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REVIEWS

Data-Driven Control: Solving Some Classical Problems of Control Theory

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Abstract—This paper is devoted to data-driven control methods for discrete linear time-invariant systems based on controller design from finite sets of input-output measurements of a system with unknown matrices. The discussed approach is also characterized by the use of linear matrix inequalities. We consider methods based on the Fundamental Lemma and the concept of Data Informativity for quadratic stabilization of a system (with exact data and in the case of bounded exogenous disturbances affecting the system), optimal linear quadratic regulator design, and $\mathcal{H}_2$- and $\mathcal{H}_\infty$-optimal control design. The application features of the methods are considered; their advantages and disadvantages are discussed; and numerical examples are provided to illustrate their performance.

Keywords: data-driven control, Fundamental Lemma, Data Informativity, discrete linear time-invariant system, linear matrix inequalities (LMIs), bounded exogenous disturbances, optimal control

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1. INTRODUCTION

The past 10–15 years have seen a considerable expansion of a relatively young branch of control theory, known as data-driven control (DDC). Here, the main idea is to design control for a dynamic system with a partially or completely unknown model using measurements of the input and output signals of the system, without its identification.

At present, numerous different DDC methods are known for solving a great variety of control problems, and new ones are constantly appearing. The history and philosophy of DDC in general, as well as of its individual aspects, approaches, and methods, were discussed, e.g., in the surveys [1–7], books [8–12], and other publications.

Classical control problem formulations remain relevant. First of all, these include stabilization problems, particularly under exogenous disturbances, optimal linear quadratic regulator (LQR) design, and $\mathcal{H}_2$- and $\mathcal{H}_\infty$-optimal control design. Data-driven versions of control methods exist for all of them as well; and they will be considered in detail below.

This survey focuses on direct DDC methods for solving the above problems with various system parameterizations under incomplete information about its model that involve well-established conventional model-based control (MBC) methods, e.g., those with Lyapunov function construction. This approach unites DDC methods based on Jan Willems' behavioral systems theory [13, 14] and his Fundamental Lemma (FL) [15–17] with the methods involving the concept of Data Informativity [18–20].

According to the FL (also called Willems' lemma in the literature), for a controllable linear system, all input-output trajectories of finite length can be parameterized using a single finite-length trajectory generated by a persistently exciting input sequence, i.e., through finite-length vectors of signal measurements. To date, various modifications and extensions of the FL are known as follows: a constructive proof with an estimated lower bound for the persistency of excitation (PE) condition [22]; analysis and design of continuous-time systems [23, 24]; the FL for stochastic systems [25–27] and for controllability verification [28]; relaxation of controllability and PE conditions [29]; the FL for stability verification [30]; the robust FL [31, 32]; the FL for time-varying systems [33–35], systems with input saturation [36, 37], affine [38], switched [39], bilinear [40], and nonlinear [41, 42] systems, and autoregressive models [43]; in predictive control [44–49], optimal control with constraints under data with measurement errors [50], tracking [51, 52], and consensus design [53]; the extended FL in the case of multiple datasets [54]; the augmented FL for constructing alternative non-minimal state vectors [55] and universal input signals [56]; and many others.

In the alternative concept, the data must satisfy certain requirements for each particular problem that are generally milder compared to the FL, i.e., the so-called Data Informativity (DI) conditions. There are diverse ways to use the DI concept as well: analysis of the structural properties of linear systems based on noisy data [57]; observability investigation [58]; analysis and design of continuous-time systems [59]; distributed positive stabilization [60]; analysis of linear systems with conic constraints [61]; design of dissipative [62], affine [63], and parameter-varying [64] systems, and autoregressive models [65]; algebraic controller design in tracking problems [66]; deadbeat control [18]; Data Informativity in the case of multiple datasets and extensions of the FL [67]; applications in data-driven optimization problems [68]; and many other results.

The motivation to survey this particular group of DDC methods stems from the following considerations. First, according to the publications above, this group is quite universal, and the corresponding methods solve a wide variety of control problems. Second, these methods have a rigorous theoretical foundation, including minimal requirements on the amount and quality of the initial data. Third, they are very convenient from a practical standpoint: the controller design is performed using finite sets of experimental data; for theoretical derivations and computations, well-developed and efficient methods of linear algebra, convex optimization, and semidefinite programming, the technique of linear matrix inequalities (LMIs), etc., are used. Fourth, methods from this group can be combined with fundamentally different DDC approaches (e.g., with PID controllers under unknown system model [21]). Naturally, there exist some drawbacks: for example, the use of LMIs may limit the amount of data that can be manipulated without serious computational challenges.

The scope of a single survey is insufficient to cover all existing results. Here, we focus only on *discrete linear time-invariant systems* and the case of a finite set of measurements available for the input and output signals (of known dimensions) of the *open-loop system*. In addition, we restrict the presentation to linear static state-feedback control and use only experimental data for controller design, assuming the data to be exact. The cases of errors in the input and/or output measurements were addressed in [19, 39, 69–73], etc. Other classes of systems and controllers are not described below, nor are alternative methods, e.g., those based on various types of learning, online control, etc.

In some of the presented problems, by assumption, the system is subject to persistent *exogenous disturbances*, often unmeasurable directly but with known general bounds.

In all the problems under consideration, the systems have no other uncertainties apart from unknown matrices and exogenous disturbances. However, note significant results established recently for systems with structural and/or parametric uncertainty based on the combined use of experimental data and a priori information. For example, systems with structured uncertainty

were studied in [74]. In a number of publications by Russian researchers, the proposed methods for time-varying systems [75] and systems with parametric uncertainty [76–78] involve a Lyapunov function: the original uncertain system is "immersed" into an extended completely defined system with an additional artificial disturbance, without requiring the FL conditions or even, in some problems, the DI conditions.

This survey aims to present the diversity of DDC methods that are united by the common ideology of behavioral systems theory (the application of the FL and DI conditions) and the use of LMIs for solving a series of classical control problems, to investigate the features of their application in each problem, and to discuss their advantages and drawbacks.

The remainder of the paper is organized as follows. Section 2 is devoted to the quadratic stabilization problem and data-driven approaches to its solution. Section 3 deals with methods for solving the optimal LQR problem. In Section 4, we present DDC methods for systems with exogenous disturbances, namely, quadratic stabilization (Section 4.2), $\mathcal{H}_2$-optimal control design (Section 4.3), and $\mathcal{H}_\infty$-optimal control design (Section 4.4). For each of the problems, illustrative numerical examples are provided. The findings of this survey are summarized in Section 5.

We adopt the following notation: $\mathbb{R}^{m\times n}$ is the set of real matrices of dimensions $m\times n$; $\mathbb{S}^n$ is the set of symmetric matrices of dimensions $n\times n$; $\mathbb{S}^n_{++}$ is the set of symmetric positive definite matrices of dimensions $n\times n$; $\|\cdot\|$ stands for the spectral norm of a matrix; $^\top$ indicates transpose; † means the right pseudo-inverse; finally, I is an identity matrix of appropriate dimensions. All matrix inequalities are understood in the sense of sign definiteness of corresponding matrices; in LMIs, the symbol $*$ represents blocks that are symmetric to those explicitly specified.

For signals $x(k)\in\mathbb{R}^n$ and $u(k)\in\mathbb{R}^m$, $k=0,1,2,\dots$, we denote finite matrices of length $T\geqslant 0$ composed of their measurements as follows:

$$\begin{aligned}
U :=&[u(0) \quad u(1) \quad \dots \quad u(T-1)] \in \mathbb{R}^{m\times T},\\
X :=&[x(0) \quad x(1) \quad \dots \quad x(T)] \in \mathbb{R}^{n\times (T+1)},\\
X_- :=&[x(0) \quad x(1) \quad \dots \quad x(T-1)] \in \mathbb{R}^{n\times T},\\
X_+ :=&[x(1) \quad x(2) \quad \dots \quad x(T)] \in \mathbb{R}^{n\times T}.
\end{aligned}$$

2. QUADRATIC STABILIZATION

Consider a discrete linear time-invariant system of the form

$$x(k+1)=Ax(k)+Bu(k),\quad k=0,1,2,\dots, \tag{1}$$

where $x(k)\in\mathbb{R}^n$, $u(k)\in\mathbb{R}^m$, $A\in\mathbb{R}^{n\times n}$, and $B\in\mathbb{R}^{n\times m}$.

From this point onwards, let the pair (A,B) be controllable, and let the measured output coincide with the state of the system:

$$y(k)=x(k),\quad k=0,1,2,\dots.$$

The problem is to design a quadratically stabilizing static linear state-feedback controller of the form

$$u(k)=Kx(k),\quad K\in\mathbb{R}^{m\times n}.$$

When the matrices A and B are given (the conventional MBC approach), the problem reduces to finding matrices $P\in\mathbb{S}^n_{++}$ and $K\in\mathbb{R}^{m\times n}$ satisfying the well-known Lyapunov matrix inequality

$$(A+BK)P(A+BK)^\top - P\prec 0. \tag{2}$$

It can be equivalently written as [79]

$$\begin{bmatrix} P & AP + BY \\ * & P \end{bmatrix} \succ 0. \tag{3}$$

The solution $\hat{P}$, $\hat{Y}$ yields a controller with the gain matrix $\hat{K} = \hat{Y}\hat{P}^{-1}$ that quadratically stabilizes system (1).

Now we represent the system using measured data vectors: if (U, X) are given, then system (1) with exactly known matrices A_{tr} and B_{tr} can be written as

$$X_+ = \begin{bmatrix} B_{tr} & A_{tr} \end{bmatrix} \begin{bmatrix} U \\ X_- \end{bmatrix}. \tag{4}$$

The set of *all* systems with matrices A and B capable of generating such data (U, X) will be denoted by

$$\Sigma = \left\{ (A, B) : X_+ = \begin{bmatrix} B & A \end{bmatrix} \begin{bmatrix} U \\ X_- \end{bmatrix} \right\}. \tag{5}$$

Since the data are obtained from system (4), we have $(A_{tr}, B_{tr}) \in \Sigma$. However, based on these data, it is impossible to distinguish the system with matrices (A_{tr}, B_{tr}) from any other system in Σ. Consequently, in the control problem, it is required to design a controller that quadratically stabilizes *all* systems in Σ.

From the Lyapunov function perspective in the DDC approach, one needs to find a common matrix P satisfying (2) for all $(A, B) \in \Sigma$.

The following subsections present the most well-known methods for designing a quadratically stabilizing data-driven controller involving the FL (Section 2.1), the DI concept (Section 2.2), and quadratic matrix inequalities (QMIs, Section 2.3).

2.1. Parameterization Based on the FL

In [69], the conventional model-based formulation of the quadratic stabilization problem for system (1) was modified to the data-driven case using a nonparametric representation of the system from measured data and the FL (Willems et al.) [15, 16]. According to this lemma, the data (measurements) (U, X) must be sufficiently "*rich*" for system identifiability and, hence, for controller design.

Definition 1 [15]. An input signal $[u(0)\ \ u(1)\ \dots\ u(T-1)] \in \mathbb{R}^{m\times T}$ of a system is said to be persistently exciting of order $n + L$, $1 \leqslant L \leqslant T$, if the matrix composed of the shifts of input measurements and output (state) measurements

$$H_L = \begin{bmatrix} u(0) & u(1) & \dots & u(T-L) \\ \vdots & \vdots & & \vdots \\ u(L-1) & u(L) & \dots & u(T-1) \\ x(0) & x(1) & \dots & x(T-L) \end{bmatrix} \tag{6}$$

has full rank:

$$\operatorname{rank} H_L = n + mL.$$

Actually, this means that the set of input signals "covers" the entire space of possible system dynamics, thereby forcing all states and input-output interactions to manifest themselves. Hereinafter, we assume that $L = 1$.

Note that the PE conditions, which underlie the FL, originally appeared in the 1970s, i.e., long before the emergence of DDC theory, and were used in adaptive control. For example, in [80, Theorem 2.4.2] and [81], identifiability properties in adaptive systems were established under the

PE condition. In general, the ideology of DDC is internally closely related to adaptive and robust control [76] since it rests on the combined use of experimental and a priori data in control design for systems with model uncertainties and disturbances; for details, see [75–78].

Lemma 1 (fundamental Lemma, [15, Theorem 1], [16, Lemma 1]). *If in system* (1) *the input* U *of length* T *is persistently exciting of order* $n+1$, *then:*

1. *The rank condition of the FL holds:*

$$rank \begin{bmatrix} U \\ X_- \end{bmatrix} = n + m. \tag{7}$$

2. *Any trajectory* $\begin{bmatrix} \tilde{U} \\ \tilde{X}_- \end{bmatrix}$ *of length* T *can be represented as a linear combination of the measured data:*

$$\begin{bmatrix} \tilde{U} \\ \tilde{X}_- \end{bmatrix} = \begin{bmatrix} U \\ X_- \end{bmatrix} g \quad \forall g \in \mathbb{R}^T;$$

and conversely, any linear combination $\begin{bmatrix} U \\ X_- \end{bmatrix} g$, $g \in \mathbb{R}^T$, *is a trajectory of length* T *for system* (1).

In several methods, including the one presented here, condition (7) is crucial for data-driven controller design without prior model identification.

Let *the measurement vectors* U *and* X_- *be obtained by measuring signals of the open-loop system.* In the method under consideration, see [69, Theorem 2], the gain matrix K satisfies the relation

$$\begin{bmatrix} K \\ I \end{bmatrix} = \begin{bmatrix} U \\ X_- \end{bmatrix} G \tag{8}$$

with some (generally non-unique) unknown matrix $G \in \mathbb{R}^{T \times n}$, i.e., the controller has the form

$$K = UG, \tag{9}$$

and the closed-loop system matrix is expressed from the data as

$$A_K = A + BK = X_+ G \tag{10}$$

with the same matrix G.

Thus, for a stabilizing controller, the matrix $A_K = X_+ G$ must satisfy the classical Lyapunov condition (2). As a result, a stabilizing data-driven controller can be designed using a quadratic Lyapunov function and LMIs as follows.

Theorem 1 [69, Theorem 3]. *Let the rank condition* (7) *of the FL hold. Then any matrix* $Q \in \mathbb{R}^{T \times n}$ *satisfying the inequality*

$$\begin{bmatrix} X_- Q & X_+ Q \\ * & X_- Q \end{bmatrix} \succ 0 \tag{11}$$

yields the gain matrix

$$\hat{K} = U\hat{Q}(X_-\hat{Q})^{-1}. \tag{12}$$

The converse is also valid: if K *is a stabilizing controller, then it can be represented in the form* (12) *with a matrix* Q *satisfying* (11).

Remark 1. What is important, this theorem provides a characterization of the *entire set of stabilizing controllers* of the form $u = Kx$ for all systems with matrices $(A, B) \in \Sigma$ consistent with the original measurements.

Note that in this problem, the dimension of the matrix variable Q depends on the amount of experimental data T, and, in view of (7), $T \geqslant n + m$.

Remark 2. From this point onwards, we consider measurements obtained from the open-loop system. If this system is unstable, then the considered methods may lead to challenges: it may be physically impossible to obtain the required minimum amount of measurements due to instability. In such a situation, other controller design approaches are needed.

2.2. Controller Design Based on the Concept of Data Informativity

Willems' behavioral systems theory and the FL strongly stimulated the development of DDC and demonstrated broad possibilities for parameterizing the system and the control law using measured input and output signals. However, this lemma imposes significant restrictions on the minimum required amount and quality of the data used. Therefore, it was logical to develop this approach toward relaxing the measurement requirements, so that direct data-driven controller design no longer needs system identifiability (i.e., the fulfillment of the FL). This gave rise to the concept of Data Informativity [18, 19].

In [18], the PE condition was replaced by the milder Data Informativity conditions, under which the finite-length trajectory $\begin{bmatrix} U^\top & X_-^\top \end{bmatrix}^\top$ carries as much information as the state-space model (1) with known matrices. As a rule, necessary and sufficient DI conditions for the solvability of a particular problem are formulated individually as a linear matrix equality or inequalities (or systems thereof).

As applied to the stabilization problem for system (1) with a state-feedback control law $u = Kx$, the data (U, X) are informative if there exists a matrix $K \in \mathbb{R}^{m\times n}$ such that the closed-loop matrix $A + BK$ is Schur stable for all $(A, B) \in \Sigma$ consistent with the experimental data. As it turned out, a data-driven control law can be designed without the PE property of the data. More specifically, the following result was proved.

Theorem 2 [18, Theorem 16]. *The experimental data (U, X) are informative for stabilization by a state-feedback control law if and only if*

$$\operatorname{rank} X_- = n \tag{13}$$

and there exists a right pseudo-inverse $X_-^\dagger$ of X_- such that $X_+X_-^\dagger$ is stable.

Based on this theorem, a constructive method was formulated to design a stabilizing controller using the DI concept.

Theorem 3 [18, Theorem 17]. *The experimental data (U, X) are informative for stabilization by a state-feedback control law if and only if there exists a matrix $Q \in \mathbb{R}^{T\times n}$ such that*

$$\begin{bmatrix} X_-Q & X_+Q \\ * & X_-Q \end{bmatrix} \succ 0, \tag{14}$$
$$X_-Q = (X_-Q)^\top.$$

Moreover, K is a stabilizing controller if and only if for some $\hat{Q}$ satisfying (14),

$$\hat{K} = U\hat{Q}(X_-\hat{Q})^{-1}.$$

Obviously, here the LMI and the parameterization of the controller K have the same form (11)–(12) as in the FL-based design (Section 2.1). However, in the case of the DI conditions, the rank condition (7) is not required: the milder condition $X_-Q = (X_-Q)^\top$ appears instead.

Another important difference follows from (13): in the controller design based on Data Informativity, only $T \geqslant n$ measurements are needed, whereas the FL conditions involve at least $T \geqslant n+m$ measurements.

Thus, the DI concept expands the availability of experimental data suitable for controller design: for some (U, X) the FL conditions may fail (the system is not identifiable), but the method with the DI conditions successfully solves the control problem.

2.3. Controller Design via Quadratic Matrix Inequalities

Another parameterization method with Data Informativity conditions was proposed in [82–84]; it is based on Finsler's matrix lemma applied to constraints in the form of QMIs ([84, Theorem 1], see also Lemma 3 in Section 4.1). This result is a special case of a controller design theorem for systems with exogenous disturbances; the corresponding approach is presented in more detail in the second part of the survey, particularly in Section 4.2.2. Here, for the sake of brevity, we provide the final result, which is of interest from the computational point of view.

Theorem 4 [84, Theorem 2]. *The data (U, X) generated by system (1) are informative for stabilization by a state-feedback control law if and only if there exist matrices $P \in \mathbb{S}^n_{++}$, $Y \in \mathbb{R}^{m\times n}$ and a scalar $\beta > 0$ such that*

$$\begin{bmatrix} P-\beta I & * & * & * \\ 0 & -P & * & * \\ 0 & -Y & 0 & * \\ 0 & 0 & Y^\top & P \end{bmatrix} + \begin{bmatrix} X_+ \\ -X_- \\ -U \\ 0 \end{bmatrix} \begin{bmatrix} X_+ \\ -X_- \\ -U \\ 0 \end{bmatrix}^\top \succcurlyeq 0. \tag{15}$$

For such $\hat{P}$ and $\hat{Y}$ with some $\hat{\beta}$, the stabilizing controller has the form $\hat{K} = \hat{Y}\hat{P}^{-1}$ for all $(A, B) \in \Sigma$.

The principal distinction between this method and the previous two is the different structure of the variables in the LMI: in (11) and (14), the unknown matrix Q has dimensions $T \times n$, i.e., it directly depends on the experiment duration (measurement length) and the amount of data obtained, whereas the variables in the matrix inequality (15) are independent of T.

For this problem, the DI condition restricts the experiment duration, $T \geqslant n$, since the existence of a right pseudo-inverse $X_-^\dagger$ is required for a stabilizing controller design (see Theorem 2 in Section 2.2). In other words, X_- has full rank n [18].

Thus, one can choose the computationally most advantageous method depending on the system dimension and the available amount of measurements.

Example 1. In all examples, including this one, the plant is the discretized model of an unstable batch chemical reactor [69, 85] (with a sampling step of 0.1 s) with the matrices

$$A = \begin{bmatrix} 1.178 & 0.001 & 0.511 & -0.403 \\ -0.051 & 0.661 & -0.011 & 0.061 \\ 0.076 & 0.335 & 0.560 & 0.382 \\ 0 & 0.355 & 0.089 & 0.849 \end{bmatrix}, \quad B = \begin{bmatrix} 0.004 & -0.087 \\ 0.467 & 0.001 \\ 0.213 & -0.235 \\ 0.213 & -0.016 \end{bmatrix}. \tag{16}$$

In the data-driven methods, these matrices are supposed to be unknown; they are used only for simulating measurements and solving the problem by the MBC approach, for comparison purposes.

We designed quadratically stabilizing controllers by each of the above methods and compared the results with the MBC approach and with each other. Let the initial condition $x(0)$ be uniformly distributed on the unit sphere: $\|x(0)\| = 1$. We chose $T \geqslant n + m$ to satisfy the conditions of all the methods. By setting $T = 10$, we applied control inputs $u(k)$ uniformly distributed on $[-1, 1]$ to obtain the measurement vectors $U = \begin{bmatrix} u(0) & \dots & u(9) \end{bmatrix}$, $X_- = \begin{bmatrix} x(0) & \dots & x(9) \end{bmatrix}$, and $X_+ = \begin{bmatrix} x(1) & \dots & x(10) \end{bmatrix}$. All computations described in the paper were carried out in MATLAB using the CVX package [86].

