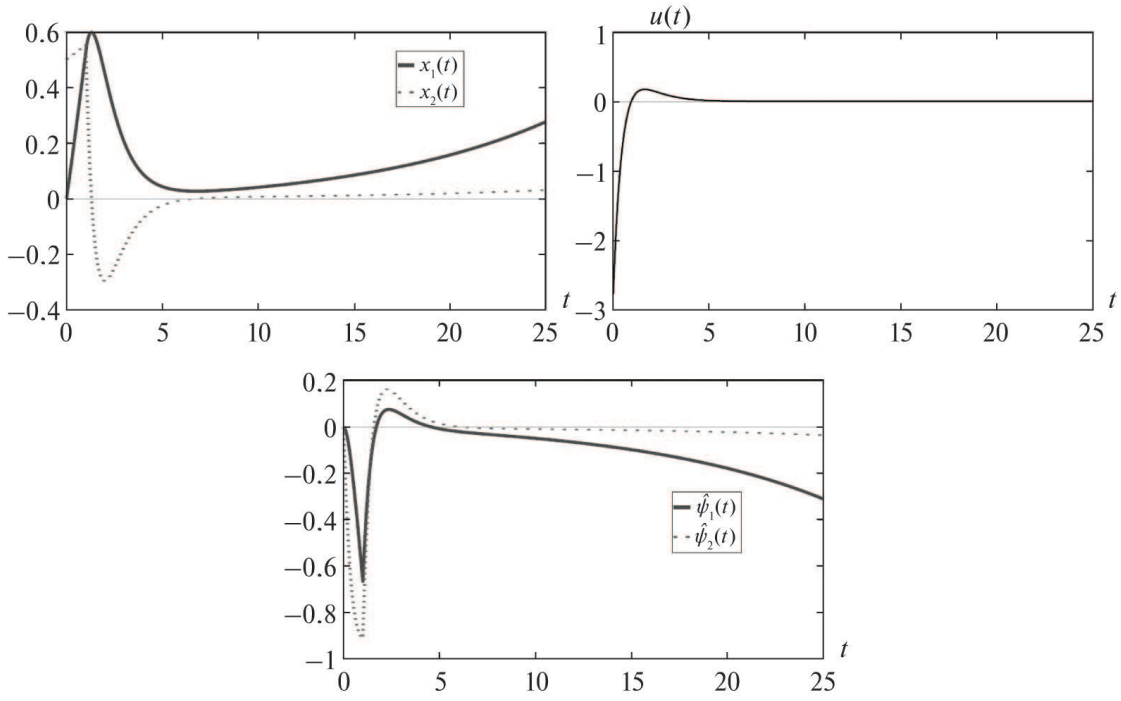


Fig. 7. (Example 3) Transients in the system obtained using the classical open-loop predictor with the parameters (48) and $h = 1$ s and the method of numerical integration `ode1 (Euler)` with the fixed sampling time 0.0001 s.

6. CONCLUSION

A new simple closed-loop state predictor is designed and its robustness with respect to parametric perturbation as well as numerical realization is proved. The key idea of the closed-loop predictor is to estimate the difference between the inaccessible true state prediction $x(t+h)$ and the state prediction estimate $\hat{x}(t+h)$ generated by the predictor and use this estimate for online correction of $\hat{x}(t+h)$. The estimation problem is solved with the use of the observer designed for autonomous systems with output delay. In order to prevent the unbounded growth of some internal signals, a special resetting rule is introduced.

FUNDING

The work is supported by the Ministry of Science and Higher Education of the Russian Federation (project no. FSER-2025-0002).

APPENDIX A

Derivation of the Model (45). Substituting (44) into (40) and (1), we receive

$$\dot{x} = \bar{A}_k x + b\hat{k}^\top \tilde{\varepsilon}(t-h), \tag{A.1}$$

where $\bar{A}_k = A - b\hat{k}^\top$.