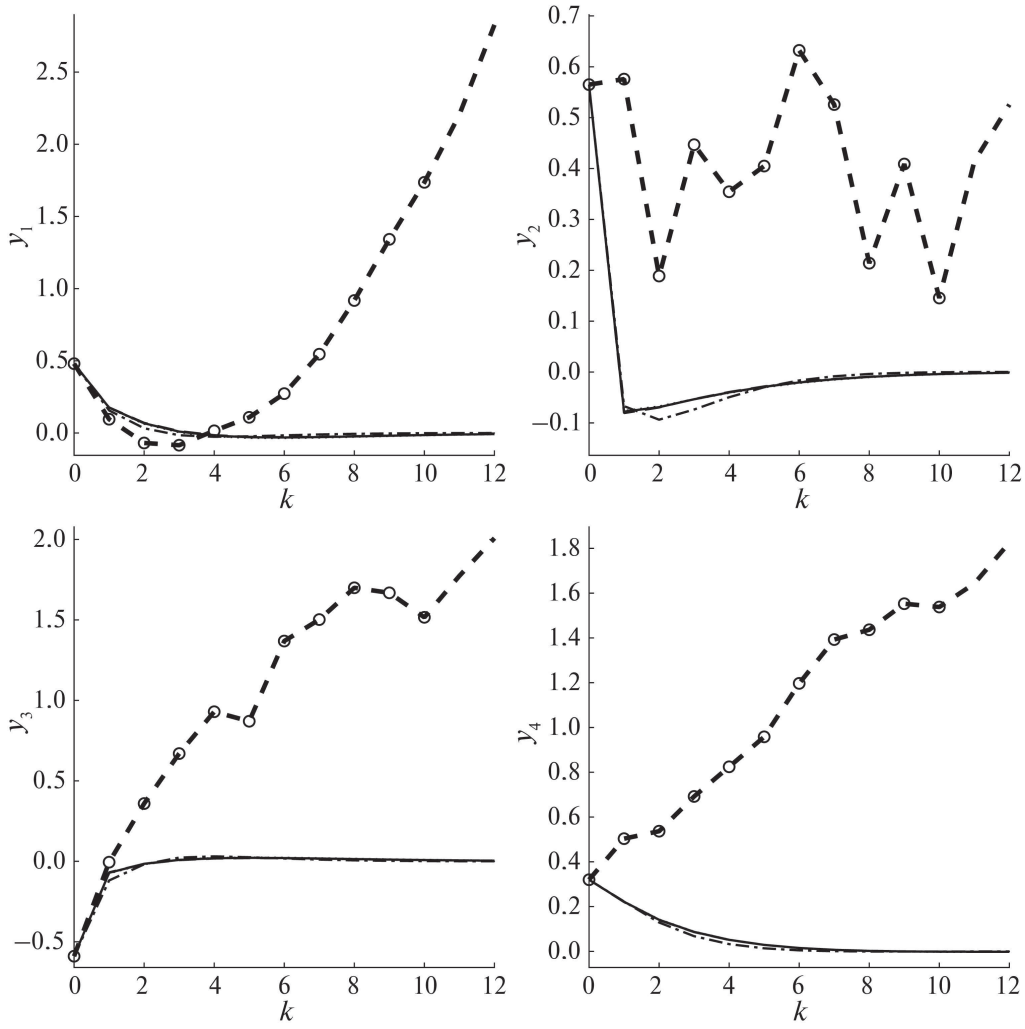


Fig. 1. System outputs in the quadratic stabilization problem.

The conventional MBC solution using formula (3) yielded the controller

$$\hat{K}_{mbc} = \begin{bmatrix} 0.6377 & -1.4696 & 0.0944 & -1.2149 \\ 3.0471 & 0.0622 & 2.9947 & -0.7239 \end{bmatrix}.$$

Before applying the FL-based method, the rank condition was checked again for the data. Computations using Theorem 1 of Section 2.1 gave the controller

$$\hat{K}_{fl} = \begin{bmatrix} 0.6320 & -1.4699 & 0.0940 & -1.2254 \\ 3.0371 & 0.0612 & 3.0009 & -0.7457 \end{bmatrix},$$

and the solution by the DI-based method (Theorem 3 of Section 2.2) was

$$\hat{K}_{di} = \begin{bmatrix} 0.6455 & -1.4713 & 0.1016 & -1.2018 \\ 3.0770 & 0.0563 & 3.0210 & -0.6770 \end{bmatrix}.$$

The controller

$$\hat{K}_{qmi} = \begin{bmatrix} 0.9537 - 1.45460.1305 - 1.5784 \\ 4.5344 - 0.15273.1986 - 1.5027 \end{bmatrix}$$

was obtained by solving the problem based on QMIs (Theorem 4 of Section 2.3) with $\hat{\beta} = 7.8 \cdot 10^{-6}$.

Figure 1 shows the following trajectories: the outputs $y(k) = x(k)$, $k = 0, 1, 2, \dots$, of the open-loop system (thick dashed line, with circles marking the measurements X used) and those of the closed-loop system, namely, the outputs $y_{mbc}(k)$ (the MBC solution, thin solid line), $y_{fl}(k)$ (the FL-based method, thin dashed line), $y_{di}(k)$ (the DI-based method, thin dotted line), and $y_{qmi}(k)$ (the DI-based method with QMIs, thin dash-dotted line).

The results were compared with the MBC solution by assessing the closeness of the corresponding trajectories:

$$\|y_{mbc}(k) - y_{fl}(k)\|_2 = 0.0073, \quad \|y_{mbc}(k) - y_{di}(k)\|_2 = 0.0088,$$
$$\|y_{mbc}(k) - y_{qmi}(k)\|_2 = 0.0696, \quad k = 0, 1, 2, \dots.$$

As the measurement length T increases, the trajectory in the method with QMIs approaches the MBC one more closely.

In fact, the problem formulation has no control requirements other than the stabilizing property. Therefore, the closeness of the matrices $\hat{K}$ and the trajectories does not characterize the quality of the solutions at all: the effectiveness of the methods is confirmed by the fact that they all successfully stabilized the system. However, an additional advantage of data-driven control is the observed degree of closeness of the MBC and DDC solutions.

3. THE OPTIMAL LINEAR QUADRATIC REGULATOR PROBLEM

Consider a system of the form

$$x(k+1) = Ax(k) + Bu(k) \tag{17}$$

with a fixed initial condition $x(0) = x_0$; $u(k) \in \mathbb{R}^m$, $x(k) \in \mathbb{R}^n$, $A \in \mathbb{R}^{n\times n}$, and $B \in \mathbb{R}^{n\times m}$.

Unless otherwise stated, the performance output is

$$z(k) = x(k), \quad k = 0, 1, 2, \dots.$$

The LQR problem is to design a state-feedback control law

$$u(k) = Kx(k), \quad K \in \mathbb{R}^{m\times n},$$

that minimizes the performance criterion (also termed the performance index or cost function in the literature)

$$J(x_0, u) = \sum_{k=1}^{\infty} (x(k)^\top Q x(k) + u(k)^\top R u(k)) \tag{18}$$

and makes the closed-loop matrix $A + BK$ stable. By assumption, the matrices of compatible dimensions $Q = Q^\top \succcurlyeq 0$ and $R = R^\top \succ 0$ are given, and the pair (Q, A) is observable.

As is well known (e.g., see [87]), the solution of this problem is provided by the controller

$$K = -(R + B^\top P^+ B)^{-1} B^\top P^* A, \tag{19}$$

where $P^* \in \mathbb{S}^n_{++}$ is the unique solution of the discrete Riccati equation

$$A^\top PA - P - A^\top PB(R + B^\top PB)^{-1} B^\top PA + Q = 0. \tag{20}$$

The minimum value of the performance criterion is

$$J^{LQR}_{MBC} = x_0^\top P^* x_0. \tag{21}$$

If the matrices A and B are unknown, the DDC framework currently suggests several solution methods. Most of them involve the Lyapunov function, the Riccati inequality, and various parameterizations of the system from the data. Below we discuss the approaches based on the Fundamental Lemma (Section 3.1) and the Data Informativity concept (Section 3.2). Also, the results for the suboptimal LQR problem with the DI condition (Section 3.3) and the finite-horizon LQR problem with the Fundamental Lemma (Section 3.4) will be presented.

3.1. Parameterization Based on the Fundamental Lemma

Let the rank condition (7) of the FL hold.

In [69, 88], the original LQR problem was written as

$$\min_{P,K} \operatorname{trace}(QP + K^\top RKP)$$

subject to the constraints

$$(A + BK)P(A + BK)^\top - P + I \preccurlyeq 0, \quad P \succcurlyeq I.$$

Using the parameterization (9)–(10), the cited authors passed directly to a data-driven formulation of the problem as follows:

$$\min_{P,K,G} \operatorname{trace}(QP + K^\top RKP)$$

subject to the constraints

$$(X_+G)P(X_+G)^\top - P + I \preccurlyeq 0,$$
$$\begin{bmatrix} K \\ I \end{bmatrix} = \begin{bmatrix} U \\ X_- \end{bmatrix} G, \quad P \succcurlyeq I.$$

After the change of variables $Y = KP$ and a series of standard transformations, one obtains a semidefinite programming problem with experimental data:

$$\min_{Y \in \mathbb{R}^{T\times n},\, Z \in \mathbb{R}^{m\times m}} \operatorname{trace}(QX_-Y) + \operatorname{trace}(Z) \tag{22}$$

subject to the constraints

$$\begin{bmatrix} X_-Y - I & X_+Y \\ * & X_-Y \end{bmatrix} \succcurlyeq 0, \quad \begin{bmatrix} Z & R^{1/2}UY \\ * & X_-Y \end{bmatrix} \succcurlyeq 0. \tag{23}$$

Its solution is provided by the optimal linear quadratic regulator

$$\hat{K} = U\hat{Y}(X_-\hat{Y})^{-1}. \tag{24}$$

To ensure the unique matrix G in the parameterization $K = UG$, $A + BK = X_+G$, it was proposed in [88] to augment the constraints (23) with the condition

$$\left(I - \begin{bmatrix} U \\ X_- \end{bmatrix}^\dagger \begin{bmatrix} U \\ X_- \end{bmatrix}\right) Y = 0;$$

in this case, the optimal regulator still has the form (24).

According to [69, Theorem 4], under the exact data (U, X) and the rank condition of the FL, the regulator (24) coincides with the exact MBC solution (19).

Obviously, the applicability of this method requires $T \geqslant n + m$ measurements.

3.2. Parameterization Based on the Data Informativity Concept

In the LQR case, the DI condition coincides with the FL. It was formulated in [18, Theorem 28]: an optimal linear quadratic regulator can be designed from the available measured data if and only if the rank condition (7) of the FL holds; consequently, one has the restriction $T \geqslant m + n$ on the measurement length. The following constructive algorithm was described therein to design the optimal feedback control law.

Algorithm 1 ([18], Theorem 29).

1. Solve the semidefinite programming problem

$$\max_{P \in \mathbb{S}^n_{++}} \operatorname{trace} P \tag{25}$$

subject to the constraint

$$L(P) \preccurlyeq 0, \tag{26}$$

where the linear operator $P \mapsto L(P)$ is

$$L(P) := {X_-}^\top P X_- - X_+^\top P X_+ - X_-^\top Q X_- - U^\top R U.$$

Denote by P^* the resulting unique solution.

2. Find a right pseudo-inverse $\hat{X}_-^\dagger$ such that

$$L(P^*)\hat{X}_-^\dagger = 0. \tag{27}$$

3. Compute the optimal linear quadratic regulator

$$\hat{K} = U\hat{X}_-^\dagger, \tag{28}$$

yielding the minimum value

$$J_{ID}^{LQR} = {x_0}^\top P^* x_0 \tag{29}$$

of the performance criterion (18).

When solving the LQR problem with this algorithm, the dimensions of the unknown matrix P are only $n \times n$ and do not depend on the measurement length, whereas the unknown matrices Y and Z in problem (22)–(23) have dimensions $T \times n$ and $m \times m$, respectively. An undoubted advantage of the DI-based solution is the ability to compute the minimum value of the criterion (29) directly from the measured data, which cannot be done using the FL-based method (see the previous subsection).

A fundamentally different approach to the infinite-horizon LQR problem was proposed in [89, 90]. According to this approach, finite-length measurement vectors are used to estimate the extended (one-step-ahead) observability matrix and an auxiliary matrix, and both matrices are employed to construct the regulator. As noted in [18, 89], the resulting control *tends to the optimal* linear quadratic solution only in the case of infinitely many measurements, and optimality cannot be ensured for experiments of insufficient length. This feature distinguishes the mentioned method from the data-driven design of the optimal LQR using the Fundamental Lemma (Section 3.1) and Data Informativity (Section 3.2), where the *exact optimal* solution is obtained from a finite dataset. Therefore, we will not dwell on this method further.

3.3. Suboptimal LQR Design with the Data Informativity Condition

The DI concept has also proved very useful for designing suboptimal linear quadratic control. For system (17) with a fixed initial condition $x(0) = x_0$ and the performance output

$$z(k) = Cx(k) + B_2u(k)$$

with *known* matrices $C \in \mathbb{R}^{l\times n}$ and $B_2 \in \mathbb{R}^{l\times m}$, a state-feedback control law $u = Kx$ such that the closed-loop matrix $A + BK$ is Schur stable and the quadratic performance criterion (18) takes values below a *prescribed level* $\gamma > 0$, $J(x_0, u) < \gamma$, was constructed from finite-length measured data in [91].

The authors' derivations stemmed from [91, Proposition 1]: if the matrices A and B are known, then K is a suboptimal linear quadratic regulator if and only if there exists a matrix $P \in \mathbb{S}^n_{++}$ such that

$$\begin{gathered}(A + BK)^\top P(A + BK) - P + Q + K^\top RK \prec 0,\\ {x_0}^\top P x_0 \prec \gamma.\end{gathered}$$

In the case of unknown system matrices, the suboptimal LQR design involves the DI condition for (U, X) to ensure stabilizability by a state-feedback control law $u = Kx$ (see Theorem 2 in Section 2.2).

As a consequence, the following result was established (see [91, Theorem 9]: for given x_0 and $\gamma > 0$, it is possible to construct a suboptimal linear quadratic regulator from measured data if and only if there exist a matrix $P \in \mathbb{S}^n_{++}$ and a right pseudo-inverse $X^\dagger_-$ of X_- such that

$$\begin{gathered}(X_+X^\dagger_-)^\top P X_+X^\dagger_- - P + Q + (UX^\dagger_-)^\top RUX^\dagger_- \prec 0,\\ {x_0}^\top P x_0 \prec \gamma.\end{gathered}$$

For the found $\hat{X}^\dagger_-$, the suboptimal linear quadratic regulator has the form $\hat{K} = U\hat{X}^\dagger_-$.

From a computational point of view, it is convenient to pass to an equivalent problem in an LMI form using standard transformations. To this end, in (18) let $Q = C^\top C$, $R = {B_2}^\top B_2$, $C^\top B_2 = 0$, $x_0 \in \mathbb{R}^n$, and $\gamma > 0$; after straightforward manipulations, one arrives at the following result.

Proposition 1 [91, Proposition 10]. *The data (U, X) are informative for suboptimal linear quadratic control if and only if there exist matrices $P \in \mathbb{S}^n_{++}$ and $Y \in \mathbb{R}^{T\times n}$ such that*

$$\begin{gathered}\begin{bmatrix} P & * & * \\ X_+Y & P & * \\ (CX_- + B_2U)Y & 0 & I \end{bmatrix} \succ 0, \quad \begin{bmatrix} \gamma & * \\ x_0^\top & P \end{bmatrix} \succ 0, \\ P = X_-Y.\end{gathered} \tag{30}$$

Moreover, K is a suboptimal linear quadratic regulator for all *systems with matrices $(A, B) \in \Sigma$ from* (5) *if and only if $\hat{K} = U\hat{Y}\hat{P}^{-1}$ for some matrices $\hat{P}$, $\hat{Y}$ found from* (30).

To satisfy these conditions, the matrix X_- must have full rank, i.e., only $T \geqslant n$ input-output measurements are required for designing a *suboptimal* linear quadratic regulator by the above method. Recall that $T \geqslant n + m$ measurements are needed for the unique identification of the system matrices A and B and the *optimal* LQR design (Sections 3.1 and 3.2).

Below we present one method for solving the finite-horizon optimal LQR problem from a finite set of measured data and compare its accuracy with the infinite-horizon methods in the numerical example.

3.4. Finite-Horizon LQR Design Based on the Fundamental Lemma

For system (17), the optimal LQR problem on a finite horizon $N \geqslant 0$ with the performance criterion

$$J = x(N)^\top Q_f x(N) + \sum_{k=1}^{N-1} (x^\top Q x + u^\top R u), \quad Q_f \succcurlyeq 0,$$

was considered in [92].

When the matrices A and B are known, the optimal solution in the form of a sequence $u^*(0), \dots, u^*(N-1)$ with $u^*(k) = K(k)x(k)$, $k = 1, \dots, N-1$, is given by

$$K(k) = -(R + B^\top P(k+1)B)^{-1} B^\top P(k+1)A, \tag{31}$$

where $P(k)$ is the solution of the discrete Riccati equation

$$P(k) = A^\top P(k+1)A - A^\top P(k+1)B(R + B^\top P(k+1)B)^{-1} B^\top P(k+1)A + Q,$$
$$P(N) = Q_f.$$

Obviously, the control law (31) depends on time and is defined on the time interval $[0, N]$. Assuming the observability of the pair (Q, A), the sequence of the matrices $P(k)$ converges to the matrix P^+ as $N \to \infty$, which is the solution of the discrete Riccati equation (20). Thus, one obtains the optimal infinite-horizon linear quadratic regulator (19).

If the matrices A, B are unknown, the data-driven parameterization (8) with the Fundamental Lemma (7) is employed.

Theorem 5 [92, Theorem 3]. *The optimal solution of the finite-horizon LQR problem is a sequence $K = (K_0, \dots, K_{N-1})$ such that*

$$\hat{K}_k = U\hat{Y}_k \hat{P}_k^{-1}, \tag{32}$$

where the matrices $\hat{P}_k \in \mathbb{S}^n_{++}$ and $\hat{Y}_k \in \mathbb{R}^{T\times n}$ are the solution of the semidefinite programming problem

$$\min_{P_k, Y_k, Z_k} \operatorname{trace}(Q_f P_N) + \sum_{k=1}^{N-1} \Big(\operatorname{trace} Q P_k + \operatorname{trace} Z_k \Big) \tag{33}$$

subject to the constraints

$$P_0 \succcurlyeq I_n, \quad P_k = X_- Y_k, \tag{34}$$
$$\begin{bmatrix} P_{k+1} - I_n & X_+ Y_k \\ * & P_k \end{bmatrix} \succcurlyeq 0, \quad \begin{bmatrix} Z_k & R^{1/2} U Y_k \\ * & P_k \end{bmatrix} \succcurlyeq 0, \quad k = 0, \dots, N-1,$$
$$P = (P_1, \dots, P_N), \quad Y = (Y_0, \dots, Y_{N-1}), \quad Z = (Z_0, \dots, Z_{N-1}).$$

For an observable pair (Q, A), as $N \to \infty$, this solution converges to the optimal infinite-horizon LQR solution (19), and the minimum required number of measurements is dictated by the Fundamental Lemma: $T \geqslant n + m$. Importantly, the dimensions of the matrix variables Y_k depend on T. Another distinguishing feature is inherent in the problem formulation and solution method: the solution is not a single matrix K but a set of matrices.

Let us mention another method for designing the optimal finite-horizon LQR from measured data [93], which combines stabilization conditions and the classical finite-horizon LQR. (For the sake of brevity, it is not presented in the survey.)

Example 2. To compare the above LQR design methods, we considered a more general reactor model from Example 1 with the matrices (16) and the performance output

$$z(k) = Cx(k) + B_2u(k),$$

setting $C = I$ and $B_2 = 0$ so that $C^\top B_2 = 0$. Let $Q = I$ and $R = I$. The input-output measurements were generated for $T = 10$ in the same way as in Example 1, so the applicability conditions of all the methods were satisfied.

We computed the model-based optimal LQR using formulas (19)–(20) (or in MATLAB using the *idare* function):

$$\hat{K}_{mbc}^{LQR} = \begin{bmatrix} 0.0811 & -0.7246 & -0.1444 & -0.6792 \\ 2.1509 & 0.0667 & 1.4934 & -0.9709 \end{bmatrix}, \quad \hat{J}_{mbc}^{LQR} = 3.9116.$$

The data-driven solutions using the Fundamental Lemma (22)–(24) (Section 3.1) and the Data Informativity conditions (25)–(28) (Section 3.2) were, respectively,

$$\hat{K}_{fl}^{LQR} = \begin{bmatrix} 0.0811 & -0.7246 & -0.1444 & -0.6791 \\ 2.1508 & 0.0667 & 1.4934 & -0.9709 \end{bmatrix},$$

$$\hat{K}_{di}^{LQR} = \begin{bmatrix} 0.0811 & -0.7246 & -0.1444 & -0.6792 \\ 2.1509 & 0.0667 & 1.4934 & -0.9709 \end{bmatrix}, \quad \hat{J}_{di}^{LQR} = 3.9116.$$

Obviously, these regulators almost coincide with the MBC solution.