Now, let us consider (39):

$$\begin{aligned} y(t) &= e^{Ah}x(t-h) + \psi(t-h) - e^{\hat{A}h}x(t-h) - \hat{\psi}^*(t-h) \\ &= \tilde{E}x(t-h) + \psi(t-h) - \hat{\psi}^*(t-h) = \varepsilon(t-h). \end{aligned}$$

In view of (42), (8) and (36), we obtain

$$\begin{aligned}\dot{\varepsilon} = \tilde{E}\dot{x} + \dot{\psi} - \dot{\hat{\psi}} = \tilde{E}Ax + \tilde{E}bu(t-h) + A\psi + bu - e^{Ah}bu(t-h)\\ - \hat{A}\hat{\psi} - bu + e^{\hat{A}h}bu(t-h) = \hat{A}\varepsilon + \tilde{E}_A x + \tilde{A}\psi,\end{aligned} \tag{A.2}$$

where $\tilde{A} = A - \hat{A}$ and $\tilde{E}_A = \tilde{A}e^{Ah} - e^{\hat{A}h}\tilde{A}$.

In view (A.2) we have (see Section 3.3 in [2])

$$\varepsilon(t-h) \;=\; e^{-\hat{A}h}\varepsilon(t) - \int\limits_{t-h}^{t} e^{\hat{A}(t-h-\tau)}\left(\tilde{E}_A x(\tau) + \tilde{A}\psi(\tau)\right)d\tau.$$

Let us define

$$\tilde{\eta}(x,\psi) = -\int\limits_{t-h}^{t} e^{\hat{A}(t-h-\tau)}\left(\tilde{E}_A x(\tau) + \tilde{A}\psi(\tau)\right)d\tau. \tag{A.3}$$

Then

$$y(t) = \varepsilon(t-h) = e^{-\hat{A}h}\varepsilon(t) + \tilde{\eta}(x,\psi), \tag{A.4}$$

and (in view of (A.2) and (38)) we have

$$\begin{aligned}\dot{\tilde{\varepsilon}} = \dot{\varepsilon} - \dot{\hat{\varepsilon}} = \hat{A}\varepsilon + \tilde{E}_A x + \tilde{A}\psi - \hat{A}\hat{\varepsilon} + \hat{L}e^{\hat{A}h}\left(e^{-\hat{A}h}\varepsilon + \tilde{\eta}(x,\psi) - e^{-\hat{A}h}\hat{\varepsilon}\right)\\ = \hat{A}_L\tilde{\varepsilon} + \tilde{E}x + \tilde{A}\psi + \hat{L}e^{\hat{A}h}\tilde{\eta}(x,\psi).\end{aligned} \tag{A.5}$$

In view of (8), (40) and (44), we have

$$\begin{aligned}\dot{\psi} = A\psi - b\hat{k}^\top\left(e^{\hat{A}h}x + \hat{\psi} + \hat{\varepsilon} \pm \psi\right) + e^{Ah}b\hat{k}^\top x - e^{Ah}b\hat{k}^\top\tilde{\varepsilon}(t-h)\\ = \bar{A}_k\psi + \hat{D}x + b\hat{k}^\top\tilde{\varepsilon} - e^{Ah}b\hat{k}^\top\tilde{\varepsilon}(t-h),\end{aligned} \tag{A.6}$$

where $\hat{D} = e^{Ah}b\hat{k}^\top - b\hat{k}^\top e^{Ah}$.

Taking into account (A.5), (A.1) and (A.6), we have (45).

For $\hat{\varepsilon}$ in view of (38) and (A.4), we have

$$\dot{\hat{\varepsilon}} \;=\; \hat{A}\hat{\varepsilon} + \hat{L}e^{\hat{A}h}\left(e^{-\hat{A}h}\varepsilon + \tilde{A}\eta - e^{-\hat{A}h}\hat{\varepsilon}\right) = \hat{A}_L\hat{\varepsilon} + \hat{L}\varepsilon + \hat{L}e^{\hat{A}h}\tilde{A}\eta. \tag{A.7}$$

Substituting (40) into (36) instead of $u(t)$, and (40), (44) instead of $u(t-h)$, we finally have

$$\begin{aligned}\dot{\hat{\psi}} = \hat{A}\hat{\psi} - b\hat{k}^\top\left(e^{\hat{A}h}x + \hat{\psi} + \hat{\varepsilon}\right) + e^{\hat{A}h}b\hat{k}^\top x - e^{\hat{A}h}b\hat{k}^\top\tilde{\varepsilon}(t-h)\\ = \hat{A}_k\hat{\psi} - b\hat{k}^\top\hat{\varepsilon} + \bar{D}x - e^{\hat{A}h}b\hat{k}^\top\tilde{\varepsilon}(t-h),\end{aligned} \tag{A.8}$$

where $\bar{D} = e^{\hat{A}h}b\hat{k}^\top - b\hat{k}^\top e^{\hat{A}h}$.

APPENDIX B

Useful mathematical background.

In this appendix, we summarize some mathematical tools used in the proof of the main result in Section 3.

Lemma 3. *If a matrix $A \in \mathbb{R}^{n\times n}$ is Hurwitz, then a matrix $\hat{A} \in \mathbb{R}^{n\times n}$ is also Hurwitz for sufficiently small $\|\tilde{A}\|$, where $\tilde{A} = A - \hat{A}$ and $\|A\|$ is the spectral norm of the matrix.*

The validity of this lemma directly follows from the robustness of the exponential stability with respect to multiplicative (i.e., parametric) perturbations (see Section 2.1.4 in [32]).

Corollary 2. *If a matrix $\hat{A}_k = \hat{A} - b\hat{k}^\top$ is Hurwitz, where $\hat{A} \in \mathbb{R}^{n\times n}$ and $b, \hat{k} \in \mathbb{R}^n$, then a matrix $\bar{A}_k = A - b\hat{k}^\top$ is also Hurwitz for sufficiently small $\|\tilde{A}\|$.*

Lemma 4. *If the linear system*

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}A_1 & 0\\ 0 & A_2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix},$$

where $A, A_1, A_2, B \in \mathbb{R}^{n\times n}$ and $x_1, x_2 \in \mathbb{R}^n$, is exponentially stable, then the system

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}A_1 & \tilde{A}\\ 0 & A_2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\\ B\end{bmatrix} x_1(t-h)$$

is also exponentially stable for sufficiently small $\|\tilde{A}\|$.

Proof. For x_1 we have

$$\dot{x}_1 = A_1 x_1 + \tilde{A}W(s)[x_1(t-h)], \tag{B.1}$$

where $W(s) = (sI_2 - A_2)^{-1}B$. It can be shown that the characteristic polynomial $p(s)$ of (B.1) can be represented in the form [12]:

$$p(s) = P(s) + \sum_{i=1}^{n} Q_i(\tilde{A}, s)e^{-ihs},$$

where $P(s) = \det(sI_1 - A_1)\det(sI_2 - A_2)$,

$$Q_i(\tilde{A}, s) = \sum_{k=0}^{2(n-1)} q_{ik}(\tilde{A})s^k,$$

and the coefficients $q_{ik}(\tilde{A})$ are such that

$$\lim_{\|\tilde{A}\|\to 0} q_{ik}(\tilde{A}) = 0.$$

Therefore, for sufficiently small $\|\tilde{A}\|$ and all $\omega \in \mathbb{R}_+$ we have

$$\frac{\sum_{i=1}^{n}\left|Q_i(\tilde{A}, j\omega)\right|}{P(j\omega)} < 1,$$

where j is the imaginary unit. Then, in accordance with Lemma 1 of [12, page 45] we conclude that the system (B.1) is asymptotically stable independently from delay (see also [11, Proposition 2.1]).

Lemma 5. *If a matrix $A_1 \in \mathbb{R}^{n\times n}$ is Hurwitz, then the origin of the system*

$$\dot{x} = A_1 x + B_1\eta(x), \tag{B.2}$$

where $x \in \mathbb{R}^n$, $B_1 \in \mathbb{R}^{n\times m}$,

$$\eta(x) = \int_{t-h}^{t} e^{A_2(t-h-\tau)}B_2 x(\tau)d\tau, \qquad \eta \in \mathbb{R}^n,$$

$A_2 \in \mathbb{R}^{m\times m}$, $B_2 \in \mathbb{R}^{m\times m}$, and $h > 0$ is a constant, is asymptotically stable for sufficiently small $\|B_2\|$.