Since this solution and the data-driven one with the DI conditions were very close: $\|\hat{K}_{mbc}^{LQR} - \hat{K}_{di}^{LQR}\| \approx 4.1350 \cdot 10^{-7}$, we also obtained very close values of the performance criteria (21) and (29).

In view of the strict inequality $J(x_0, u) < \gamma$ in the suboptimal control problem (30) from Section 3.3, we set a slightly larger value of $\gamma = 3.9117$. The formulation of this problem is somewhat different, so the result for the same parameters, where possible, was expected to vary from the MBC solution and the first two data-driven methods. For instance, for the above C and B_2, we obtained the suboptimal regulator

$$\hat{K}_{sub}^{LQR} = \begin{bmatrix} -0.6008 & -1.5819 & -0.2738 & -2.4524 \\ 3.0702 & -0.0725 & 3.0633 & -2.2513 \end{bmatrix}.$$

For designing the finite-horizon optimal linear quadratic regulator, we set $N = 20$ and $Q_f = I$ and solved problem (33)–(34) (Section 3.4). The solution was a sequence of control gains of the form (32): $\hat{K}_{fin}^{LQR} = (\hat{K}_{fin\,0}^{LQR}, \ldots, \hat{K}_{fin\,19}^{LQR})$. (The values are omitted here for brevity.) Applying them to the input yielded the system trajectory on the time interval $[0, 20]$.

Figure 2 shows the following trajectories: the performance outputs $z_i(k) = x_i(k)$, $i = 1, 2, 3, 4$, of the open-loop system (thick dashed line, with circles marking the initial measurements X) and those of the closed-loop system, namely, the outputs $z_{mbc}(k)$ (the MBC solution, thin solid line), $z_{fl}(k)$ (the FL-based method, thin dashed line), $z_{di}(k)$ (the DI-based method, thin dotted line), $z_{sub}(k)$ (the suboptimal control case, thin dash-dotted line), and $z_{fin}(k)$ (the finite-horizon LQR, thin solid line).

We numerically estimated the closeness of the trajectories:

$$\|z_{mbc}(k) - z_{fl}(k)\|_2 = 2.3563 \cdot 10^{-5}, \quad \|z_{mbc}(k) - z_{di}(k)\|_2 = 1.4229 \cdot 10^{-7},$$
$$\|z_{mbc}(k) - z_{sub}(k)\|_2 = 0.1030, \quad \|z_{mbc}(k) - z_{fin}(k)\|_2 = 0.0066, \quad k = 0, 1, 2, \ldots.$$

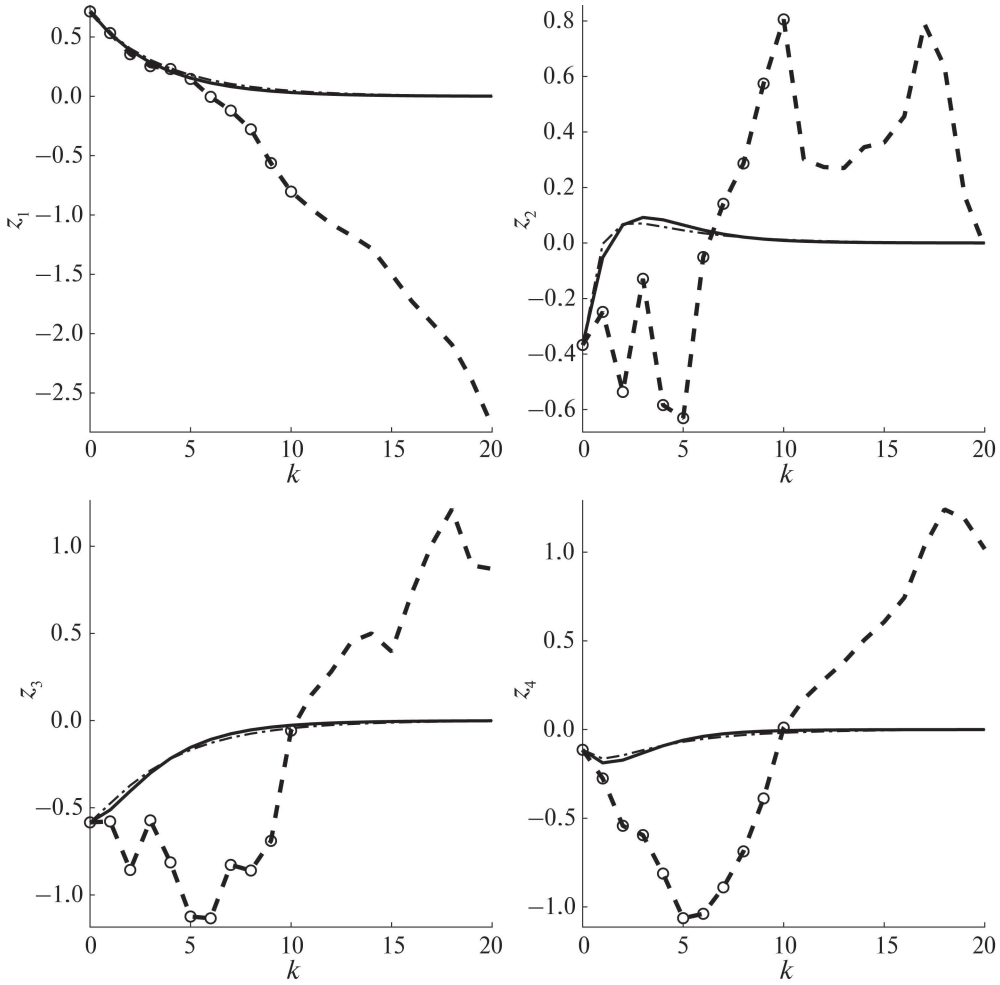


Fig. 2. System outputs in the LQR problem.

In this example, a performance criterion is specified and a unique optimal solution exists; therefore, the closeness of the solutions (the found regulators and the resulting output trajectories) to the MBC one directly characterizes the accuracy of the DDC methods under consideration. Clearly, the methods based on the Fundamental Lemma and the Data Informativity conditions provided high accuracy. The suboptimal LQR design gave a generally acceptable solution, taking into account the differences in problem formulations. For the finite-horizon LQR problem, already with 20 steps we obtained a solution very close to the exact one (in terms of the distance between the trajectory $x_{fin}(k)$ and the exact MBC counterpart $x_{mbc}(k)$); and the accuracy will improve when increasing the number of steps N.

4. CONTROL OF SYSTEMS WITH BOUNDED EXOGENOUS DISTURBANCES

In controller design for systems with exogenous disturbances, two alternative approaches are possible. According to the first approach, called Certainty Equivalence in the literature, all constructs are carried out under the assumption of no disturbances, but estimates of its level are provided under which the control law exists and stabilizes the system with the required performance. This approach seems less interesting from the data-driven perspective: in practice, from measured input and state/output values it is impossible to estimate the bounds within which this (generally speaking, unknown) disturbance can be neglected, owing to the frequent unavailability

of its measurements. As noted, such control methods are effective for small disturbances [88, 94]. From a computational viewpoint, methods of this group reduce to exact-data methods, e.g., those presented in Sections 2.1, 3.1, etc., and the main focus is on estimating the admissible disturbance level [69]. In this paper, we do not dwell on the Certainty Equivalence approach; interested readers are referred, e.g., to [18, 69, 88, 95–97].

The second approach directly accounts for the presence of disturbances, attempting to limit their effect. It is necessary to describe the set of admissible disturbances in some way, so as to design a control law (possibly optimal in terms of a certain criterion) that stabilizes the system for any possible disturbances from this set. This approach will be addressed in the following sections with regard to the quadratic stabilization problem for systems with exogenous disturbances and the problems of $\mathcal{H}_2$- and $\mathcal{H}_\infty$-optimal control design.

4.1. Description of the System and the Set of Disturbances

Consider a general discrete-time system with exogenous disturbances:

$$\begin{aligned} x(k+1) &= Ax(k) + B_1u(k) + Dw(k), \\ z(k) &= Cx(k) + B_2u(k), \end{aligned} \tag{35}$$

where $x(k) \in \mathbb{R}^n$, $u(k) \in \mathbb{R}^m$, and $z(k) \in \mathbb{R}^l$; the matrices $A \in \mathbb{R}^{n\times n}$ and $B_1 \in \mathbb{R}^{n\times m}$ are unknown in the DDC setting; the matrices $C \in \mathbb{R}^{l\times n}$, $B_2 \in \mathbb{R}^{l\times m}$, and $D \in \mathbb{R}^{n\times p}$ are given.

The values of a disturbance $w(k) \in \mathbb{R}^p$ are supposed to be unknown and unmeasurable. Their particular realization of length T will be denoted by

$$W = [w(0) \quad w(1) \quad \dots \quad w(T-1)] \in \mathbb{R}^{p\times T}.$$

A system of the form

$$x(k+1) = Ax(k) + B_1u(k) + Dw(k) \tag{36}$$

can be written in measurement terms:

$$X_+ = AX_- + B_1U + DW = \begin{bmatrix} B_1 & A & D \end{bmatrix} \begin{bmatrix} U \\ X_- \\ W \end{bmatrix}.$$

The complete set of systems with all such matrices (A, B_1, D) that can generate the data (U, X, W) will be denoted by

$$\Sigma_W = \left\{ (A, B_1, D) : X_+ = \begin{bmatrix} B_1 & A & D \end{bmatrix} \begin{bmatrix} U \\ X_- \\ W \end{bmatrix} \right\}; \tag{37}$$

in particular, system (36) with exactly known matrices $(A_{tr}, B_{1tr}, D_{tr})$ also belongs to this set: $(A_{tr}, B_{1tr}, D_{tr}) \in \Sigma_W$.

In many publications devoted to DDC of systems with exogenous disturbances (e.g., [19, 20, 82–84, 94]), the set of admissible exogenous disturbances is parameterized using QMIs. More specifically, the unknown disturbance matrix W is supposed to belong to the set

$$\mathbf{W} = \left\{ W \in \mathbb{R}^{p\times T} : \begin{bmatrix} I \\ W^\top \end{bmatrix}^\top \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{12}^\top & \Phi_{22} \end{bmatrix} \begin{bmatrix} I \\ W^\top \end{bmatrix} \succcurlyeq 0 \right\} \tag{38}$$

with given matrices $\Phi_{11} \in \mathbb{S}^p$, $\Phi_{12} \in \mathbb{R}^{p\times T}$, and $0 \succ \Phi_{22} \in \mathbb{S}^T$. Let us denote $\Phi = \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{12}^\top & \Phi_{22} \end{bmatrix} \in \mathbb{S}^{(n+T)\times(n+T)}$.

This inequality is quadratic in the disturbance matrix W, implying that all admissible disturbances lie *in a matrix ellipsoid*. A QMI of this form is convenient for describing the disturbance set because it can express various constraints on the disturbances.

For example, with $\Phi_{12} = 0$ and $\Phi_{22} = -I$, we obtain a disturbance W with bounded energy over the entire experiment duration $[0, \dots, T-1]$:

$$WW^\top = \sum_{k=0}^{T-1} w_k w_k^\top \preccurlyeq \Phi_{11}. \tag{39}$$

1. For W with an upper bound on the maximum eigenvalue $\sigma_{\max}(W) \leqslant \hat{\sigma}$, one should set $\Phi_{11} = \hat{\sigma} I$ in (39).

2. For W with bounded energy, i.e., $\|w_k\|_2 \leqslant \varepsilon$, $k = 0, \dots, T-1$, the constraints are written as (39) with $\Phi_{11} = \varepsilon^2 T I$. This constraint introduces conservatism, and its degree grows with T [19].

3. Upper bounds on the maximum deviations of each component of W, $\|w_{k,j}\|_2 \leqslant \hat{w}_j$, $k = 0, \dots, T-1$, $j = 1, \dots, p$, can be written as (39) with $\Phi_{11} = \sum_{j=1}^p \hat{w}_j{}^2 T I$.

4. The case of exact measurements ($w = 0$) is characterized by the matrix $\Phi = \begin{bmatrix} 0 & 0 \\ 0 & -I \end{bmatrix}$.

In the subsequent derivations, the description of the admissible disturbance set (38) allows using the matrix S-procedure [82–84] and Finsler's matrix lemma [84]. We present them in a form convenient for the problems under consideration.

Lemma 2 (strict matrix S-procedure, [84, Proposition 3]). *Consider given matrices $M, N \in \mathbb{S}^{k+l}$ of the form*

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{12}^\top & M_{22} \end{bmatrix} \quad \text{and} \quad N = \begin{bmatrix} N_{11} & N_{12} \\ N_{12}^\top & N_{22} \end{bmatrix}, \tag{40}$$

where $M_{22} \preccurlyeq 0$, $N_{22} \preccurlyeq 0$, and $\ker N_{22} \subseteq \ker N_{12}$.

Suppose the existence of a matrix $\tilde{Z} \in \mathbb{R}^{l\times k}$ such that

$$\begin{bmatrix} I \\ \tilde{Z} \end{bmatrix}^\top N \begin{bmatrix} I \\ \tilde{Z} \end{bmatrix} \succ 0.$$

Then the quadratic matrix inequality

$$\begin{bmatrix} I \\ Z \end{bmatrix}^\top M \begin{bmatrix} I \\ Z \end{bmatrix} \succ 0$$

holds for all $Z \in \mathbb{R}^{l\times k}$ such that

$$\begin{bmatrix} I \\ Z \end{bmatrix}^\top N \begin{bmatrix} I \\ Z \end{bmatrix} \succcurlyeq 0$$

if and only if there exist scalars $\alpha \geqslant 0$ and $\beta > 0$ satisfying

$$M - \alpha N \succcurlyeq \begin{bmatrix} \beta I & 0 \\ 0 & 0 \end{bmatrix}.$$

Lemma 3 (matrix Finsler's lemma, [84, Theorem 1]). *Let matrices $M, N \in \mathbb{S}^{k+l}$ be specified by (40), and let the following conditions be valid:*

1) $M_{11} = 0$ *and* $M_{22} \preccurlyeq 0$.

2) $N_{22} \preccurlyeq 0$ *and* $N_{11} - N_{12}N_{22}^{\dagger}N_{12}^{\top} = 0$.

3) *There exists a matrix* G *such that* $M_{11} + G^{\top}M_{22}G \succ 0$ *and* $N_{22}G = N_{12}^{\top}$.

Then the inequality

$$\begin{bmatrix} I \\ Z \end{bmatrix}^{\top} M \begin{bmatrix} I \\ Z \end{bmatrix} \succcurlyeq 0$$

holds for all $Z \in \mathbb{R}^{l\times k}$ *such that*

$$\begin{bmatrix} I \\ Z \end{bmatrix}^{\top} N \begin{bmatrix} I \\ Z \end{bmatrix} = 0$$

if and only if there exists a scalar $\alpha \in \mathbb{R}$ *satisfying*

$$M - \alpha N \succcurlyeq 0.$$

Now we proceed to the known data-driven methods for systems with bounded exogenous disturbances.

4.2. Quadratic Stabilization of Systems with Exogenous Disturbances

If the matrices A and B_1 in the system with exogenous disturbances

$$x(k+1) = Ax(k) + B_1u(k) + Dw(k), \quad \|w_k\|_2 \leqslant w^*, \quad k = 0, 1, 2, \ldots,$$

are known, then the quadratic stabilization problem of the closed-loop system with the controller $u = Kx$ reduces to finding matrices $P \in \mathbb{S}^n_{++}$ and $K \in \mathbb{R}^{m\times n}$ in the Lyapunov inequality (3) [79], by analogy with the disturbance-free case (Section 2).

In the case of unknown matrices A and B_1, one needs to construct a *common* quadratic Lyapunov matrix P for all systems with matrices $(A, B_1) \in \Sigma_W$ (37).

Below we will show the solution of this problem by data-driven methods using the FL and the linear fractional transformation (Section 4.2.1), the DI concept and QMIs (Section 4.2.2), the FL and Petersen's lemma (Section 4.2.3), and QMIs for each input-output measurement instant (Section 4.2.4). Then we will provide a numerical example to compare these methods.

4.2.1. Fundamental Lemma and Linear Fractional Transformation

A data-driven quadratically stabilizing controller design involving the FL-based parameterization (Section 2.1) and also a linear fractional transformation (LFT) was proposed in [98]. Let the original system be given by

$$x(k+1) = Ax(k) + B_1u(k) + Dw(k), \tag{41}$$

where the notation is the same as in (35).

The authors used an equivalent description of the disturbance set (38) in the form

$$\tilde{\mathbf{W}} = \left\{ W \in \mathbb{R}^{p\times T} : \begin{bmatrix} W \\ I \end{bmatrix}^{\top} \begin{bmatrix} \tilde{\Phi}_{11} & \tilde{\Phi}_{12} \\ \tilde{\Phi}_{12}^{\top} & \tilde{\Phi}_{22} \end{bmatrix} \begin{bmatrix} W \\ I \end{bmatrix} \succcurlyeq 0 \right\}, \tag{42}$$

where $\tilde{\Phi}_{11} \in \mathbb{S}^p$, $\tilde{\Phi}_{12} \in \mathbb{R}^{p\times T}$, and $\tilde{\Phi}_{22} \in \mathbb{S}^T_{++}$.

First, it is necessary to parameterize the set of all systems closed by the controller $u = Kx$. For the system written in measurement terms

$$X_+ = AX_- + B_1U + DW,$$

the set of quadratically stabilizing controllers can be parameterized using the measured data exactly as in (8):

$$\begin{bmatrix} K \\ I \end{bmatrix} = \begin{bmatrix} U \\ X_- \end{bmatrix} G, \quad G \in \mathbb{R}^{T\times n}.$$

Then the gain matrix is computed as $K = UG$ with the condition $X_-G = I$.

Therefore, we obtain the following parameterization of the set of system matrices (41) closed by a controller with the gain matrix $K = UG$ based on the measured data:

$$A_G = (X_+ - DW)G \tag{43}$$

for *some* $W \in \tilde{\mathbf{W}}$ such that $(X_+ - DW)\begin{bmatrix} U \\ X_- \end{bmatrix}^{\perp} = 0$. Here, $^{\perp}$ indicates the kernel basis matrix of an appropriate matrix.

Obviously, in contrast to (10), this parameterization of the closed-loop system matrices contains the unknown disturbance matrix W as a parameter, i.e., it is not unique.

Next, it is necessary to consider a broader set of all stable matrices of the form (43) with *any* admissible disturbances from the set $\tilde{\mathbf{W}}$:

$$A_G^{st} = \left\{ A_K : A_K = (X_+ - DW)G,\ W \in \tilde{\mathbf{W}} \right\}. \tag{44}$$

Although this transition leads to more conservative results, it avoids computing the kernel of the matrix $\begin{bmatrix} U^\top & X_-^\top \end{bmatrix}^\top$ and allows applying a more convenient parameterization and standard control methods. For instance, matrices from (44) are represented as a lower linear fractional transformation of the closed-loop original system (41) with the disturbance $W \in \tilde{\mathbf{W}}$:

$$\begin{bmatrix} x(k+1) \\ \tilde{z}(k) \end{bmatrix} = \begin{bmatrix} X_+G & D \\ -G & 0 \end{bmatrix} \begin{bmatrix} x(k) \\ \tilde{w}(k) \end{bmatrix},$$
$$\tilde{w}(k) = W\tilde{z}(k),$$
$$X_-G = I.$$

With the above representation, the following is the case.

Proposition 2 [98, Proposition 6]. *If there exist matrices* $\Psi \succ 0$ *and* $G \in \mathbb{R}^{T\times n}$ *satisfying*

$$\begin{bmatrix} * & * \\ * & * \\ * & * \\ * & * \end{bmatrix}^\top \begin{bmatrix} -\Psi & 0 & 0 & 0 \\ * & -\Psi & 0 & 0 \\ * & * & \tilde{\Phi}_{11} & \tilde{\Phi}_{12} \\ * & * & * & \tilde{\Phi}_{22} \end{bmatrix} \begin{bmatrix} I & 0 \\ X_+G & D \\ 0 & I \\ -G & 0 \end{bmatrix} \prec 0,$$
$$X_-G = I,$$

then the controller with the gain matrix $K = UG$ *is stabilizing for all* $(A, B_1) \in \Sigma_W$.

After a congruence transformation, application of the Schur complement lemma, and a change of variables, one arrives at the final result.

Proposition 3 [98, Remark 7]. *If the data (U, X) satisfy the rank condition of the FL, then the solution of the problem*

$$\begin{bmatrix} -P & * & * & * \\ -\tilde{\Phi}_{12}Y & \tilde{\Phi}_{11} & * & * \\ X_+Y & D & -P & * \\ Y & 0 & 0 & -\tilde{\Phi}_{22}^{-1} \end{bmatrix} \prec 0,$$

$$X_-Y = P$$

with respect to the matrix variables $P \in \mathbb{S}^n_{++}$ and $Y \in \mathbb{R}^{T\times n}$ yields the quadratically stabilizing controller $\hat{K} = U\hat{Y}\hat{P}^{-1}$ for all systems in Σ_W.

As noted above, this approach is conservative: the parameterization (43) contains the parameter W, and moreover, a common Lyapunov function is constructed for the entire set of systems $(A, B_1) \in \Sigma_W$ consistent with the measurements (U, X).

Here, the dimensions of the matrix variable Y depend on the amount of data T, and since the method is based on the Fundamental Lemma, the rank condition (7) must hold, i.e., $T \geqslant n + m$.