Proof. Let us consider the Lyapunov function candidate $V(x) = \frac{1}{2}x^\top Px$, where the matrix $P = P^\top > 0$ is the solution of the matrix equation $A_1^\top P + PA_1 = -I_n$. Let λ_1 and λ_2 be the minimum and maximum eigenvalues of P, respectively. Therefore,

$$\frac{\lambda_1}{2}|x(t)|^2 \leqslant V(t) \leqslant \frac{\lambda_2}{2}|x(t)|^2. \tag{B.3}$$

Then, the time derivative of V along solutions of (B.2) obeys

$$\dot V = -|x|^2 + x^\top PB_1\eta(x) \leqslant -\frac{2}{\lambda_2}V + \|PB_1\||x||\eta(x)|.$$

Via the properties of defined integrals, we obtain [41]

$$|\eta(x)| = \left|\int\limits_{t-h}^{t} e^{A_2(t-h-\tau)}B_2x(\tau)d\tau\right| \leqslant \int\limits_{t-h}^{t}\left|e^{A_2(t-h-\tau)}B_2x(\tau)\right|d\tau \leqslant \|B_2\|\int\limits_{t-h}^{t}\left\|e^{A_2(t-h-\tau)}\right\||x(\tau)|d\tau.$$

Since for any $\tau \in [t-h, t]$ the function $\left\|e^{A_2(t-h-\tau)}\right\|$ is bounded, e.g. $r \leqslant \left\|e^{A_2(t-h-\tau)}\right\| \leqslant R < \infty$, and the constants r and R do not depend on t, we can write [41]

$$|\eta(x)| \leqslant \mu\|B_2\|\int\limits_{t-h}^{t}|x(\tau)|d\tau,$$

where $\mu > 0$ is some constant depending on A_2 and h. Therefore, we have

$$\dot V \leqslant -\frac{2}{\lambda_2}V + \mu\|PB_1\|\|B_2\||x|\int\limits_{t-h}^{t}|x(\tau)|d\tau. \tag{B.4}$$

Now let us check if it is possible to find such a value of $\|B_2\|$ that (B.2) be asymptotically stable and $\dot V = -\alpha V$, where $\alpha > 0$ is a positive constant. In this case, $V(t) = e^{-\alpha t}V(0)$ and in view of (B.3) $|x(t)| \leqslant \sqrt{\frac{\lambda_2}{\lambda_1}}e^{-\frac{\alpha}{2}t}|x(0)|$. Therefore,

$$\dot V \leqslant -\frac{2}{\lambda_2}V + \mu\sqrt{\frac{\lambda_2}{\lambda_1}}e^{-\frac{\alpha}{2}t}\|PB_1\|\|B_2\||x(0)|\int\limits_{t-h}^{t}|x(\tau)|d\tau,$$

where

$$\int\limits_{t-h}^{t}|x(\tau)|d\tau \leqslant \sqrt{\frac{\lambda_2}{\lambda_1}}\int\limits_{t-h}^{t}e^{-\frac{\alpha}{2}\tau}|x(0)|\,d\tau = \frac{2}{\alpha}\sqrt{\frac{\lambda_2}{\lambda_1}}\left(e^{\frac{\alpha}{2}h}-1\right)e^{-\frac{\alpha}{2}t}|x(0)|.$$

Then, we can write

$$\begin{aligned} r\dot V &\leqslant -\frac{2}{\lambda_2}V + \frac{2\mu}{\alpha}\frac{\lambda_2}{\lambda_1}\left(e^{\frac{\alpha}{2}h}-1\right)\|PB_1\|\|B_2\|e^{-\alpha t}|x(0)|^2 \\ &\leqslant -\frac{2}{\lambda_2}V + \frac{4\mu}{\alpha}\frac{\lambda_2}{\lambda_1^2}\left(e^{\frac{\alpha}{2}h}-1\right)\|PB_1\|\|B_2\|e^{-\alpha t}V(0). \end{aligned}$$