4.2.2. Data Informativity Condition Based on Quadratic Matrix Inequalities

The problem of designing a state-feedback control law $u = Kx$ that quadratically stabilizes the system with exogenous disturbances

$$x(k+1) = Ax(k) + B_1u(k) + w(k) \tag{45}$$

(in the same notation as (35)) was considered in [83, 84]. The authors addressed the case $D = I$; other cases require a simple modification of the method [19, 82, 83].

First, it is necessary to write the problem constraints as QMIs. For this purpose, using the system representation with unknown matrices A, B_1, and W and the measured data (U, X), one expresses

$$W = X_+ - AX_- - B_1U$$

and substitutes this formula into the QMI describing the set of admissible disturbances (38) to get

$$\begin{bmatrix} I \\ A^\top \\ B_1^\top \end{bmatrix}^\top \begin{bmatrix} I & X_+ \\ 0 & -X_- \\ 0 & -U \end{bmatrix} \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{12}^\top & \Phi_{22} \end{bmatrix} \begin{bmatrix} I & X_+ \\ 0 & -X_- \\ 0 & -U \end{bmatrix}^\top \begin{bmatrix} I \\ A^\top \\ B_1^\top \end{bmatrix} \succcurlyeq 0. \tag{46}$$

Next, the quadratic stabilization condition for all systems $(A, B_1) \in \Sigma$ using a common Lyapunov function is reduced to a similar form:

$$(A + B_1K)P(A + B_1K)^\top - P \prec 0, \quad P \in \mathbb{S}^n_{++}.$$

This yields the QMI with respect to the unknown matrices A and B_1 :

$$\begin{bmatrix} I \\ A^\top \\ B_1^\top \end{bmatrix}^\top \begin{bmatrix} P & * & * \\ 0 & -P & * \\ 0 & -KP & -KPK^\top \end{bmatrix} \begin{bmatrix} I \\ A^\top \\ B_1^\top \end{bmatrix} \succ 0. \tag{47}$$

Finally, the strict matrix S-procedure (Lemma 2, Section 4.1) with the matrices M from (47) and N from (46) is applied to the above QMIs. As a result, one arrives at the following necessary and sufficient condition for Data Informativity for the quadratic stabilization of the system with exogenous disturbances.

Proposition 4 [84, Proposition 4]. *If the Slater condition*

$$\begin{bmatrix} I \\ Z \end{bmatrix}^\top \begin{bmatrix} I & X_+ \\ 0 & -X_- \\ 0 & -U \end{bmatrix} \Phi \begin{bmatrix} I & X_+ \\ 0 & -X_- \\ 0 & -U \end{bmatrix}^\top \begin{bmatrix} I \\ Z \end{bmatrix} \succ 0$$

holds for some matrix $Z \in \mathbb{R}^{(n+m)\times n}$, then the data (U, X) generated by system (45) are informative for quadratic state-feedback stabilization if and only if there exist matrices $P \in \mathbb{S}^n_{++}$ and $Y \in \mathbb{R}^{m\times n}$ and a scalar $\beta > 0$ such that

$$\begin{bmatrix} P-\beta I & * & * & * \\ 0 & -P & * & * \\ 0 & -Y & 0 & * \\ 0 & 0 & Y^\top & P \end{bmatrix} - \begin{bmatrix} I & X_+ \\ 0 & -X_- \\ 0 & -U \\ 0 & 0 \end{bmatrix} \Phi \begin{bmatrix} I & X_+ \\ 0 & -X_- \\ 0 & -U \\ 0 & 0 \end{bmatrix}^\top \succcurlyeq 0. \tag{48}$$

For such $\hat{P}$ and $\hat{Y}$ with some $\hat{\beta}$, the stabilizing controller has the form $\hat{K} = \hat{Y}\hat{P}^{-1}$ for all $(A, B_1) \in \Sigma_W$.

In this approach, the measurement length is restricted by the full rank condition for X_-, which stems from the DI condition for the existence of a controller stabilizing the entire set of systems $(A, B_1) \in \Sigma_W$ [18]. Thus, $T \geqslant n$.

Moreover, if the hypotheses of Proposition 4 are supplemented by $\Phi_{22} \prec 0$ and $\operatorname{rank}[U^\top \ X_-^\top]^\top = n + m$ (i.e., the Fundamental Lemma holds), then condition (48) becomes a strict inequality [19]:

$$\begin{bmatrix} P & * & * & * \\ 0 & -P & * & * \\ 0 & -Y & 0 & * \\ 0 & 0 & Y^\top & P \end{bmatrix} - \begin{bmatrix} I & X_+ \\ 0 & -X_- \\ 0 & -U \\ 0 & 0 \end{bmatrix} \Phi \begin{bmatrix} I & X_+ \\ 0 & -X_- \\ 0 & -U \\ 0 & 0 \end{bmatrix}^\top \succ 0. \tag{49}$$

Note that its solution yields a controller of the same form: $\hat{K} = \hat{Y}\hat{P}^{-1}$.

Remark 3. In the above formulas, for the sake of simplicity, the original scalar α [83] has been included in the linear matrix variables P, Y, and β.

Remark 4. According to [83, 84], quadratic stabilization with exact data (see Theorem 4 in Section 2) is a special case of the above proposition (for $\Phi_{11} = 0$, $\Phi_{12} = 0$, and $\Phi_{22} = -I$). However, the Slater condition fails here because the matrix

$$N = \begin{bmatrix} I & X_+ \\ 0 & -X_- \\ 0 & -U \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} I & X_+ \\ 0 & -X_- \\ 0 & -U \end{bmatrix}^\top$$

is not positive definite. Nevertheless, one can establish a connection between these two cases using Finsler's matrix lemma (see Lemma 3 in Section 4.1), which does not require the Slater condition [84]. This result, discussed in detail in [84], has been presented in Section 2.3.

As noted by the authors, the method is non-conservative and convenient for large datasets since the dimension of the LMIs does not depend on T. Indeed, with the constraints $P \succ 0$ and $\beta > 0$, the linear matrix inequality (48) has dimension $4n + m + 1$ and contains $\frac{n(n+1)}{2} + nm + 1$ unknowns. This result can be improved in terms of computational complexity, so that the LMI dimension becomes $4n + 1$ with $\frac{n(n+1)}{2} + 1$ unknowns; see [20, Theorem 3].

4.2.3. Fundamental Lemma and Petersen's lemma

The system considered in [94] has the same form as in the previous subsection:

$$x(k+1) = Ax(k) + B_1u(k) + w(k), \tag{50}$$

where the set of bounded-energy disturbances $w(k) \in \mathbb{R}^n$ is described by the QMI

$$\bar{\mathbf{W}} = \left\{W \in \mathbb{R}^{n\times T} : \begin{bmatrix} I & W \end{bmatrix} \begin{bmatrix} -\Delta\Delta^\top & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} I \\ W^\top \end{bmatrix} \preccurlyeq 0\right\}, \quad \Delta > 0. \tag{51}$$

Obviously, this description is equivalent to (39) with $\Phi_{11} = \Delta\Delta^\top$, but a more general form (38) is also possible (omitted here).

In contrast to the results presented in Section 4.2.2, the authors imposed the FL condition (7) and proposed a *matrix ellipsoid* description for the set of all systems (A, B_1) consistent with the original data (U, X) under disturbances $W \in \bar{\mathbf{W}}$. For instance, using the QMI from (51), we obtain the set of systems

$$\begin{aligned}\Sigma_E &= \left\{(A, B_1) : \begin{bmatrix} I & A & B_1 \end{bmatrix} \begin{bmatrix} \mathbf{C} & * \\ \mathbf{B} & \mathbf{A} \end{bmatrix} \begin{bmatrix} I \\ A^\top \\ B_1^\top \end{bmatrix} \preccurlyeq 0\right\} \\ &= \{[A\ B_1] = Z^\top : \mathbf{C} + \mathbf{B}^\top Z + Z^\top\mathbf{B} + Z^\top\mathbf{A}Z \preccurlyeq 0\},\end{aligned} \tag{52}$$

where

$$\mathbf{A} = \begin{bmatrix} X_- \\ U \end{bmatrix} \begin{bmatrix} X_- \\ U \end{bmatrix}^\top, \quad \mathbf{B} = -X_+ \begin{bmatrix} X_- \\ U \end{bmatrix}^\top, \quad \text{and} \quad \mathbf{C} = X_+X_+^\top - \Delta\Delta^\top.$$

The FL condition ensures $\mathbf{A} \succ 0$, and this matrix ellipsoid can therefore first be written in a more convenient form:

$$\Sigma_E = \{[A\ B_1] = Z^\top : (Z - \zeta)^\top\mathbf{A}(Z - \zeta) \preccurlyeq Q\},$$

where $\zeta := -\mathbf{A}^{-1}\mathbf{B}$ and $\mathbf{Q} := \mathbf{B}^\top\mathbf{A}^{-1}\mathbf{B} - \mathbf{C}$. After proving that $\mathbf{Q} \succcurlyeq 0$ [94], one can pass to an equivalent parameterization of the system set Σ_E (52) as a matrix ellipsoid as well:

$$\mathcal{E} = \left\{(\zeta + \mathbf{A}^{-1/2}\Upsilon\mathbf{Q}^{1/2})^\top : \|\Upsilon\| \leqslant 0\right\}. \tag{53}$$

Next, it is necessary to use the Lyapunov function-based quadratic stabilization condition

$$P - (A + B_1K)P(A + B_1K)^\top \succ 0$$

with respect to the unknown matrices $P \in \mathbb{S}^n_{++}$ and Y, which is reduced to a QMI with respect to the unknown matrices A and B_1 :

$$P - \begin{bmatrix} A & B_1 \end{bmatrix} \begin{bmatrix} I \\ K \end{bmatrix} PP^{-1}P \begin{bmatrix} I \\ K \end{bmatrix}^\top \begin{bmatrix} A & B_1 \end{bmatrix}^\top \succ 0. \tag{54}$$

The considerations below employ Petersen's lemma [99, 100] in its strict form.

Lemma 4 (strict Petersen's lemma). *Let $G \in \mathbb{S}^n$, $M \neq 0$, $N \neq 0$ be matrices of appropriate dimensions, $\tilde{\Delta} \in \mathbb{S}^q$, $\tilde{\Delta} \succcurlyeq 0$, and let*

$$\mathcal{F} := \{\Delta \in \mathbb{R}^{p\times q} : \Delta^\top \Delta \preccurlyeq \tilde{\Delta}\}.$$

Then the inequality

$$G + M\Delta N + N^\top \Delta^\top M^\top \prec 0$$

holds for all $\Delta \in \mathcal{F}$ if and only if there exists $\varepsilon > 0$ such that

$$G + \varepsilon M M^\top + \varepsilon^{-1} N^\top \tilde{\Delta} N^\top \prec 0.$$

The parameterization (53), condition (54), and strict Petersen's lemma were jointly applied to establish the following necessary and sufficient condition for a data-driven design of a quadratically stabilizing controller. (Here, for the sake of convenience and brevity, we directly express this condition in terms of the original data (U, X).)

Theorem 6 [94, Theorem 1]. *For the measured data (U, X) of system (50) satisfying the rank condition of the FL, the solution of the LMI (if it exists)*

$$\begin{bmatrix} P + X_+X_+^\top - \Delta\Delta^\top & * & * \\ 0 & P & * \\ X_+ \begin{bmatrix} X_- \\ U \end{bmatrix}^\top & -\begin{bmatrix} P \\ Y \end{bmatrix} & \begin{bmatrix} X_- \\ U \end{bmatrix}\begin{bmatrix} X_- \\ U \end{bmatrix}^\top \end{bmatrix} \succ 0 \tag{55}$$

with respect to the unknown matrices $P \in \mathbb{S}^n_{++}$ and $Y \in \mathbb{R}^{m\times n}$ yields the quadratically stabilizing controller $\hat{K} = \hat{Y}\hat{P}^{-1}$ for all systems $(A, B_1) \in \Sigma_E$.

Upon applying the Schur complement lemma, condition (55) becomes very similar to condition (48) of the previous subsection. However, there are substantial differences. For instance, in the derivation of (48), the Slater condition was used without the rank condition of the FL; hence, the method from Section 4.2.2 can cover the case of an unbounded set Σ_E, but it does not directly include the exact-data case. (However, as already noted, this becomes possible by using some modifications and Finsler's matrix lemma [84, Theorem 2].)

Since the proposed method is based on the Fundamental Lemma, the rank condition requires $T \geqslant n + m$ measurements. Nevertheless, an undoubted advantage is that the sizes of the variables in (55) do not depend on the length of the data used.

4.2.4. Quadratic Matrix Inequalities for Each Input-Output Measurement Instant

In the paper [101], which appeared slightly earlier than [94] (see the previous subsection), the same authors also used a matrix-ellipsoid representation of the reachable set of system (50) and a quadratically stabilizing controller design very close to that considered in Section 4.2.2 [83, 84]. However, note essential differences of the two approaches that significantly reduce the degree of conservatism: the constraints in [101] apply to exogenous disturbances at each time instant, $\|w_k\|_2 \leqslant \varepsilon$, $k = 0, 1, \ldots, T-1$, whereas the previous methods typically assume bounded-energy disturbances over the entire sequence $W = [w(0) \quad w(1) \quad \ldots \quad w(T-1)] : WW^\top \preccurlyeq \varepsilon^2 TI$, which increases the degree of conservatism with T.

Due to such constraints, the set of systems (A, B_1) capable of generating the data (U, X) can be described by separate relations for each time instant:

$$\Sigma_i = \{(A, B_1) : x_{i+1} = Ax_i + B_1u_i + w_i, \quad w_i w_i^\top \preccurlyeq \varepsilon^2 I\} \quad i = 0, \ldots, T-1.$$

Therefore, we obtain a quadratic inequality of the form (46) for each measurement instant:

$$\begin{bmatrix} I \\ A^\top \\ B_1^\top \end{bmatrix}^\top \begin{bmatrix} I & x_{i+1} \\ 0 & -x_i \\ 0 & -u_i \end{bmatrix} \Phi \begin{bmatrix} I & x_{i+1} \\ 0 & -x_i \\ 0 & -u_i \end{bmatrix}^\top \begin{bmatrix} I \\ A^\top \\ B_1^\top \end{bmatrix} \succcurlyeq 0, \quad i = 0, \dots, T-1,$$

where the matrices (38) describe the set of admissible disturbances:

$$\Phi = \begin{bmatrix} \varepsilon^2 I & 0 \\ 0 & -1 \end{bmatrix} \in \mathbb{S}^{(n+1)\times(n+1)}.$$

In this case, the intersection

$$\Sigma_c = \bigcap_{i=0}^{T-1} \Sigma_i$$

yields the set of *all* systems (A, B_1) that can generate *all the obtained measurements* at time instants $i = 0, 1, \dots, T-1$.

According to [101, Proposition 3], the set given by the above intersection is a *subset* of all systems of the form

$$\begin{bmatrix} I \\ A^\top \\ B_1^\top \end{bmatrix}^\top \begin{bmatrix} I & X_+ \\ 0 & -X_- \\ 0 & -U \end{bmatrix} \begin{bmatrix} \varepsilon^2 TI & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} I & X_+ \\ 0 & -X_- \\ 0 & -U \end{bmatrix}^\top \begin{bmatrix} I \\ A^\top \\ B_1^\top \end{bmatrix} \succcurlyeq 0$$

(i.e., those using vectors composed of the measurements, assuming bounded-energy exogenous disturbances, see (46)) and, consequently, a subset of the system set Σ_E (52). Thus, the method considered in this subsection is less conservative than those in Sections 4.2.2 and 4.2.3 and gives more accurate results for a larger number of measurements T. (For comparisons and proofs, we refer to [101].)

The authors proposed a method for directly representing the reachable set from measured data, which ultimately leads to a quadratically stabilizing controller design. We omit the main derivations here: they are analogous to those presented in Section 4.2.2, taking into account the relations for each measurement instant. In fact, the following result was established.

Proposition 5 [101, Proposition 1]. *Let the rank condition* (7) *of the FL hold. Then the solution of the problem*

$$\begin{bmatrix} P - \beta I & * & * & * \\ 0 & -P & * & * \\ 0 & -Y & 0 & * \\ 0 & 0 & Y^\top & P \end{bmatrix} - \sum_{i=0}^{T-1} \tau_i \begin{bmatrix} I & x_{i+1} \\ 0 & -x_i \\ 0 & -u_i \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \varepsilon^2 I & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} I & x_{i+1} \\ 0 & -x_i \\ 0 & -u_i \\ 0 & 0 \end{bmatrix}^\top \succcurlyeq 0,$$
$$i = 0, \dots, T-1,$$

with respect to the unknown matrices $P \in \mathbb{S}^n_{++}$ and $Y \in \mathbb{R}^{m\times n}$ and scalar variables $\beta > 0$ and $\tau_i > 0$, $i = 0, \dots, T-1$, yields the controller $\hat{K} = \hat{Y}\hat{P}^{-1}$ that quadratically stabilizes all systems $(A, B_1) \in \Sigma_c$.

Obviously, the dimensions of the matrix variables in this problem do not depend on the length of the experimental data vectors, but the number of scalar variables grows with the number of measurements used. Generally, the FL conditions are not required initially, so the measurement

length restriction follows from the Data Informativity condition for stabilization: $T \geqslant n$. However, according to [101, Remark 1], $T \geqslant n + m$ ensures the closedness of the reachable set and facilitates the solution procedure of the LMIs. Moreover, increasing the number of measurements improves the solution accuracy but may also cause computational challenges. Therefore, when dealing with high-dimensional problems and large datasets, one should choose a trade-off between accuracy and computational complexity, or apply other methods.

Generally speaking, the use of individual measurements instead of their vectors distinguishes this method from most approaches covered by this survey.

Example 3. Consider the batch chemical reactor from Example 1. To model bounded exogenous disturbances $\|w_k\|_2 \leqslant \varepsilon$, $k = 0, \dots, T-1$, we set $D = I$ and $T = 10$.

Since all examples in this survey are illustrative, the results will be presented for a single realization of the exogenous disturbance W with a very small ε to obtain a solution by all methods. We chose $\varepsilon = 0.001$, generated a sequence of values $\check{w}(k) \in \mathbb{R}^n$ with the uniform distribution on $[-1, 1]$, and took

$$w(k) = \varepsilon \frac{\check{w}(k)}{\|\check{w}(k)\|_2}, \quad k = 0, 1, 2, \dots.$$

The input vector U and the initial condition $x(0)$ were the same as in Example 1.

In the conventional MBC approach, in the presence of bounded exogenous disturbances in system (36) with $\|w(k)\|_2 \leqslant 1$ (and with $D = I$, as in the current case), the controller is given by the solution of the same problem (3) as in the disturbance-free case:

$$\hat{K}^{st}_{mbc} = \begin{bmatrix} 0.6377 & -1.4696 & 0.0944 & -1.2149 \\ 3.0471 & 0.0622 & 2.9947 & -0.7239 \end{bmatrix}.$$

Solving the problem based on the linear fractional transformation (Proposition 3 of Section 4.2.1) yielded the controller

$$\hat{K}^{st}_{lft} = \begin{bmatrix} 1.0018 & -1.4901 & 0.1370 & -1.8788 \\ 4.1643 & -0.0818 & 3.2851 & -2.5132 \end{bmatrix}.$$

In this case, the experiment duration and the input signal were appropriately chosen to fulfill the FL; therefore, in the DI-based method with QMIs (Proposition 4 of Section 4.2.2), we used inequality (49). As a result,

$$\hat{K}^{st}_{qmi} = \begin{bmatrix} 0.9949 & -1.5262 & 0.1471 & -1.9208 \\ 4.1617 & -0.0413 & 3.2424 & -2.4442 \end{bmatrix}.$$

For stabilization using Petersen's lemma, we obtained the uncertainty $\Delta = \varepsilon\sqrt{T}I = 0.0032 I_4$, and the solution of the problem from Theorem 6 (Section 4.2.3) yielded the controller

$$\hat{K}^{st}_{pet} = \begin{bmatrix} 0.9894 & -1.5534 & 0.1013 & -1.9177 \\ 4.1600 & 0.0096 & 3.2913 & -2.4432 \end{bmatrix}.$$

The method with individual QMIs for each measurement instant (Proposition 5 of Section 4.2.4) gives a more accurate result for a larger number of measurements, so we set $T = 20$ to obtain twenty scalar variables $\hat{\tau}_i$ (omitted here for brevity) and $\hat{\beta} = 9.9971 \cdot 10^{-7}$. With these values, the controller had the gain matrix

$$\hat{K}^{st}_{ins} = \begin{bmatrix} 0.6576 & -1.4449 & 0.1234 & -1.1973 \\ 3.0850 & 0.0522 & 3.0250 & -0.7318 \end{bmatrix}.$$

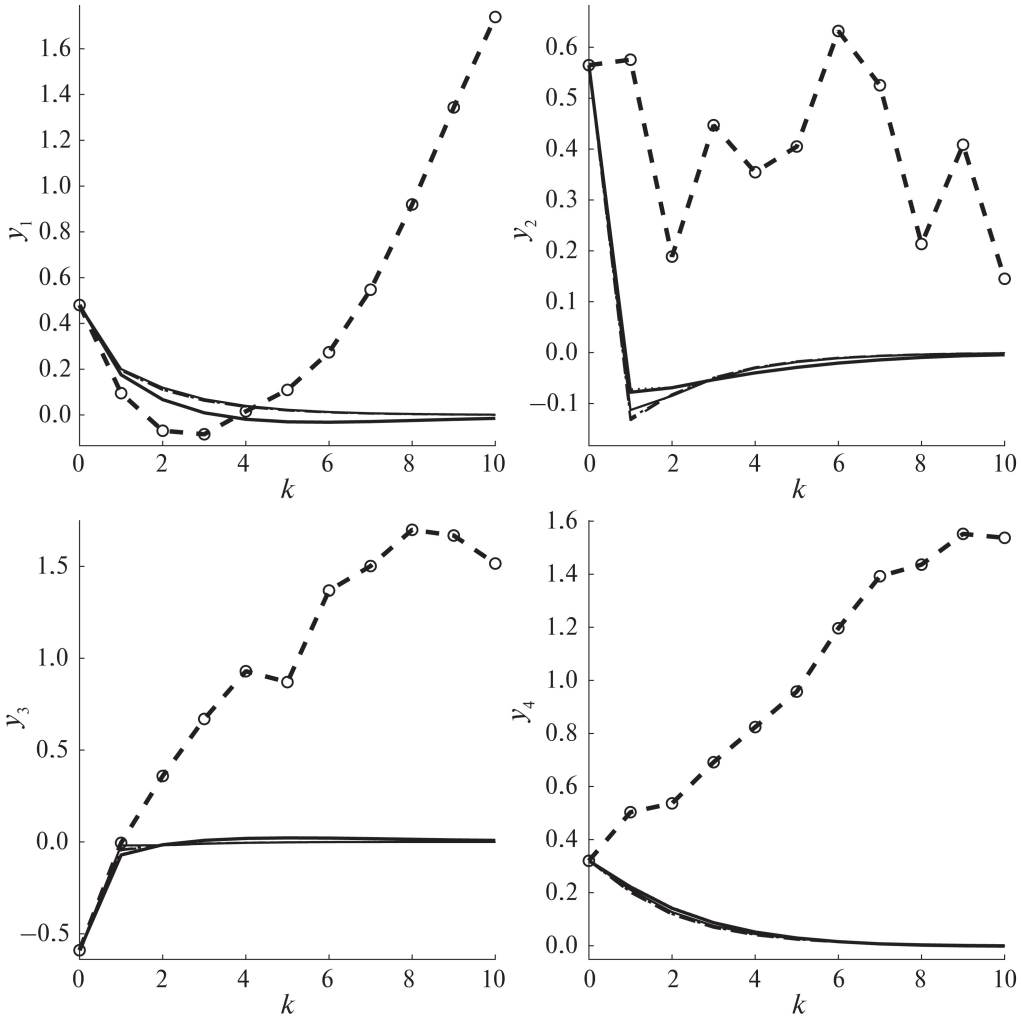


Fig. 3. System outputs in the quadratic stabilization problem under exogenous disturbances.

Figure 3 shows the following trajectories: the outputs $y_i(k)$, $i = 1, 2, 3, 4$, of the open-loop system (thick dashed line, with circles marking the measurements X used) and those of the closed-loop system, namely, the outputs $y_{mbc}(k)$ (the MBC solution, thick solid line), $y_{lft}(k)$ (the LFT-based method, thin solid line), $y_{qmi}(k)$ (the DI-based method with QMIs, thin dashed line), $y_{pet}(k)$ (Petersen's lemma, thin dash-dotted line), and $y_{ins}(k)$ (QMIs for individual measurements, thin dash-dotted line).

The closeness estimates of the trajectories were as follows:

$$\|y_{mbc}(k) - y_{lft}(k)\|_2 = 0.1409, \quad \|y_{mbc}(k) - y_{qmi}(k)\|_2 = 0.1328,$$
$$\|y_{mbc}(k) - y_{pet}(k)\|_2 = 0.1338, \quad \|y_{mbc}(k) - y_{ins}(k)\|_2 = 0.0079, \quad k = 0, 1, 2, \ldots.$$

Clearly, all the methods stabilized the system, i.e., the control objective was achieved in all cases: the problem formulation has no accuracy requirements (in terms of the closeness of controller matrices/trajectories to their MBC counterparts). However, in this example, the most accurate solution was provided by the method with individual measurements, which matches expectations owing to its conservatism-reducing structure.

The readers interested in a deeper performance analysis of the methods for systems with exogenous disturbances under larger uncertainty regions and various disturbance realizations (particularly, in the dependence of guaranteed performance bounds on disturbance levels and experiment duration) are referred to the original publications.

4.3. $\mathcal{H}_2$-Optimal Control

Consider a linear discrete time-invariant system with exogenous disturbances of the form

$$\begin{aligned} x(k+1) &= Ax(k) + B_1u(k) + Dw(k), \\ z(k) &= Cx(k) + B_2u(k), \end{aligned} \tag{56}$$

where $x(k) \in \mathbb{R}^n$, $u(k) \in \mathbb{R}^m$, $z(k) \in \mathbb{R}^l$, and $w(k) \in \mathbb{R}^p$; $A \in \mathbb{R}^{n\times n}$ and $B_1 \in \mathbb{R}^{n\times m}$; the matrices $C \in \mathbb{R}^{l\times n}$, $B_2 \in \mathbb{R}^{l\times m}$, and $D \in \mathbb{R}^{n\times p}$ are given.

The system is closed by a state-feedback controller $u = Kx$. The $\mathcal{H}_2$ norm of the transfer function from w to z is defined as

$$G(z) = (C + B_2K)(zI - (A + B_1K))^{-1}D.$$

The $\mathcal{H}_2$-suboptimal control problem is to find a control law $u = Kx$ such that $\|G(z)\|_{\mathcal{H}_2} \leqslant \gamma$ for some $\gamma > 0$. In the case of the smallest value $\gamma = \gamma^*$, we obtain $\mathcal{H}_2$-optimal control and $\|G(z)\|_{\mathcal{H}_2} = \gamma^*$.

When the matrices A and B_1 are known, the solution is given by the following statement: the matrix $A + B_1K$ is Schur stable and $\|G(z)\|_{\mathcal{H}_2} = \gamma$ if and only if there exists a matrix $P \in \mathbb{S}^n_{++}$ such that

$$\begin{gathered} P \succ (A + B_1K)^\top P(A + B_1K) + (C + B_2K)^\top (C + B_2K), \\ \operatorname{trace} D^\top PD < \gamma^2. \end{gathered} \tag{57}$$

In terms of linear matrix inequalities, the problem is formulated as follows [102].

For system (56), there exists an $\mathcal{H}_2$-optimal control law of the form $u = Kx$ if and only if the convex programming problem

$$\begin{gathered} \min_{P,Y,Z} \gamma, \\ \begin{bmatrix} P & AP + B_1Y & D \\ * & P & 0 \\ * & * & I \end{bmatrix} \succ 0, \quad \begin{bmatrix} Z & CP + B_2Y \\ * & P \end{bmatrix} \succ 0, \\ \operatorname{trace} Z < \gamma^2, \quad \gamma > 0 \end{gathered} \tag{58}$$

with respect to the matrix variables $P \in \mathbb{S}^n_{++}$, $Y \in \mathbb{R}^{m\times n}$, and $Z \in \mathbb{S}^p_{++}$ is solvable. In this case, $\hat{K} = \hat{Y}\hat{P}^{-1}$ and $\|G(z)\|_{\mathcal{H}_2} = \hat{\gamma}$.

If the matrices A, B_1 are unknown, we obtain an analogous formulation: from the measurements U and X of the system signals, it is required to design a static state-feedback control law so that the closed-loop system is stable and $\|G(z)\|_{\mathcal{H}_2} \leqslant \gamma$ for the entire set of systems consistent with these measurements.

In the following subsections, we present methods for solving this problem based on the DI condition and QMIs (Section 4.3.1), suboptimal control methods with the DI condition (Section 4.3.2), a method based on the FL with separation of system matrices and the Lyapunov matrix in the LMI (Section 4.3.3), and a method involving QMIs for individual input-output measurement instants (Section 4.3.4).

4.3.1. Data Informativity Condition Based on Quadratic Matrix Inequalities

A method for designing an $\mathcal{H}_2$-suboptimal controller using a parameterization of the closed-loop system matrices consistent with experimental data was proposed in [19, 83]. The main idea is to employ, within the DI framework, QMIs to describe both the set of corresponding systems and control objectives.

Consider system (56) with $D = I$ and exogenous disturbances w described by (38). The case $D \neq I$ requires minor modifications, see [19, 82, 83].

First, condition (57) is transformed to a QMI in the unknowns A and B_1, and then the strict matrix S-procedure (Lemma 2 of Section 4.1) is applied to the resulting expression. This approach leads to the following result.

Theorem 7 [83, Theorem 17]. *If the Slater condition*

$$\begin{bmatrix} I \\ \tilde{Z} \end{bmatrix}^\top \begin{bmatrix} I & X_+ \\ 0 & -X_- \\ 0 & -U \end{bmatrix} \Phi \begin{bmatrix} I & X_+ \\ 0 & -X_- \\ 0 & -U \end{bmatrix}^\top \begin{bmatrix} I \\ \tilde{Z} \end{bmatrix} \succ 0$$

holds for some matrix $\tilde{Z} \in \mathbb{R}^{(n+m)\times n}$, then the data (U, X) generated by system (56) are informative for $\mathcal{H}_2$-suboptimal control with $\gamma > 0$ if and only if there exist matrices $P \in \mathbb{S}^n_{++}$, $Y \in \mathbb{R}^{m\times n}$, and $Z \in \mathbb{S}^n_{++}$ and scalars $\alpha \geqslant 0$ and $\beta > 0$ such that

$$\begin{bmatrix} P-\beta I & * & * & * & * \\ 0 & 0 & * & * & * \\ 0 & 0 & 0 & * & * \\ 0 & P & Y^\top & P & * \\ 0 & 0 & 0 & CP+B_2Y & I \end{bmatrix} - \alpha \begin{bmatrix} I & X_+ \\ 0 & -X_- \\ 0 & -U \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \Phi \begin{bmatrix} I & X_+ \\ 0 & -X_- \\ 0 & -U \\ 0 & 0 \\ 0 & 0 \end{bmatrix}^\top \succcurlyeq 0,$$

$$\begin{bmatrix} P & * \\ CP+B_2Y & I \end{bmatrix} \succ 0, \quad \begin{bmatrix} Z & * \\ I & P \end{bmatrix} \succcurlyeq 0,$$

$$\operatorname{trace} Z < \gamma^2.$$

For such $\hat{P}$ and $\hat{Y}$ with some $\hat{\alpha}$ and $\hat{\beta}$, the $\mathcal{H}_2$-suboptimal controller has the form $\hat{K} = \hat{Y}\hat{P}^{-1}$ and $\|G(z)\|_{\mathcal{H}_2} < \hat{\gamma}$ for all $(A, B_1) \in \Sigma_W$.

Note that the dimensions of the matrix variables here do not depend on the data amount, which is limited only by the DI condition for stabilization of system (13), i.e., $T \geqslant n$.

4.3.2. $\mathcal{H}_2$-Suboptimal Control Based on the Data Informativity Concept

A necessary and sufficient condition for Data Informativity to design an $\mathcal{H}_2$-suboptimal control law $u = Kx$ for system (56) in the case where *it is possible to measure the values of the exogenous disturbances* (i.e., the vector $W = [w(0) \quad w(1) \quad \dots \quad w(T-1)] \in \mathbb{R}^{p\times T}$ is known) was provided in [91]. This condition fundamentally distinguishes the result discussed in this subsection from the others described by the survey.

Proposition 6 [91, Proposition 15]. *The data (U, X, W) are informative for $\mathcal{H}_2$-suboptimal control if and only if at least one of the following conditions holds for a given value $\gamma > 0$:*

1. *There exists a matrix $Q \in \mathbb{R}^{T\times n}$ such that $X_-Q = (X_-Q)^\top$,*

$$\begin{bmatrix} W_- \\ CX_- + B_2U \end{bmatrix} Q = 0, \quad \textit{and} \quad \begin{bmatrix} X_-Q & * \\ X_+Q & X_-Q \end{bmatrix} \succ 0.$$

2. *There exist a right pseudo-inverse* $W_-^\dagger$ *and matrices* $Y \in \mathbb{S}^n_{++}$ *and* $Q \in \mathbb{R}^{T\times n}$ *such that* $X_-Q = (X_-Q)^\top$, $W_-Q = X_-W_-^\dagger = UW_-^\dagger = 0$, *and*

$$\begin{bmatrix} X_-Q & * & * \\ X_+Q & X_-Q & * \\ (CX_- + B_2U)Q & 0 & I \end{bmatrix} \succ 0, \quad \begin{bmatrix} Y & * \\ X_+W_-^\dagger & X_-Q \end{bmatrix} \succ 0,$$

$$\operatorname{trace}(Y) < \gamma.$$

Moreover, K is an $\mathcal{H}_2$-suboptimal controller for all $(A, B_1) \in \Sigma_W$ and $\|G(z)\|_{\mathcal{H}_2} < \gamma$ if and only if the controller can be written as $\hat{K} = U\hat{Q}(X_-\hat{Q})^{-1}$ for any matrix $\hat{Q}$ satisfying conditions 1 or 2.

Obviously, with this solution method, the dimension of the matrix variable in the LMI depends on the experiment duration T and, hence, on the amount of data used. For this problem, the DI condition imposes the restriction $T \geqslant n$ (Theorem 2 of Section 2.2). Thus, the requirements on the minimum data amount are milder than the FL condition, but computational challenges may arise for large T.

Remark 5. The suboptimal LQR problem from Section 3.3 is a special case of the $\mathcal{H}_2$-suboptimal control problem presented here for $D = x_0$, $CC^\top = Q$, $B_2B_2^\top = R$, and $C^\top B_2 = 0$; however, the two methods slightly differ in the use of the original data in the solution.

The main drawback of this method lies in the assumption that the measurement vector of the exogenous disturbances W is known, which is not often realized in practice. However, when this condition holds, the method shows good results (see Example 4 below).

4.3.3. $\mathcal{H}_2$-Suboptimal Control Based on Quadratic Matrix Inequalities and an Auxiliary Matrix

In the case of a fully known system, the degree of conservatism can be reduced by the well-proved method [103, 104]: an LMI is constructed with separation of the system matrix A and the Lyapunov matrix by introducing an auxiliary matrix variable. For instance, this condition was applied to design a γ-suboptimal $\mathcal{H}_2$ controller, including a fixed-structure (sparse) controller (by adding extra constraints in the system) [105]. Here, the separation procedure is combined with the use of QMIs, and the method rests on the DI concept.

Consider system (56) closed by the controller $u = Kx$.

If the matrices A and B_1 are known, then the parameters of the γ-suboptimal $\mathcal{H}_2$ controller are obtained by solving the following problem [104].

Theorem 8 [104, Theorem 5]. *A controller K is a γ-suboptimal $\mathcal{H}_2$ controller for system* (56) *if and only if there exist matrices $P \in \mathbb{S}^n_{++}$, $Q \in \mathbb{S}^l$, $G \in \mathbb{R}^{n\times n}$, and $Y \in \mathbb{R}^{m\times n}$ such that*

$$\begin{bmatrix} P - DD^\top & AG + B_1Y \\ * & G + G^\top - P \end{bmatrix} \succ 0, \quad \begin{bmatrix} Q & CG + B_2Y \\ * & G + G^\top - P \end{bmatrix} \succ 0,$$

$$\operatorname{trace} Q < \gamma^2,$$

with $K = YG^{-1}$ and $\|G(z)\|_{\mathcal{H}_2} < \gamma$.

Passing to the design problem from measured data (U, X), we assume that the disturbance matrix W belongs to the set (38):

$$\mathbf{W} = \left\{ W \in \mathbb{R}^{n\times T} : \begin{bmatrix} I \\ W^\top \end{bmatrix}^\top \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{12}^\top & \Phi_{22} \end{bmatrix} \begin{bmatrix} I \\ W^\top \end{bmatrix} \succcurlyeq 0 \right\}.$$

Using [104, Theorem 1] and the matrix S-procedure [82], we arrive at the following γ-suboptimal $\mathcal{H}_2$ controller design.

Theorem 9 [105, Theorem 3.2]. *If for the measurements (U, X) of system (56) with exogenous disturbances (38) the problem*

$$\min_{P,Q,G,L,\alpha,\beta,\gamma} \gamma, \tag{59}$$
$$\begin{bmatrix} P - DD^\top - \beta I & 0 & 0 & 0 \\ * & 0 & 0 & G \\ * & * & 0 & Y \\ * & * & * & G + G^\top - P \end{bmatrix} - \alpha \begin{bmatrix} I & X_+ \\ 0 & -X_- \\ 0 & -U \\ 0 & 0 \end{bmatrix}^\top \Phi \begin{bmatrix} I & X_+ \\ 0 & -X_- \\ 0 & -U \\ 0 & 0 \end{bmatrix} \succcurlyeq 0,$$
$$\begin{bmatrix} Q & CG + B_2 Y \\ * & G + G^\top - P \end{bmatrix} \succcurlyeq 0,$$
$$\operatorname{trace} Q \leqslant \gamma^2$$

with respect to the matrices $P \in \mathbb{S}^n_{++}$, $Q \in \mathbb{S}^l$, $G \in \mathbb{R}^{n\times n}$, and $Y \in \mathbb{R}^{m\times n}$ and the scalars $\alpha \geqslant 0$, $\beta > 0$, and $\gamma > 0$ has a solution $\hat\gamma$ for some $\hat Y$ and $\hat G$, then the controller $\hat K = \hat Y \hat G^{-1}$ is a $\hat\gamma$-suboptimal $\mathcal{H}_2$ controller for all systems $(A, B_1) \in \Sigma_W$ and $\|G(z)\|_{\mathcal{H}_2} \leqslant \gamma$.

Note that the multiplier α in the first constraint cannot be eliminated here by scaling the other unknowns because the first term contains the constant matrix $DD^\top$.

Besides conservatism reduction (see above), an advantage of this γ-suboptimal $\mathcal{H}_2$ controller design method is that the computational complexity does not depend on the amount of data used; this amount must satisfy (13), i.e., $T \geqslant n$.

4.3.4. Quadratic Matrix Inequalities for Individual Input-Output Measurement Instants

Here, we outline the approach from [106] for system (56). Its form is largely similar to that of the one discussed in the previous subsection, although the authors do not separate the variable matrices. The main idea is to use constraints on the disturbances at each time instant and then apply QMIs for each measurement instant. Thus, all assumptions of Section 4.2.4 remain valid; in particular, we have

$$\Phi = \begin{bmatrix} \varepsilon^2 I & 0 \\ 0 & -1 \end{bmatrix} \in \mathbb{S}^{(n+1)\times(n+1)},$$
$$\begin{bmatrix} I \\ A^\top \\ B_1^\top \end{bmatrix}^\top \Psi_i \begin{bmatrix} I \\ A^\top \\ B_1^\top \end{bmatrix} \succcurlyeq 0,$$
$$\Psi_i = \begin{bmatrix} I & x_{i+1} \\ 0 & -x_i \\ 0 & -u_i \end{bmatrix} \Phi \begin{bmatrix} I & x_{i+1} \\ 0 & -x_i \\ 0 & -u_i \end{bmatrix}^\top, \quad i = 0, \dots, T-1.$$

Thus, the intersection

$$\Sigma_c = \bigcap_{i=0}^{T-1} \Sigma_i$$

yields the set of *all* systems of the form (56) with matrices (A, B_1) that can generate *all the measurements* at time instants $i = 0, 1, \dots, T-1$.

As claimed by the authors [106], the unique set obtained by this intersection is a *subset* of all systems from [105]. In other words, their method is less conservative and leads to more accurate results.

The subsequent derivations are standard, including the matrix S-procedure, the Schur complement, etc. The resulting problem with respect to the matrix variables $P \in \mathbb{S}^n_{++}$, $Q \in \mathbb{S}^n_{++}$, and $Y \in \mathbb{R}^{m\times n}$ and the scalar variables $\alpha_i \geqslant 0$, $i = 0, 1, \ldots, T-1$, $\beta > 0$, and $\gamma > 0$ has the following form:

$$\min_{P,Y,Q,\alpha_i,\beta,\gamma} \gamma \tag{60}$$

subject to the constraints

$$\begin{bmatrix} P - DD^\top - \beta I & 0 & 0 & 0 \\ * & 0 & 0 & P \\ * & * & 0 & Y \\ * & * & * & P \end{bmatrix} - \begin{bmatrix} \sum_{i=0}^{T-1} \alpha_i \Psi_i & 0 \\ 0 & 0 \end{bmatrix} \succcurlyeq 0, \tag{61}$$

$$\begin{bmatrix} Q & CP + B_2 Y \\ * & P \end{bmatrix} \succ 0,$$

$$\operatorname{trace} Q \leqslant \gamma^2.$$

Its solution gives the $\mathcal{H}_2$-suboptimal controller $\hat{K} = \hat{Y}\hat{P}^{-1}$, with $\|G(z)\|_{\mathcal{H}_2} \leqslant \hat{\gamma}$ for all $(A, B_1) \in \Sigma_c$.

Obviously, here the dimensions of the variables do not depend on the data amount T, but the latter directly determines the number of scalar variables. This may cause computational challenges for large measurement lengths.

The restriction on the experiment duration T follows from (13): $T \geqslant n$.