Consider the worst case, when

$$\dot{V} = -\frac{2}{\lambda_2}V + \frac{4\mu}{\alpha}\frac{\lambda_2}{\lambda_1^2}\left(e^{\frac{\alpha}{2}h} - 1\right)\|PB_1\|\|B_2\|e^{-\alpha t}V(0)$$

and, therefore,

$$\dot{V} = -\left(\frac{2}{\lambda_2} - \frac{4\mu\lambda_2}{\alpha\lambda_1^2}(e^{\frac{\alpha}{2}h} - 1)\|PB_1\|\|B_2\|\right)V.$$

From the inequality

$$\alpha = \frac{2}{\lambda_2} - \frac{4\mu\lambda_2}{\alpha\lambda_1^2}\left(e^{\frac{\alpha}{2}h} - 1\right)\|PB_1\|\|B_2\|$$

we can conclude that for any preselected $0 < \alpha < \frac{2}{\lambda_2}$ we can calculate corresponding $\|B_2\| > 0$:

$$\|B_2(\alpha)\| = \frac{\alpha\lambda_1^2}{4\mu\lambda_2}\left(e^{\frac{\alpha}{2}h} - 1\right)^{-1}\|PB_1\|^{-1}\left(\frac{2}{\lambda_2} - \alpha\right).$$

Moreover, if $\alpha \to \frac{2}{\lambda_2}$, then $\|B_2(\alpha)\| \to 0$, and if $\alpha \to 0$, then $\|B_2(\alpha)\| \to \frac{\lambda_1^2}{\mu\lambda_2^2 h}\|PB_1\|^{-1}$. Therefore, for all $\|B_2\| < \frac{\lambda_1^2}{\mu\lambda_2^2 h}\|PB_1\|^{-1}$ we have $\alpha > 0$, and the system (B.2) is exponentially stable.

APPENDIX C

Proof of Lemma 2. Let us consider a linear time-invariant system

$$\dot{\chi}(t) = A\chi(t) + b\nu(m), \tag{C.1}$$

where $\chi \in \mathbb{R}^n$, $\nu(m)$ is the discrete input , δ is the sampling period and $m = 1, 2, 3, \ldots$. We prove the lemma considering the particular case of the Euler numerical solver of the equation (C.1):

$$\bar{\chi}(m+1) = (I + \delta A)\bar{\chi}(m) + \delta b\nu(m). \tag{C.2}$$

The continuous majorant model is given by

$$\dot{\bar{\chi}}^*(t) = \bar{A}\bar{\chi}^*(t) + \bar{b}\nu(m). \tag{C.3}$$

To define the matrix $\bar{A}$, we equate the zero-input responses of the models (C.2) and (C.3) at the time instants $t = m\delta$:

$$e^{\bar{A}m\delta}x_0 = (I + \delta A)^m x_0.$$

From the latter equality we immediately obtain that

$$\bar{A}(\delta) = \frac{1}{\delta}\ln(I + \delta A),$$

where $\ln(\cdot)$ is the matrix natural logarithm. Then, with the use of the L'Hopital's rule, we can show that

$$\lim_{\delta\to 0}\bar{A}(\delta) = \lim_{\delta\to 0}\frac{1}{\delta}\log(I + \delta A) = A.$$

For invertible matrix A, we can show that $\bar{b}(\delta) = \bar{A}(\delta)A^{-1}b$, and, therefore, $\lim_{\delta\to 0}\bar{b}(\delta) = b$.