Remark 6. By [106, Theorem 3], the value γ^* representing the solution of problem (60)–(61) decreases monotonically with the time T.

Also, the authors of [106] proposed an extension of the method (an iterative algorithm) to the case of fixed-structure (sparse) controllers and demonstrated the advantages of the approach compared to the method discussed in Section 4.3.3 [105].

Among other $\mathcal{H}_2$-optimal control design techniques, we could mention, e.g., the linear fractional transformation [107]. Its advantage is that the variables in the semidefinite programming problem do not depend on the amount of data used.

Example 4. We continued modeling the unstable batch chemical reactor from Example 1. Let $C = I$ and $B_2 = 0$, i.e., the performance output of the system was given by

$$z(k) = x(k), \quad k = 0, 1, \ldots.$$

We set the experiment duration $T = 20$ and chose $\varepsilon = 0.001$. (For this value, all the methods under consideration yield a solution, for illustrative purposes.) For the matrix Φ in (38), we had $\Phi_{11} = 0.2 \cdot 10^{-4} I$, $\Phi_{12} = 0$, and $\Phi_{22} = -I$. The input and output measurements and exogenous disturbances were simulated as in Example 3.

The model-based computations using (58) (Section 4.3) yielded

$$\hat{K}^{H_2}_{mbc} = \begin{bmatrix} 0.9674 & -1.5118 & 0.1646 & -1.8382 \\ 4.7270 & -0.0118 & 3.4519 & -1.9159 \end{bmatrix},$$

with the optimal value

$$\hat{\gamma}_{mbc}^{H_2} = 2.7707.$$

Logically, data-driven methods cannot surpass the optimal value $\hat{\gamma}_{mbc}^{H_2}$ obtained with exactly known model parameters, so they naturally produced somewhat larger values of γ.

For instance, computations using the DI condition based on QMIs (Theorem 7 of Section 4.3.1) with the found values of $\hat{\alpha} = 319.54$ and $\hat{\beta} = 0.001$ led to the suboptimal controller

$$\hat{K}_{qmi}^{H_2} = \begin{bmatrix} 0.9745 & -1.5120 & 0.1616 & -1.8590 \\ 4.7494 & -0.0151 & 3.4368 & -1.9681 \end{bmatrix}, \quad \hat{\gamma}_{qmi}^{H_2} = 2.7951.$$

Due to the strict inequality criterion in the suboptimal control problem of the DI-based method (Proposition 6 of Section 4.3.2), we set γ slightly larger than the model-based optimum: $\gamma = 2.7708$. Also recall that in this method, the measurement vector of exogenous disturbances W is known and used directly in the computations. Both of these factors significantly improve the solution accuracy (see Fig. 4 and the closeness estimate below). For instance, we obtained

$$\hat{K}_{sub}^{H_2} = \begin{bmatrix} 0.9673 & -1.5118 & 0.1646 & -1.8378 \\ 4.7263 & -0.0118 & 3.4517 & -1.9151 \end{bmatrix}.$$

In the suboptimal controller design method based on QMIs and an auxiliary matrix in the LMI (59) (Section 4.3.3), the value of γ is not prescribed but computed as the minimum possible during the solution procedure. For instance, we found

$$\hat{K}_{do}^{H_2} = \begin{bmatrix} 0.9521 & -1.5119 & 0.1598 & -1.8413 \\ 4.7003 & -0.0124 & 3.4428 & -1.9355 \end{bmatrix}, \quad \hat{\gamma}_{do}^{H_2} = 2.8028.$$

Considering Remark 6 of Section 4.3.4, we used all 20 available measurements in the method with QMIs for individual input-output measurement instants of system (60)–(61). We obtained the scalar variables $\hat{\alpha}_i$, $i = 1, 2, \ldots, 20$ (omitted here for brevity), $\hat{\beta} = 2.2745 \cdot 10^{-5}$, and the controller

$$\hat{K}_{ins}^{H_2} = \begin{bmatrix} 0.9556 & -1.5116 & 0.1610 & -1.8425 \\ 4.7132 & -0.0132 & 3.4475 & -1.9450 \end{bmatrix}, \quad \hat{\gamma}_{ins}^{H_2} = 2.7957.$$

Figure 4 shows the following trajectories: the performance outputs $z_i(k) = x_i(k)$, $i = 1, 2, 3, 4$, of the open-loop system (thick dashed line, with circles marking the initial measurements X) and those of the closed-loop system, namely, the outputs $z_{mbc}(k)$ (the MBC solution, thin solid line), $z_{qmi}(k)$ (the DI-based method with QMIs, thin dashed line), $z_{sub}(k)$ (the suboptimal controller with the DI conditions, thin dotted line), $z_{do}(k)$ (the auxiliary-matrix method, dash-dotted line), and $z_{ins}(k)$ (the method with QMIs for individual measurements, thin solid line).

We compared the results with the MBC solution:

$$\begin{aligned}
&\|\hat{K}_{mbc}^{H_2} - \hat{K}_{qmi}^{H_2}\|_2 = 0.0629, \quad \|z_{mbc}(k) - z_{qmi}(k)\|_2 = 0.0021, \quad |\hat{\gamma}_{mbc}^{H_2} - \hat{\gamma}_{qmi}^{H_2}| = 0.0245, \\
&\|\hat{K}_{mbc}^{H_2} - \hat{K}_{sub}^{H_2}\|_2 = 0.0011, \quad \|z_{mbc}(k) - z_{sub}(k)\|_2 = 4.9881 \cdot 10^{-5}, \quad \hat{\gamma}_{sub}^{H_2} = 2.7708 (\text{given}), \\
&\|\hat{K}_{mbc}^{H_2} - \hat{K}_{do}^{H_2}\|_2 = 0.0376, \quad \|z_{mbc}(k) - z_{do}(k)\|_2 = 0.0045, \quad |\hat{\gamma}_{mbc}^{H_2} - \hat{\gamma}_{do}^{H_2}| = 0.0321, \\
&\|\hat{K}_{mbc}^{H_2} - \hat{K}_{ins}^{H_2}\|_2 = 0.0340, \quad \|z_{mbc}(k) - z_{ins}(k)\|_2 = 0.0044, \quad |\hat{\gamma}_{mbc}^{H_2} - \hat{\gamma}_{ins}^{H_2}| = 0.0250, \\
&\qquad\qquad k = 0, 1, 2, \ldots.
\end{aligned}$$

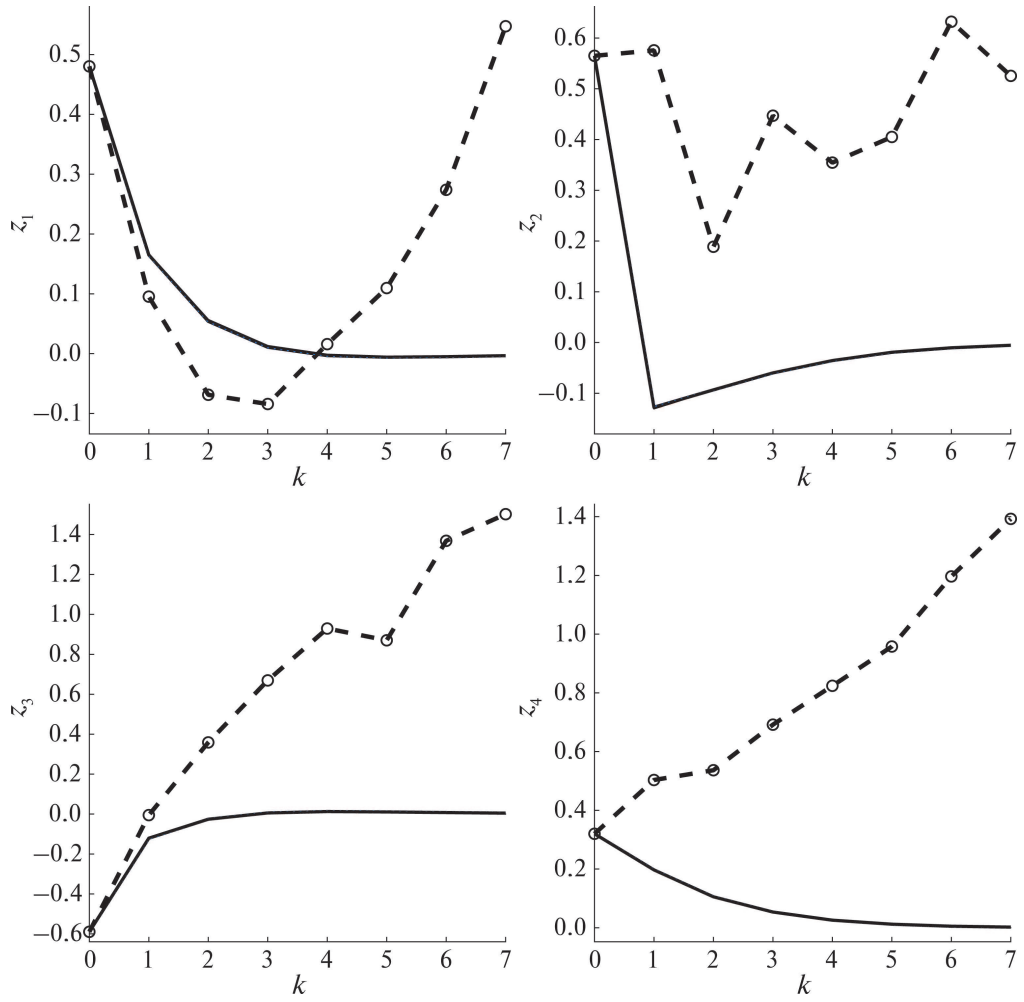


Fig. 4. System outputs in the $\mathcal{H}_2$-optimal control problem.

Evidently, all considered methods coped well with the problem: the controllers $\hat{K}$ and the values of $\hat{\gamma}$ were close to the optimal ones. In particular, in this example with unknown measurements of exogenous disturbances, the method with QMIs and the DI condition, as well as the method with individual QMIs for each time instant, gave more accurate solutions, and their accuracy will improve when increasing the initial experiment duration T. In general, one may use any of the presented methods, depending on the features of the available prior data.

4.4. $\mathcal{H}_\infty$-Optimal Control

Consider a discrete-time system of the form

$$x(k+1) = Ax(k) + B_1u(k) + Dw(k), \qquad (62)$$
$$z(k) = Cx(k) + B_2u(k),$$

where $x(k) \in \mathbb{R}^n$, $u(k) \in \mathbb{R}^m$, and $z(k) \in \mathbb{R}^l$, $w(k) \in \mathbb{R}^p$; $A \in \mathbb{R}^{n\times n}$ and $B_1 \in \mathbb{R}^{n\times m}$; the matrices $C \in \mathbb{R}^{l\times n}$, $B_2 \in \mathbb{R}^{l\times m}$, and $D \in \mathbb{R}^{n\times p}$ are given. It is required to design a control law $u = Kx$ that stabilizes the closed-loop system with the matrices

$$A_K = A + B_1K, \quad C_K = C + B_2K \qquad (63)$$

and simultaneously minimizes the $\mathcal{H}_\infty$ norm of the transfer function from the disturbance w to the output z :

$$G(z) = C_K(zI - A_K)^{-1}D.$$

If $\|G(z)\|_{\mathcal{H}_\infty} \leqslant \gamma$ for some $\gamma > 0$, we have an $\mathcal{H}_\infty$-suboptimal controller; in the case of the smallest value $\gamma = \gamma^*$, this statement leads to $\mathcal{H}_\infty$-optimal control and $\|G(z)\|_{\mathcal{H}_\infty} = \gamma^*$.

When the matrices (A, B_1) are known, the problem is written in LMI terms as follows [102].

An $\mathcal{H}_\infty$-optimal controller exists if and only if there exists a matrix $P \in \mathbb{S}^n_{++}$ such that

$$\min_{P\in\mathbb{S}^n_{++},\, Y\in\mathbb{R}^{m\times n},\, \gamma>0} \gamma, \tag{64}$$

$$\begin{bmatrix} P & AP + B_1Y & D & 0 \\ * & P & 0 & PC + Y^\top B_2^\top \\ * & * & \gamma I & 0 \\ * & * & * & \gamma I \end{bmatrix} \succ 0;$$

the solution $(\hat{P}, \hat{Y})$ yields the optimal controller $\hat{K} = \hat{Y}\hat{P}^{-1}$ and $\|G(z)\|_{\mathcal{H}_\infty} = \hat{\gamma}$.

Below we present DDC methods for solving this problem using the FL and LFT (Section 4.4.1), the FL and the discrete Bounded Real Lemma (Section 4.4.2), the DI condition and QMIs (Section 4.4.3), and QMIs for individual input-output measurement instants (Section 4.4.4).

4.4.1. Fundamental Lemma and Linear Fractional Transformation

System (62) with unknown matrices (A, B_1) was considered in [98].

Following the same methodology as in Section 4.2.1, under the rank condition (7) of the FL, the system closed by the controller $u = Kx$ can be represented as a linear fractional transformation with the channel from w to z :

$$\begin{bmatrix} x(k+1) \\ z(k) \\ \tilde{z}(k) \end{bmatrix} = \begin{bmatrix} X_+G & D & D \\ C + B_2UG & 0 & 0 \\ -G & 0 & 0 \end{bmatrix} \begin{bmatrix} x(k) \\ w(k) \\ \tilde{w}(k) \end{bmatrix},$$

$$\tilde{w}(k) = W\tilde{z}(k), \quad W \in \tilde{\mathbf{W}},$$

$$X_-G = I.$$

The authors formulated $\mathcal{H}_\infty$-suboptimal controller design conditions containing a matrix inequality and reduced them to a linear form by a change of variables and equivalent transformations. For brevity, let us present the final result.

Proposition 7 [98, Proposition 10]. *If the data (U, X) satisfy the rank condition of the FL and there exist matrices $P \in \mathbb{S}^n_{++}$ and $Y \in \mathbb{R}^{T\times n}$ and a scalar $\lambda > 0$ such that*

$$X_-Y = P,$$

then the solution of the problem

$$\begin{bmatrix} -P & * & * & * & * & * \\ 0 & -\gamma^2 I & * & * & * & * \\ -\lambda\Phi_{12}Y & 0 & \lambda\Phi_{11} & * & * & * \\ X_+Y & D & D & -P & * & * \\ CP + B_2U & 0 & 0 & 0 & -I & * \\ Y & 0 & 0 & 0 & 0 & -(\lambda\Phi_{22}^{-1}) \end{bmatrix} \prec 0 \tag{65}$$

with some $\hat{\lambda}$ yields the controller $\hat{K} = U\hat{Y}\hat{P}^{-1}$ for all $(A, B_1) \in \Sigma_W$ and $\|G(z)\|_{\mathcal{H}_\infty} \leqslant \hat{\gamma}$.

Thus, the LMI (65) can be solved by one-dimensional optimization over the parameter λ, and in this formulation, the experiment duration $T \geqslant n+m$ directly affects the problem dimension. Moreover, the solution accuracy improves with T [98, Example 1]; therefore, for large system dimensions and data amounts, one should choose a trade-off between satisfactory accuracy and the ability to solve LMIs of acceptable dimensions.

4.4.2. Fundamental Lemma and the Discrete Bounded Real Lemma

In contrast to the previous subsection and [98], the idea put forward in [108] is to design an $\mathcal{H}_\infty$-suboptimal controller for system (62) based on QMIs and the discrete Bounded Real Lemma (BRL) under the FL condition. For the closed-loop system with matrices (63), the author formulated and proved this lemma in the following form.

Lemma 5 [108, Lemma 6, Appendix A.1]. *The following statements are equivalent:*

1. *The matrix A_K is Schur stable and $\|G(z)\|_\infty < \gamma$.*
2. *The linear matrix inequality*

$$\begin{bmatrix} P & * & * & * \\ 0 & \gamma I & * & * \\ PA_K & PD & P & * \\ C_K & 0 & 0 & \gamma I \end{bmatrix} \succ 0 \tag{66}$$

holds for a Lyapunov matrix $P \in \mathbb{S}^n_{++}$ and $\gamma > 0$.

Then the known gain matrix parameterization (8) (see Sections 2.1 and 4.2.1) is applied to the LMI (66), which requires the rank condition (7) of the FL. In this case, $K = UG$ with $X_- G = I$, and the closed-loop system matrices can be expressed from the measured data as

$$A_K = (X_+ - DW)G, \quad C_K = C + B_2 UG \tag{67}$$

for an unknown (non-unique) matrix G and some disturbance matrix $W \in \mathbf{W}$ satisfying

$$(X_+ - DW)\begin{bmatrix} U \\ X_- \end{bmatrix}^\dagger = 0.$$

By substituting (67) into (66) and using the more general characterization of the disturbance set (42) (see Section 4.2.1), we obtain the data-driven $\mathcal{H}_\infty$-optimal control problem. After standard transformations and a change of the variables, it is reduced to the following LMI with the parameter $W \in \tilde{\mathbf{W}}$ and the variables $P \in \mathbb{S}^n_{++}$ and $Y \in \mathbb{R}^{T\times n}$:

$$\begin{bmatrix} P & * & * & * \\ 0 & \gamma I & * & * \\ (X_+ - DW)Y & D & P & * \\ CP + B_2 UY & 0 & 0 & \gamma I \end{bmatrix} \succ 0.$$

According to the approach of [108], the LFT, the matrix S-procedure (with the inequality in (42)) [19, 83], and the Schur complement lemma are successively applied to this constraint. And the data-driven $\mathcal{H}_\infty$-suboptimal controller design problem takes the final form

$$\min_{P,\,Y} \gamma \tag{68}$$

subject to the constraint

$$\begin{bmatrix} P & * & * & * & * & * \\ 0 & \gamma I & * & * & * & * \\ X_+Y & D & P & * & * & * \\ CP+B_2UY & 0 & 0 & \gamma I & * & * \\ Y\Phi_{12} & 0 & -D^\top & 0 & \Phi_{11} & * \\ Y & 0 & 0 & 0 & 0 & -\Phi_{22}^{-1} \end{bmatrix} \succ 0, \tag{69}$$

$$P = X_- Y.$$

Its solution yields the controller $\hat{K} = U\hat{Y}\hat{P}^{-1}$ and $\|G(z)\|_{\mathcal{H}_\infty} < \hat{\gamma}$.

Note that the parameterization of the matrix A_K in (67) is non-unique, and the control objective is achieved for all systems with the matrix $A_K = (X_+ - DW)G$ for any $W \in \tilde{\mathbf{W}}$. Therefore, the controller design in this case gives potentially conservative results.

The dimensions of the matrix variables in this problem directly depend on the amount of data used, which must satisfy $T \geqslant n + m$ by the FL.

4.4.3. Data Informativity Condition Based on Quadratic Matrix Inequalities

Following [19], consider a system of the form

$$\begin{aligned} x(k+1) &= Ax(k) + B_1u(k) + w(k), \\ z(k) &= Cx(k) + B_2u(k), \end{aligned} \tag{70}$$

where $x(k) \in \mathbb{R}^n$, $u(k) \in \mathbb{R}^m$, $z(k) \in \mathbb{R}^l$, and $w(k) \in \mathbb{R}^p$; $A \in \mathbb{R}^{n\times n}$ and $B_1 \in \mathbb{R}^{n\times m}$; the matrices $C \in \mathbb{R}^{l\times n}$, $B_2 \in \mathbb{R}^{l\times m}$, and $D \in \mathbb{R}^{n\times p}$ are given. The matrix W must satisfy (38):

$$\mathbf{W} = \left\{ W \in \mathbb{R}^{n\times T} : \begin{bmatrix} I \\ W^\top \end{bmatrix}^\top \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{12}^\top & \Phi_{22} \end{bmatrix} \begin{bmatrix} I \\ W^\top \end{bmatrix} \succcurlyeq 0 \right\}.$$

As in the case of $\mathcal{H}_2$-suboptimal control (Section 4.3.1), the authors used the DI concept and the technique of QMIs (the matrix S-procedure, the Slater condition, etc.) to establish control optimality conditions. As a result, the following result was formulated.

Theorem 10 [19, Theorem 13]. *The data (U, X) obtained from system (70) with disturbances described by the set $\mathbf{W}$ are informative for $\mathcal{H}_\infty$-suboptimal control design with $\gamma > 0$ if and only if there exist matrices $P \in \mathbb{S}^n_{++}$ and $Y \in \mathbb{R}^{m\times n}$ and scalars $\alpha \geqslant 0$ and $\beta > 0$ such that*

$$\begin{bmatrix} P - \frac{1}{\gamma^2}I - \beta I & 0 & 0 & 0 & 0 \\ * & 0 & 0 & P & 0 \\ * & * & 0 & Y & 0 \\ * & * & * & P & (CP+B_2Y)^\top \\ * & * & * & * & I \end{bmatrix} - \alpha \begin{bmatrix} I & X_+ \\ 0 & -X_- \\ 0 & -U \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \Phi \begin{bmatrix} I & X_+ \\ 0 & -X_- \\ 0 & -U \\ 0 & 0 \\ 0 & 0 \end{bmatrix}^\top \succcurlyeq 0,$$

$$\begin{bmatrix} P & * \\ CP+B_2Y & I \end{bmatrix} \succ 0.$$

Then the solution $\hat{P}$, $\hat{Y}$ of this problem yields the controller $\hat{K} = \hat{Y}\hat{P}^{-1}$ and $\|G(z)\|_{\mathcal{H}_\infty} < \hat{\gamma}$ for all $(A, B_1) \in \Sigma_W$.

In comparison with the methods from the previous two subsections, an undoubted advantage here is the independence of the dimensions of the matrix variables in the LMI from the length of the input-output measurements, which is limited only by the DI condition for stabilization of system (13): $T \geqslant n$.

An iterative method for designing an $\mathcal{H}_\infty$-optimal control law from multiple datasets was proposed in [109]. As claimed by the authors, their approach is less conservative than the one discussed in the current subsection; moreover, it constructs control for a wider set of systems because the matrices C and B_2 are supposed to be *unknown*. For brevity, we do not present the method of [109] here.

4.4.4. Quadratic Matrix Inequalities for Individual Input-Output Measurement Instants

By analogy with the solution of the $\mathcal{H}_2$-suboptimal control design from Section 4.3.4, the authors of [106] proposed to construct an $\mathcal{H}_\infty$-suboptimal control law by using individual measurements at each time instant. As proved in the paper, this approach reduces the degree of conservatism.

Since the initial data and subsequent constructs are the same as in Section 4.3.4, we directly provide the final result.

For system (62), the solution of the problem

$$\min_{P,Y,Q,\alpha_i,\beta,\gamma} \gamma \tag{71}$$

subject to the constraints

$$\begin{bmatrix} P-DD^\top-\beta I & * & * \\ 0 & -Q & * \\ 0 & \begin{bmatrix} C & B_2 \end{bmatrix} Q & \gamma^2 I - \begin{bmatrix} C & B_2 \end{bmatrix} Q \begin{bmatrix} C & B_2 \end{bmatrix}^\top \end{bmatrix} - \begin{bmatrix} \sum\limits_{i=0}^{T-1} \alpha_i \Psi_i & 0 \\ 0 & 0 \end{bmatrix} \succcurlyeq 0, \tag{72}$$

$$\Psi_i = \begin{bmatrix} I & x_{i+1} \\ 0 & -x_i \\ 0 & -u_i \end{bmatrix} \Phi \begin{bmatrix} I & x_{i+1} \\ 0 & -x_i \\ 0 & -u_i \end{bmatrix}^\top, \quad i = 0, \ldots, T-1,$$

$$\begin{bmatrix} Q - \begin{bmatrix} P & * \\ Y & 0 \end{bmatrix} & \begin{bmatrix} 0 \\ Y \end{bmatrix} \\ * & P \end{bmatrix} \succ 0$$

with respect to the matrix variables $P \in \mathbb{S}^n_{++}$, $Q \in \mathbb{S}^{n+m}_{++}$, and $Y \in \mathbb{R}^{m \times n}$ and the scalar variables $\alpha_i \geqslant 0$, $i = 0, 1, \ldots, T-1$, $\beta > 0$, and $\gamma > 0$ yields the H_∞-suboptimal controller $\hat{K} = \hat{Y}\hat{P}^{-1}$ with $\|G(z)\|_{\mathcal{H}_\infty} \leqslant \hat{\gamma}$ for all $(A, B_1) \in \Sigma_c$.

Remark 7. By [106, Theorem 5], the value γ^* representing the solution of problem (71)–(72) decreases monotonically with the time $T \geqslant n$. (This restriction follows from (13).)

In [106], the method was extended to the case of fixed-structure (sparse) controllers (an iterative algorithm), and its advantages were demonstrated theoretically and by an example.

Example 5. Under the conditions of the previous example, we designed $\mathcal{H}_\infty$-optimal control by the above methods.

First, we constructed a control law in the MBC approach by solving problem (64) (see the beginning of Section 4.4)

$$\hat{K}^{H_\infty}_{mbc} = \begin{bmatrix} 1.7141 & -1.5677 & 0.3249 & -3.2775 \\ 6.8823 & -0.1974 & 3.8621 & -4.6649 \end{bmatrix}, \quad \hat{\gamma}^\infty_{mbc} = 2.5607.$$

Next, with the LFT-based method (Proposition 7 of Section 4.4.1) and $\hat{\lambda} = 346.98$, we obtained

$$\hat{K}_{lft}^{H_\infty} = \begin{bmatrix} 1.6453 & -1.6010 & 0.2668 & -3.3096 \\ 6.8780 & -0.1381 & 3.9277 & -4.5610 \end{bmatrix}, \quad \hat{\gamma}_{lft}^{H_\infty} = 2.6102.$$

The solution of problem (68)–(69) based on the FL and the discrete BRL from Section 4.4.2 gave

$$\hat{K}_{brl}^{H_\infty} = \begin{bmatrix} 1.6642 & -1.6052 & 0.2650 & -3.3001 \\ 6.9270 & -0.1262 & 3.9583 & -4.5406 \end{bmatrix}, \quad \hat{\gamma}_{brl}^{H_\infty} = 2.7206.$$

Computations according to Theorem 10 (Section 4.4.3) using the DI conditions with QMIs led to $\hat{\alpha} = 149.81$, $\hat{\beta} = 0.001$, and

$$\hat{K}_{di}^{H_\infty} = \begin{bmatrix} 1.6645 & -1.5930 & 0.2825 & -3.2925 \\ 6.8529 & -0.1515 & 3.9036 & -4.5931 \end{bmatrix}, \quad \hat{\gamma}_{di}^{H_\infty} = 2.6132.$$

In view of Remark 7 (Section 4.4.4), the method with QMIs for individual input-output measurement instants (71)–(72) yielded $\hat{\alpha}_i$, $i = 1, 2, \ldots, 20$ (omitted for brevity), $\hat{\beta} = 2.2745 \cdot 10^{-5}$, and the controller

$$\hat{K}_{ins}^{H_\infty} = \begin{bmatrix} 1.6459 & -1.5946 & 0.2779 & -3.3306 \\ 6.9028 & -0.1491 & 3.9194 & -4.5696 \end{bmatrix}, \quad \hat{\gamma}_{ins}^{H_\infty} = 2.6031.$$

Figure 5 shows the following trajectories: the outputs $y_i(k)$, $i = 1, 2, 3, 4$, of the open-loop system (thick dashed line, with circles marking the measurements X used) and those of the closed-loop system, namely, the outputs $z_{mbc}(k)$ (the MBC solution, thick solid line), $z_{lft}(k)$ (the LFT-based method, thin dashed line), $z_{brl}(k)$ (the method based on the FL and discrete BRL, thin dotted line), $z_{di}(k)$ (the DI-based method with QMIs, thin dash-dotted line), and $z_{ins}(k)$ (QMIs for individual measurements, thin solid line).

We compared the results with the MPC solution:

$$\begin{gathered} \|\hat{K}_{mbc}^{H_\infty} - \hat{K}_{lft}^{H_\infty}\|_2 = 0.1555, \quad \|z_{mbc}(k) - z_{lft}(k)\|_2 = 0.0197, \quad |\hat{\gamma}_{mbc}^{H_\infty} - \hat{\gamma}_{lft}^{H_\infty}| = 0.0495, \\ \|\hat{K}_{mbc}^{H_\infty} - \hat{K}_{brl}^{H_\infty}\|_2 = 0.1946, \quad \|z_{mbc}(k) - z_{brl}(k)\|_2 = 0.0187, \quad |\hat{\gamma}_{mbc}^{H_\infty} - \hat{\gamma}_{brl}^{H_\infty}| = 0.1599, \\ \|\hat{K}_{mbc}^{H_\infty} - \hat{K}_{di}^{H_\infty}\|_2 = 0.1047, \quad \|z_{mbc}(k) - z_{di}(k)\|_2 = 0.0122, \quad |\hat{\gamma}_{mbc}^{H_\infty} - \hat{\gamma}_{di}^{H_\infty}| = 0.0524, \\ \|\hat{K}_{mbc}^{H_\infty} - \hat{K}_{ins}^{H_\infty}\|_2 = 0.1532, \quad \|z_{mbc}(k) - z_{ins}(k)\|_2 = 0.0254, \quad |\hat{\gamma}_{mbc}^{H_\infty} - \hat{\gamma}_{ins}^{H_\infty}| = 0.0423, \\ k = 0, 1, 2, \ldots. \end{gathered}$$

Clearly, all the methods successfully solved the stated problem. In terms of the accuracy of the estimate $\hat{\gamma}$ and the closeness of the trajectories, the method with individual measurements in the matrix inequality performed best in this example, and its accuracy increases with the number of initial measurements T. The method based on the joint application of the FL and the discrete BRL turned out to be the least accurate in the estimate $\hat{\gamma}$, although this difference is not critical. The other two methods are roughly similar.

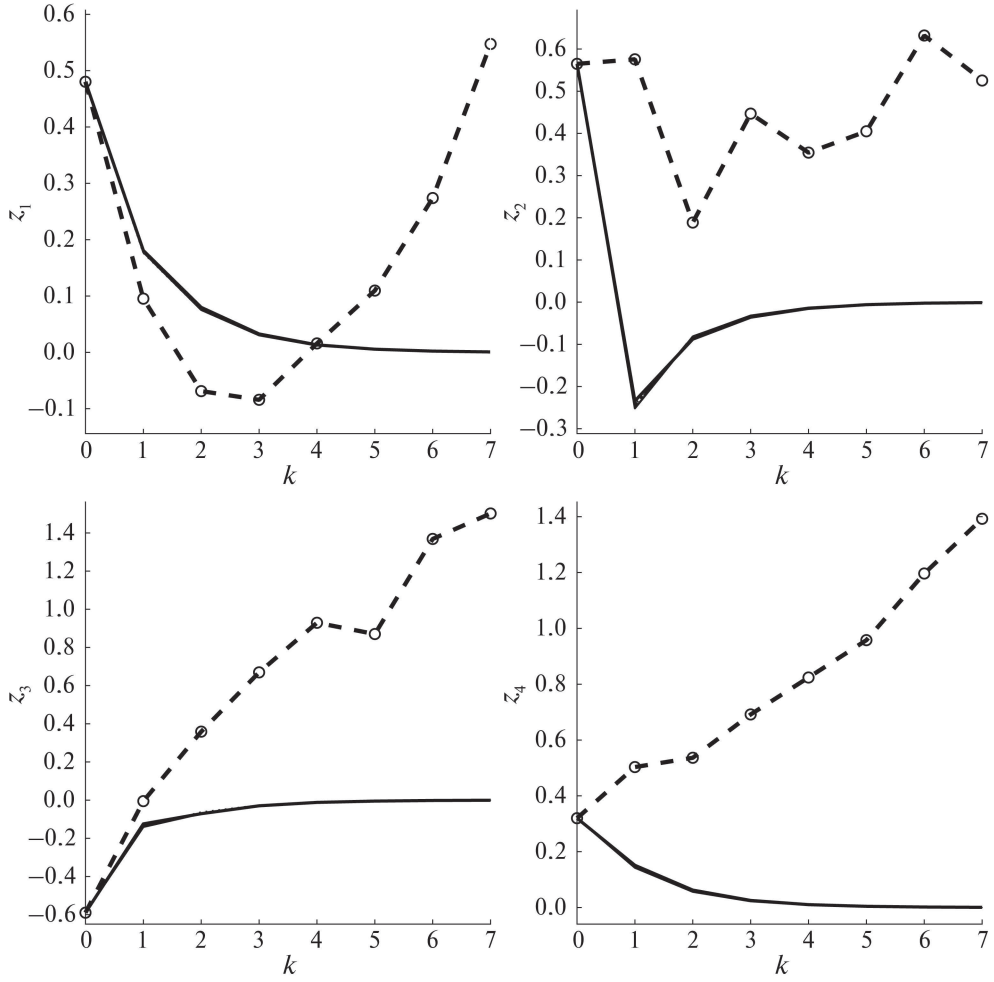


Fig. 5. System outputs in the $\mathcal{H}_\infty$-optimal control problem.

5. CONCLUSIONS

This survey has covered direct data-driven control methods based on finite sets of measurements, with the Fundamental Lemma or the Data Informativity concept at their core, and the use of linear matrix inequalities with application to quadratic stabilization (with exact measurements and under bounded exogenous disturbances), optimal linear quadratic regulator design, and $\mathcal{H}_2$- and $\mathcal{H}_\infty$-optimal control design for discrete linear time-invariant systems. Based on the above analysis, we draw the following conclusions:

— The methods involve well-known conventional MBC approaches and solution techniques, i.e., Lyapunov inequalities, the linear fractional transformation, Petersen's lemma, the S-procedure and Finsler's lemma (including their matrix versions), the discrete Bounded Real Lemma, and the techniques of linear and quadratic matrix inequalities are used.

— The only a priori knowledge is the linear time-invariant character of the system and the dimensions of its inputs and outputs (or some estimates of the dimensions).

— By assumption, all outputs are observable and measurements of *all states* are available, i.e., $y(k) = x(k)$, which is not always possible in real-world systems. In such cases, other DDC methods have to be applied, including those not included in this survey owing to differences in problem formulations.

— In all the presented methods, input-output measurements of the *open-loop system* are supposed to be available. First, this is not always feasible in practice; second, it may cause challenges in the case of unstable systems; in both cases mentioned, other DDC approaches are required.

— The Fundamental Lemma enables parameterizing the closed-loop system matrix directly, particularly in the presence of exogenous disturbances (at the cost of a higher degree of conservatism); however, its conditions require a larger amount of initial data.

— The Data Informativity conditions are milder and allow using less conservative methods, including those based on individual measurements, and smaller amounts of initial data.

— The dimensions of the matrix variables in different methods may/may not depend on the experiment duration, which significantly affects the applicability of a particular method: if the variable dimensions depend on the data amount, increasing this amount will improve the solution accuracy (an advantage of such methods), but computational challenges may arise when solving the LMIs.

— Persistent exogenous disturbances affecting the system (if any) are usually supposed to be unknown, and a description of the set of all admissible disturbances is used; bounded-energy disturbances or those with bounded norm values at individual time instants are considered (the latter approach reduces the degree of conservatism). However, there are methods with known disturbance measurements, and such measurements are directly employed in the solution (see Section 4.3.2).

— Methods have also been developed for partial problems, e.g., finite-horizon LQR design (Section 3.4).

Thus, the survey has presented a sufficiently diverse set of DDC methods for solving several classical control problems, discussed the features of their structure and application, and demonstrated their performance on numerical examples.

The above methods are part of one of the most actively developing directions within DDC. They are used, in the original form or numerous modifications, in the analysis, modeling, and control design for different real objects, systems, and processes: magnetic levitation systems [19]; single-link robotic arms [63]; simple distillation columns [70]; fighter aircraft [83]; pre-compensated surge subsystems of axial compressors [110]; stable active vehicle suspension systems and unstable rotational inverted pendulum systems [71]; inverted pendulums on fixed bases [69, 96] and carts [65]; two-mass spring systems [69]; decentralized control of DC microgrid systems with distributed generation units connected by transmission lines [111]; motion of robot groups [112]; motion of vehicles groups with a leader [60] and two vehicles at a safe distance [72]; other multi-agent systems [29, 53, 91]; population growth models [93], and other applications.

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REFERENCES

1. Hou, Z.S. and Wang, Z., From Model-Based Control to Data-Driven Control: Survey, Classification and Perspective, *Information Sciences*, 2013, vol. 235, pp. 3–35. https://doi.org/10.1016/j.ins.2012.07.014

2. Xu, J.X. and Hou, Z.S., Notes on Data-Driven System Approaches, *Acta Automatica Sinica*, 2009, vol. 35, no. 6, pp. 668–675. https://doi.org/10.1016/S1874-1029(08)60090-7.

3. Dörfler, F., Data-Driven Control: Part Two of Two: Hot Take: Why not go with Models?, *IEEE Control Systems Magazine*, 2023, vol. 43, no. 6, pp. 27–31. https://doi.org/10.1109/MCS.2023.3310302

4. Soudbakhsh, D., Annaswamy, A.M., Wang, Y., et al., Data-Driven Control: Theory and Applications, *Proc. American Control Conf. (ACC)*, May 31, 2023–June 2, 2023, San Diego, CA, USA, pp. 1–18. https://doi.org/10.23919/ACC55779.2023.10156081.

5. Fernández, R., Data-Driven Control: A New Important Field in Control Theory, *Proc. 15th Latin American Control Conf.*, 2012, pp. 6.

6. Bazanella, A.S., Campestrini, L., and Eckhard, D., An Introduction to Data-Driven Control, From Kernels to Behaviors, *Proc. IEEE 61st Conf. Decision and Control (CDC)*, December 6–9, 2022, Cancún, Mexico. https://doi.org/10.1109/CDC51059.2022.9992589

7. Bazanella, A.S., Campestrini, L., and Eckhard, D., The Data-Driven Approach to Classical Control Theory, *Annual Reviews in Control*, 2023, vol. 56, art. no. 100906, pp. 1–20. https://doi.org/10.1016/j.arcontrol.2023.100906

8. Khaki-Sedigh, A., *An Introduction to Data-Driven Control Systems*, Hoboken: John Wiley & Sons, 2024.

9. Wang, J., Ramirez-Mendoza, R.A., and Morales-Menendez, R., *Data Driven Strategies: Theory and Applications*, CRC Press, 2023.

10. Bazanella, A.S., Campestrini, L., and Eckhard, D., *Data-Driven Controller Design: The H_2 Approach*, Dordrecht: Springer, 2012.

11. Brunton, S.L. and Kutz, J.N., *Data-Driven Science and Engineering. Machine Learning, Dynamical Systems, and Control*, Cambridge University Press, 2021.

12. Precup, R.-E., Roman, R.-C., and Safaei, A., *Data-Driven Model-Free Controllers*, Boca Raton: CRC Press, 2022.

13. Willems, J.C., From Time Series to Linear System: Part I. Finite Dimensional Linear Time Invariant Systems, *Automatica*, 1986, vol. 22, no. 5, pp. 561–580; Part II. Exact Modelling, *Automatica*, 1986, vol. 22, no. 6, pp. 675–694; Part III: Approximate Modelling, *Automatica*, 1987, vol. 23, no. 1, pp. 87–115.

14. Willems, J.C. and Polderman, J.W., *Introduction to Mathematical Systems Theory: A Behavioral Approach*, Texts in Applied Mathematics, New York: Springer, 1997.

15. Willems, J.C., Rapisarda, P., Markovsky, I., et al., A Note on Persistency of Excitation, *Systems & Control Letters*, 2005, vol. 54, no. 4, pp. 325–329. https://doi.org/10.1016/j.sysconle.2004.09.003

16. Markovsky, I., Willems, J., and De Moor, B., State Representations from Finite Time Series, *Proc. 44th IEEE Conf. on Decision and Control (CDC)*, Seville, Spain, 2005, pp. 832–835. https://doi.org/10.1109/CDC.2005.1582260

17. Markovsky, I. and Dörfler, F., Behavioral Systems Theory in Data-Driven Analysis, Signal Processing, and Control, *Annual Reviews in Control*, 2021, vol. 52, pp. 42–64. https://doi.org/10.1016/j.arcontrol.2021.09.005

18. Van Waarde, H.J., Eising, J., Trentelman, H.L., et al., Data Informativity: A New Perspective on Data-Driven Analysis and Control, *IEEE Trans. on Automatic Control*, 2020, vol. 65, no. 11, pp. 4753–4768. https://doi.org/10.1109/TAC.2020.2966717

19. Van Waarde, H.J., Eising, J., Camlibel, M.K., et al., The Informativity Approach: To Data-Driven Analysis and Control, *IEEE Control Systems*, 2023, vol. 43, no. 6, pp. 32–66. https://doi.org/10.1109/MCS.2023.3310305

20. Van Waarde, H.J., Eising, J., Camlibel, M.K., et al., A Tutorial on the Informativity Framework for Data-Driven Control, *Proc. 61st IEEE Conf. on Decision and Control (CDC)*, December 2022, pp. 1085–1090. https://doi.org/10.1109/CDC51059.2022.9992611

21. Rouis, N., N'Doye, I., and Laleg-Kirati, T.-M., Model-Free LQR Based PID Controller for Trajectory Tracking of 2-DoF Helicopter: Comparison and Experimental Results, *arXiv: 2103.10988v1.*

22. Berberich, J., Iannelli, A., Padoan, A., et al., A Quantitative and Constructive Proof of Willems' Fundamental Lemma and its Implications, *Proc. American Control Conf. (ACC)*, May 31, 2023–June 2, 2023, San Diego, CA, USA. https://doi.org/10.23919/ACC55779.2023.10156227

23. Bosso, A., Borghesi, M., Iannelli, A., et al., Derivative-Free Data-Driven Control of Continuous-Time Linear Time-Invariant Systems, *arXiv:2410.24167v1.*

24. Schmitz, P., Faulwasser, T., Rapisarda, P., et al., A Continuous-Time Fundamental Lemma and its Application in Data-Driven Optimal Control, *Systems & Control Letters*, 2024, vol. 194, art. no. 105950, pp. 1–11. https://doi.org/10.1016/j.sysconle.2024.105950

25. Pan, G., Ou, R., and Faulwasser, T., On a Stochastic Fundamental Lemma and Its Use for Data-Driven Optimal Control, *IEEE Trans. on Automatic Control*, 2023, vol. 68, no. 10, pp. 5922–5937. https://doi.org/10.1109/TAC.2022.3232442

26. Ou, R., Pan, G., and Faulwasser, T., A Stochastic Fundamental Lemma with Reduced Disturbance Data Requirements, *arXiv:2502.09131v1.*