REFERENCES

1. Artstein, Z., Linear systems with delayed control: a reduction, *IEEE Trans. Automatic Control*, 1982, vol. 27, no. 8, pp. 869–879.
2. Krstić, M., Delay Compensation for Nonlinear, Adaptive and PDE Systems, Birkhauser, Boston, 2009.
3. Manitius, A. and Olbrot, A., Finite spectrum assignment problem for systems with delay, *IEEE Trans. Automatic Control*, 1979, vol. 24, no. 4, pp. 541–553.
4. Wang, Q.-G., Lee, T.H., and Tan, K.K., Finite Spectrum Assignment for Time-Delay Systems, London: Springer-Verlag, 1998.
5. Nikiforov, V.O. and Gerasimov, D.N., Adaptive Regulation: Reference Tracking and Disturbance Rejection, Springer Nature, 2022.
6. Pyrkin, A.A. and Bobtsov, A.A., Adaptive controller for linear system with input delay and output disturbance, *IEEE Trans. Automatic Control*, 2016, vol. 61, pp. 4229–4234.
7. Bui, V.H. and Margun, A.A., Compensation of output external disturbances for a class of linear systems with control delay, *Scientific and Technical J. of Information Technologies, Mechanics and Optics*, 2022, vol. 22, no. 6, pp. 1072–1077 (in Russian).
8. Nikiforov, V.O., Paramonov, A.V., and Gerasimov, D.N., Adaptive control algorithms in MIMO linear systems with control delay, *Automation and Remote Control*, 2020, vol. 81, no. 6, pp. 1091–1106.
9. Pyrkin, A.A., Bobtsov, A.A., Nikiforov, V.O., Kolyubin, S.A., Vedyakov, A.A., Borisov, O.I., and Gromov, V.S., Compensation of polyharmonic disturbance of state and output of a linear plant with delay in the control channel, *Automation and Remote Control*, 2015, vol. 76, no. 12, pp. 2124–2142.
10. Tsubakino, D., Krstić, M., and Olivera, T.R., Exact predictor feedbacks for multi-input LTI systems with distinct input delays, *Automatica*, 2016, vol. 71, pp. 143–150.
11. Fridman, E., Introduction to Time Delay Systems: Analysis and Control, Birkhauser: Boston–New York–Dordrecht–London, 2010.
12. Niculescu, S-I. and Gu, K., (Eds.) Advances in Time-Delay Systems, Springer Nature, 2004.
13. Engelborghs, K., Dambrine, M., and Roose, D., Limitations of a class of stabilization methods for delay systems, *IEEE Trans. Automatic Control*, 2001, vol. 46, no. 2, pp. 336–339.
14. Tsykunov, A.M., Adaptive and Robust Output-Feedback Control of Dynamic Systems, Moscow: Fizmatlit, 2009 (in Russian).
15. Smith, J., Closer control of loops with delay time, *Chemical Engineering Progress*, 1957, vol. 53, no. 5, pp. 217–219.
16. Furtat, I.B., Adaptive predictor-free control of a plant with delayed input signal, *Automation and Remote Control*, 2014, vol. 75, no. 1, pp. 139–151.
17. Nikiforova, L.V. and Shelenok, E.A., A system with adaptive filters for dynamic plant with state and input time delay, *Bulletin of PNU*, 2025, vol. 77, pp. 77–86 (in Russian).
18. Mondie, S. and Michiels, W., Finite spectrum assignment of unstable time-delay systems with a safe implementation, *IEEE Trans. Automatic Control*, 2003, vol. 48, no. 12, pp. 2207–2212.
19. Kharitonov, V.L., Predictor-based controls: the implementation problem, *Differential Equations*, vol. 51, no. 13, pp. 1675–1682.
20. Furtat, I., Fridman, E., and Fradkov, A., Disturbance compensation with finite spectrum assignment for plants with input delay, *IEEE Trans. Automatic Control*, 2018, vol. 63, no. 1, pp. 298–305.
21. Juarez, L., Mondie, S., and Kharitonov, V.L., Dynamic predictor for systems with state and input delay: a time-domain robust stability analysis, *Int. J. on Robust and Nonlinear Control*, 2020, vol. 30, no. 6, pp. 2204–2218.

22. Rodriguez-Guerrero, L., Kharitonov, V.L., and Mondiea, S., Robust stability of dynamic predictor based control laws for input and state delay systems, *Systems and Control Letters*, 2016, vol. 96, pp. 95–102.