27. Faulwasser, T., Ou, R., Pan, G., et al., Behavioral Theory for Stochastic Systems? A Data-driven Journey from Willems to Wiener and Back Again, *Annual Reviews in Control*, 2023, vol. 55, pp. 92–117. https://doi.org/10.1016/j.arcontrol.2023.03.005

28. Mishra, V.K., Markovsky, I., and Grossmann, B., Data-Driven Tests for Controllability, *IEEE Control Systems Letters*, 2021, vol. 5, no. 2, pp. 517–522. https://doi.org/10.1109/LCSYS.2020.3003770

29. Yu, Y., Talebi, Sh., Van Waarde, H.J., et al., On Controllability and Persistency of Excitation in Data-Driven Control: Extensions of Willems' Fundamental Lemma, *arXiv: 2102.02953v2.*

30. Yang, Y., Zheng, Z., and Li, S.E., On the Stability of Datatic Control Systems, *arXiv:2401.16793v1.*

31. Coulson, J., Van Waarde, H., and Dörfler, F., Robust Fundamental Lemma for Data-driven Control, *arXiv:2205.06636v1.*

32. Coulson, J., Van Waarde, H.J., Lygeros, J., et al., A Quantitative Notion of Persistency of Excitation and the Robust Fundamental Lemma, *IEEE Control Systems Letters*, 2023, vol. 7, pp. 1243–1248. https://doi.org/10.1109/LCSYS.2022.3232303

33. Verhoek, C., Tóth, R., Haesaert, S., et al., Fundamental Lemma for Data-Driven Analysis of Linear Parameter-Varying Systems, *Proc. 60th IEEE Conf. on Decision and Control (CDC)*, December 14–17, 2021, Austin, TX, USA. https://doi.org/10.1109/CDC45484.2021.9683151

34. Verhoek, C., Eising, J., Dörfler, F., and Tóth, R., Decoupling Parameter Variation from Noise: Bi-quadratic Lyapunov Forms in Data-Driven LPV Control, *arXiv:2403.16565v2.*

35. Nortmann, B. and Mylvaganam, T., Direct Data-Driven Control of LTV Systems, *IEEE Trans. on Automatic Control*, 2023, vol. 68, no. 8, pp. 4888–4895. https://doi.org/10.1109/TAC.2023.3276909

36. Breschi, V., Zaccarian, L., and Formentin, S., Data-Driven Stabilization of Input-Saturated Systems, *IEEE Control Systems Letters*, 2023, vol. 7, pp. 1640–1645. https://doi.org/10.1109/LCSYS.2023.3266254

37. Lu, X., Zhou, D., Chen, W.H., et al., Data-Driven Control for Linear Discrete-Time Systems with Input Saturation, *Circuits, Systems, and Signal Processing*, 2024, vol. 43, pp. 6211–6227. https://doi.org/10.1007/s00034-024-02751-w

38. Martinelli, A., Gargiani, M., Draskovic, M., et al., Data-Driven Optimal Control of Affine Systems: A Linear Programming Perspective, *IEEE Control Systems Letters*, 2022, vol. 6, pp. 3092–3097. https://doi.org/10.1109/LCSYS.2022.3180898

39. Chang, C.Y. and Bernstein, A., Robust Data-Driven Control with Noisy Data, *arXiv:2207.09587v4.*

40. Bisoffi, A., De Persis, C., and Tesi, P., Data-Based Stabilization of Unknown Bilinear Systems with Guaranteed Basin of Attraction, *Systems & Control Letters*, 2020, vol. 145, art. no. 104788, pp. 1–9. https://doi.org/10.1016/j.sysconle.2020.104788

41. De Persis, C. and Tesi, P., Designing Experiments for Data-Driven Control of Nonlinear Systems, *IFAC-PapersOnLine*, 2021, vol. 54, no. 9, pp. 285–290. https://doi.org/10.1016/j.ifacol.2021.06.085

42. Martin, T., Schönb, T.B., and Allgöwer, F., Guarantees for Data-Driven Control of Nonlinear Systems Using Semidefinite Programming: A Survey, *Annual Reviews in Control*, 2023, vol. 56, art. no. 100911. https://doi.org/10.1016/j.arcontrol.2023.100911

43. Padoan, A., Coulson, J., Van Waarde, H.J., et al., Behavioral Uncertainty Quantification for Data-Driven Control, *Proc. IEEE 61st Conf. on Decision and Control (CDC)*, December 6–9, 2022, Cancún, Mexico, pp. 4726–4731. https://doi.org/10.1109/CDC51059.2022.9993002

44. Alsalti, M., Barkey, M., Lopez, V.G., et al., Sample- and Computationally Efficient Data-Driven Predictive Control, *Proc. IEEE European Control Conf. (ECC)*, Stockholm, Sweden, 2024. https://doi.org/10.23919/ECC64448.2024.10591022

45. Coulson, J., Lygeros, J., and Dörfler, F., Data-enabled Predictive Control: In the Shallows of the DeePC, *Proc. 18th European Control Conf. (ECC)*, June 25–28, 2019, Naples, Italy, pp. 307–312. https://doi.org/10.23919/ECC.2019.8795639

46. Lazar, M. and Verheijen, P.C.N., Generalized Data-Driven Predictive Control: Merging Subspace and Hankel Predictors, *Mathematics*, 2023, vol. 11(9), art. no. 2216, 17 p. https://doi.org/10.3390/math11092216

47. Markovsky, I., Huang, L., and Dörfler, F., Data-Driven Control Based on the Behavioral Approach: From Theory to Applications in Power Systems, *IEEE Control Systems Magazine*, 2023, vol. 43, no. 5, pp. 28–68. https://doi.org/10.1109/MCS.2023.3291638

48. Mattsson, P., Bonassi, F., Breschi, V., et al., On the Equivalence of Direct and Indirect Data-Driven Predictive Control Approaches, *IEEE Control Systems Letters*, 2024, vol. 8, pp. 796–801. https://doi.org/10.1109/LCSYS.2024.3403473

49. Meijer, T.J., Scheres, K.J.A., Nouwens, S.A.N., et al., From a Frequency-Domain Willems' Lemma to Data-Driven Predictive Control, *arXiv:2501.19390v1*.

50. Dmitruk, N.M. and Manzhulina, E.A., Optimal Control of Linear Time-Invariant Discrete-Time Systems without Parametric Identification, *Autom. Remote Control*, 2022, vol. 83, no. 2, pp. 161–177.

51. Breschi, V., De Persis, C., Formentin, S., et al., Direct Data-Driven Model-Reference Control with Lyapunov Stability Guarantees, *Proc. 60th IEEE Conf. on Decision and Control (CDC)*, December 14–17, 2021, Austin, TX, USA, pp. 9212–9217. https://doi.org/10.1109/CDC45484.2021.9683437

52. Markovsky, I. and Rapisarda, P., On the Linear Quadratic Data-Driven Control, *Proc. European Control Conf. (ECC)*, Kos, Greece, July 2–5, 2007, pp. 5313–5318. https://doi.org/10.23919/ECC.2007.7068299

53. Chang, Z., Jiao, J., and Li, Z., Localized Data-driven Consensus Control, *arXiv:2401.12707v1*.

54. Van Waarde, H.J., De Persis, C., Camlibel, M.K., et al., Willems' Fundamental Lemma for State-space Systems and its Extension to Multiple Datasets, *IEEE Control Systems Letters*, 2020, vol. 4, no. 3, pp. 602–607. https://doi.org/10.1109/LCSYS.2020.2986991

55. Alsalti, M., Lopez, V.G., and Müller, M.A., Notes on Data-Driven Output-Feedback Control of Linear MIMO Systems, *IEEE Trans. on Automatic Control*, 2025, pp. 1–8. https://doi.org/10.1109/TAC.2025.3553073

56. Shakouri, A., Van Waarde, H.J., and Camlibel, M.K., A New Perspective on Willems' Fundamental Lemma: Universality of Persistently Exciting Inputs, *arXiv:2503.12489*.

57. Eising, J. and Trentelman, H.L., Informativity of Noisy Data for Structural Properties of Linear Systems, *Systems & Control Letters*, 2021, vol. 158, art. no. 105058, pp. 1–8. https://doi.org/10.1016/j.sysconle.2021.105058

58. Eising, J., Trentelman, H.L., and Camlibel, M.K., Data Informativity for Observability: An Invariance-based Approach, *Proc. European Control Conf. (ECC)*, May 12–15, 2020, St. Petersburg, Russia, pp. 1057–1059. https://doi.org/10.23919/ECC51009.2020.9143925

59. Berberich, J., Wildhagen, S., Hertneck, M., et al., Data-Driven Analysis and Control of Continuous-Time Systems under Aperiodic Sampling, *IFAC-PapersOnLine*, 2021, vol. 54, no. 7, pp. 210–215. https://doi.org/10.1016/j.ifacol.2021.08.360

60. Iwata, T., Azuma, S.-I., Ariizumi, R., et al., Data Informativity for Distributed Positive Stabilization, *IEEE Control Systems Letters*, 2024, vol. 8, pp. 826–831. https://doi.org/10.1109/LCSYS.2024.3404219

61. Eising, J. and Camlibel, M.K., Data Informativity for Analysis of Linear Systems with Convex Conic Constraints, *IEEE Control Systems Letters*, 2022, vol. 6, pp. 391–396. https://doi.org/10.1109/LCSYS.2021.3077367

62. Nguyen, E.T. and Van Waarde, H.J., Synthesis of Dissipative Systems Using Input-State Data, *Proc. IEEE European Control Conf. (ECC)*, pp. 2959–2964. https://doi.org/10.23919/ECC64448.2024.10590889

63. Hu, K. and Liu, T., Data-Driven H_∞ Control for Unknown Piecewise Affine Systems with Bounded Disturbances, *Proc. European Control Conf. (ECC)*, June 25–28, 2024, Stockholm, Sweden, pp. 2623–2628. https://doi.org/10.23919/ECC64448.2024.10590818

64. Miller, J. and Sznaier, M., Data-Driven Gain Scheduling Control of Linear Parameter-Varying Systems using Quadratic Matrix Inequalities, *IEEE Control Systems Letters*, 2022, vol. 7, pp. 835–840. https://doi.org/10.1109/LCSYS.2022.3227182

65. Van Waarde, H.J., Eising, J., Camlibel, M.K., et al., A Behavioral Approach to Data-Driven Control with Noisy Input-Output Data, *IEEE Trans. on Automatic Control*, 2024, vol. 69, no. 2, pp. 813–827. https://doi.org/10.1109/TAC.2023.3275014

66. Trentelman, H.L., Van Waarde, H.J., and Camlibel, M.K., An Informativity Approach to the Data-Driven Algebraic Regulator Problem, *IEEE Trans. on Automatic Control*, 2022, vol. 67, no. 11, pp. 6227–6233. https://doi.org/10.1109/TAC.2021.3129457

67. Cao, A. and Wang, F., Informativity Conditions for Multiple Signals: Properties, Experimental Design, and Applications, *arXiv:2501.10030v1.*

68. Eising, J. and Cortés, J., Cautious Optimization via Data Informativity, *arXiv:2307.10232v1.*

69. De Persis, C. and Tesi, P., Formulas for Data-Driven Control: Stabilization, Optimality, and Robustness, *IEEE Trans. on Automatic Control*, 2020, vol. 65, no. 3, pp. 909–924. https://doi.org/10.1109/TAC.2019.2959924

70. Bisoffi, A., Li, L., De Persis, C., et al., Controller Synthesis for Input-State Data With Measurement Errors, *IEEE Control Systems Letters*, 2024, vol. 8, pp. 1571–1576. https://doi.org/10.1109/LCSYS.2024.3402135

71. Esmzad, R., Sankar, G.S., Han, T., et al., Direct Data Driven Control Using Noisy Measurements, *arXiv:2505.06407.*

72. Bisoffi, A., De Persis, C., and Tesi, P., Controller Design for Robust Invariance From Noisy Data, *IEEE Trans. on Automatic Control*, 2023, vol. 68, no. 1, pp. 636–643. https://doi.org/10.1109/TAC.2022.3170373

73. Kaminaga, T. and Sasahara, H., Data Informativity under Data Perturbation, *arXiv:2505.01641v1.*

74. Berberich, J., Scherer, C.W., and Allgöwer, F., Combining Prior Knowledge and Data for Robust Controller Design, *IEEE Trans. Automat. Control*, 2023, vol. 68, no. 8, pp. 4618–4633. https://doi.org/10.1109/TAC.2022.3209342

75. Kogan, M.M. and Stepanov, A.V., How to Improve Robust Control of a Linear Time-Varying System by Using Experimental Data, *Autom. Remote Control*, 2024, vol. 85, no. 6, pp. 636–654.

76. Kogan, M.M., From Adaptive and Robust Control to Control Based on A Priori and Experimental Data, *Trudy XIV Vserossiiskogo soveshchaniya po problemam upravleniya (VSPU-2024)* (Proceedings of XIV All-Russian Meeting on Control Problems (AMCP-2024), June 17–20, 2024, Moscow, Novikov, D.A., Ed., Moscow: Trapeznikov Institute of Control Sciences RAS, 2024, pp. 450–454.

77. Kogan, M.M. and Stepanov, A.V., Design of Generalized H_∞-Suboptimal Controllers Based on Experimental and A Priori Data, *Autom. Remote Control*, 2024, vol. 85, no. 1, pp. 1–14.

78. Kogan, M.M. and Stepanov, A.V., Design of Suboptimal Robust Controllers Based on A Priori and Experimental Data, *Autom. Remote Control*, 2023, vol. 84, no. 8, pp. 918–932.

79. Polyak, B.T., Khlebnikov, M.V., and Shcherbakov, P.S., *Upravlenie lineinymi sistemami pri vneshnikh vozmushcheniyakh: Tekhnika lineinykh matrichnykh neravenstv* (Control of Linear Systems under Exogenous Disturbances: The Technique of Linear Matrix Inequalities), Moscow: LENAND, 2014.

80. Fomin, V.N., Fradkov, A.L., and Yakubovich, V.A., *Adaptivnoe upravlenie dinamicheskimi ob"ektami* (Adaptive Control of Dynamic Objects), Moscow: Nauka, 1981.

81. Fradkov, A.L., Miroshnik, I.V., and Nikiforov, V.O., *Nonlinear and Adaptive Control of Complex Systems*, Mathematics and Its Applications, vol. 491, Dordrecht: Kluwer Academic Publishers, 1999.

82. Van Waarde, H.J., Camlibel, M.K., Eising, J., et al., Quadratic Matrix Inequalities with Applications to Data-Based Control, *SIAM J. on Control and Optimization*, 2023, vol. 61(4), pp. 2251–2281. https://doi.org/10.1137/22M1486807

83. Van Waarde, H.J., Camlibel, M.K., and Mesbahi, M., From Noisy Data to Feedback Controllers: Nonconservative Design via a Matrix S-Lemma, *IEEE Trans. Autom. Control*, 2022, vol. 67, no. 1, pp. 162–175. https://doi.org/10.1109/TAC.2020.3047577

84. Van Waarde, H. and Camlibel, K., A Matrix Finsler's Lemma with Applications to Data-Driven Control, *Proc. 60th IEEE Conf. on Decision and Control*, 2022, pp. 5777–5782. https://doi.org/10.1109/CDC45484.2021.9683285

85. Walsh, G. and Ye, H., Scheduling of Networked Control Systems, *IEEE Control Syst. Mag.*, 2001, vol. 21, no. 1, pp. 57–65. https://doi.org/10.1109/37.898792

86. Grant, M. and Boyd, S., CVX: Matlab Software for Disciplined Convex Programming, ver. 2.1, Austin: CVX Research, Inc. URL: http://cvxr.com.

87. Chen, T. and Francis, B., *Optimal Sampled-Data Control Systems*, London: Springer, 1995.

88. Dörfler, F., Tesi, P., and De Persis, C., On the Certainty-Equivalence Approach to Direct Data-Driven LQR Design, *IEEE Trans. on Automatic Control*, 2023, vol. 68, no. 12, pp. 7989–7996. https://doi.org/10.1109/TAC.2023.3253787

89. Gonçalves Da Silva, G.R.G., Bazanella, A.S., et al., Data-Driven LQR Control Design, *IEEE Control Systems Letters*, 2018, vol. 3, no. 1, pp. 180–185. https://doi.org/10.1109/LCSYS.2018.2868183

90. Umar, A.A., Ryu, K., Back, J., et al., Stability Margin of Data-Driven LQR and Its Application to Consensus Problem, *Mathematics*, 2024, vol. 12, no. 2, art. no. 199. https://doi.org/10.3390/math12020199

91. Van Waarde, H.J. and Mesbahi, M., Data-Driven Parameterizations of Suboptimal LQR and H2 Controllers, *IFAC-PapersOnLine*, 2020, vol. 53, no. 2, pp. 4234–4239. https://doi.org/10.1016/j.ifacol.2020.12.2470

92. Rotulo, M., De Persis, C., and Tesi, P., Data-Driven Linear Quadratic Regulation via Semidefinite Programming, *IFAC-PapersOnLine*, 2020, vol. 53(2), pp. 3995–4000. https://doi.org/10.1016/j.ifacol.2020.12.2264

93. Scampicchio, A., Aravkin, A., and Pillonetto, G., LQR Design under Stability Constraints, *IFAC-PapersOnLine*, 2020, vol. 53, no. 2, pp. 5556–5560. https://doi.org/10.1016/j.ifacol.2020.12.1566

94. Bisoffi, A., De Persis, C., and Tesi, P., Data-driven Control via Petersen's Lemma, *Automatica*, 2022, vol. 145, art. no. 110537, pp. 1–14. https://doi.org/10.1016/j.automatica.2022.110537

95. Dörfler, F., Tesi, P., and De Persis, C., On the Role of Regularization in Direct Data-Driven LQR Control, *Proc. IEEE 61st Conf. on Decision and Control (CDC)*, 2022, pp. 1091–1098. https://doi.org/10.1109/CDC51059.2022.9992770

96. De Persis, C. and Tesi, P., Low-complexity Learning of Linear Quadratic Regulators from Noisy Data, *Automatica*, 2021, vol. 128, art. no. 109548. https://doi.org/10.1016/j.automatica.2021.109548

97. Zeng, X., Bako, L., and Ozay, N., Noise Sensitivity of the Semidefinite Programs for Direct Data-Driven LQR, *arXiv:2412.19705v1.*

98. Berberich, J., Koch, A., Scherer, C.W., et al., Robust Data-Driven State-Feedback Design, *Proc. American Control Conf. (ACC)*, Denver, CO, USA, 2020, pp. 1532–1538. https://doi.org/10.23919/ACC45564.2020.9147320

99. Petersen, I.R., A Stabilization Algorithm for a Class of Uncertain Linear Systems, *Systems & Control Letters*, 1987, vol. 8(4), pp. 351–357. https://doi.org/10.1016/0167-6911(87)90102-2

100. Khlebnikov, M.V. and Shcherbakov, P.S., Petersen's Lemma on Matrix Uncertainty and Its Generalizations, *Autom. Remote Control*, 2008, vol. 69, pp. 1932–1945. https://doi.org/10.1134/S000511790811009X

101. Bisoffi, A., De Persis, C., and Tesi, P., Trade-Offs in Learning Controllers from Noisy Data, *Systems & Control Letters*, 2021, vol. 154, art no. 104985. https://doi.org/10.1016/j.sysconle.2021.104985

102. Caverly, R.J. and Forbes, J.R., LMI Properties and Applications in Systems, Stability, and Control Theory, *arXiv:1903.08599.*

103. De Oliveira, M.C., Bernussou, J., and Geromel, J.C., A New Discrete-Time Robust Stability Condition, *Systems & Control Letters*, 1999, vol. 37, pp. 261–265. https://doi.org/10.1016/S0167-6911(99)00035-3

104. De Oliveira, M.C., Geromel, J.C., and Bernussou, J., Extended H_2 and H_∞ Norm Characterizations and Controller Parametrizations for Discrete-Time Systems, *Int. J. of Control*, 2002, vol. 75, no. 9, pp. 666–679. https://doi.org/10.1080/00207170210140212

105. Miller, J., Eising, J., Dörfler, F., et al., Data-Driven Structured Robust Control of Linear Systems, *arXiv: 2411.11542v1.*

106. Zhaohua, Y., Yuxing, Z., Nachuan, Y., et al., Data-Driven Structured Controller Design Using the Matrix S-Procedure, *arXiv:2503.14949v1.*

107. Brändle, F. and Allgöwer, F., A System Parametrization for Direct Data-Driven Analysis and Control with Error-in-Variables, *arXiv:2411.06787v2.*

108. De Boer, R., Robust Data-Driven State-Feedback Synthesis from Data Corrupted by Perturbations with Bounded Norms and Rates-of-Variation, *MSc Thesis*, University of Twente, 2023.

109. Hu, K. and Liu, T., A Robust Data-Driven Iterative Control Method for Linear Systems with Bounded Disturbances, *arXiv:2405.02537v1.*

110. Luppi, A., De Persis, C., and Tesi, P., On Data-Driven Stabilization of Systems with Nonlinearities Satisfying Quadratic Constraints, *Systems & Control Letters*, 2022, vol. 163, art. no. 105206. https://doi.org/10.1016/j.sysconle.2022.105206

111. Nakano, T., Aboudonia, A., Eising, J., et al., Dissipativity-Based Data-Driven Decentralized Control of Interconnected Systems, *arXiv:2509.14047.*

112. Liu, W., Li, Y., Sun, J., et al., Data-driven Internal Model Control for Output Regulation, *arXiv:2505.09255.*

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