23. Yoon, S.Y. and Lin, Z.I., Robust output regulation of linear time-delay systems: A state predictor approach, *Int. J. on Robust and Nonlinear Control*, 2016, vol. 26, no. 8, pp. 1686–1704.

24. Lin, Z. and Fang, H., On asymptotic stabilizability of linear systems with delayed input, *IEEE Trans. Automatic Control*, 2007, vol. 52, no. 6, pp. 998–1013.

25. Zhou, B., Lin, Z.I., and Duan, G., Stabilization of linear systems with input delay and saturation—A parametric Lyapunov equation approach, *Int. J. on Robust and Nonlinear Control*, 2010, vol. 20, pp. 1502–1519.

26. Zhou, B., Lin, Z., and Duan, G., Truncated predictor feedback for linear systems with long time-varying input delays, *Automatica*, 2012, vol. 48, pp. 2387–2399.

27. Zhou, B., Truncated Predictor Feedback for Time-Delay Systems, Berlin, Heidelberg: Springer-Verlag, 2014.

28. Zhou, B., Pseudo-predictor feedback stabilization of linear systems with time-varying input delays, *Automatica*, 2014, vol. 50, no. 11, pp. 2861–2871.

29. Najafi, M., Hosseinnia, S., Sheikholeslam, F., and Karimadini, M., Closed-loop control of dead time systems via sequential sub-predictors, *Int. J. of Control*, 2013, vol. 86, no. 4, pp. 599–609.

30. Dugard, L. and Verriet, E., Stability and Control of Time-delay Systems, London: Springer, 1997.

31. Cacace, F., Conte, F., Germani, A., and Pepe, P., Stabilization of strict-feedback nonlinear systems with input delay using closed-loop predictors, *Int. J. on Robust and Nonlinear Control*, 2016, vol. 26, pp. 3524–3540.

32. Fradkov, A.L., Miroshnik, I.V., and Nikiforov, V.O., Nonlinear and Adaptive Control of Complex Systems. Ser. Mathematics and Its Applications, vol. 491, Dordrecht: Kluwer Academic Publishers, 1999.

33. Furtat, I.B. and Gushchin, P.A., A control algorithm for an object with delayed input signal based on subpredictors of the controlled variable and disturbance, *Automation and Remote Control*, 2019, vol. 80, no. 2, pp. 201–216.

34. Cacace, F. and Germani, A., Output feedback control of linear systems with input, state and output delays by chains of predictors, *Automatica*, 2017, vol. 85, pp. 455–461.

35. Furtat, I. and Gushchin, P., Tracking control algorithms for plants with input time-delays based on state and disturbance predictors and sub-predictors, *J. of The Franklin Institute*, 2019, vol. 356, no. 8, pp. 4496–4512.

36. Liu, Q. and Zhou, B., Extended observer based feedback control of linear systems with both state and input delays, *J. of The Franklin Institute*, 2017, vol. 354, no. 18, pp. 8232–8255.

37. Zhou, B., Liu, Q., and Mazenc, F., Stabilization of linear systems with both input and state delays by observer-predictors, *Automatica*, 2017, vol. 83, pp. 368–377.

38. Nikiforov, V.O. and Gerasimov, D.N., Robust closed-loop state predictor for unstable systems with input delay, *62nd IEEE Conf. On Decision And Control, Singapore, Dec. 13–15*, 2023, pp. 5708–5713.

39. Léchappé, V., Moulay, E., Plestan, F., Glumineau, A., and Chriette, A., New predictive scheme for the control of LTI systems with input delay and unknown disturbances, *Automatica*, 2015, vol. 52, pp. 179–184.

40. Tam, P.K.S. and Moore, J.B., Stable realization of fixed-lag smoothing equations for continuous-time signals, *IEEE Trans. Automatic Control*, 1974, vol. 19, no. 1, pp. 84–87.

41. Korn, G. and Korn, T., Mathematical Handbook for Scientists and Engineers, Moscow: Nauka, 1970 (in Russian).

This paper was recommended for publication by M.V. Khlebnikov, a member of the Editorial Board