Volume 87, Number 9,
September 2026

ISSN 0005-1179
CODEN: AURCAT

AUTOMATION AND REMOTE CONTROL

Editor-in-Chief
Andrey A. Galyaev

http://ait.mtas.ru

Automation and Remote Control

Vol. 87, No. 9, September 2026

Available via license: CC BY 4.0

Automation and Remote Control

ISSN 0005-1179

Editor-in-Chief
Andrey A. Galyaev

Deputy Editors-in-Chief M.V. Khlebnikov and E.Ya. Rubinovich

Coordinating Editor A.S. Samokhin

Editorial Board

F.T. Aleskerov, A.V. Arutyunov, N.N. Bakhtadze, A.A. Bobtsov, P.Yu. Chebotarev, A.G. Chkhartshvili, L.Yu. Filimonyuk, A.L. Fradkov, O.N. Granichin, M.F. Karavai, E.M. Khorov, M.M. Khrustalev, S.A. Krasnova, A.P. Krishchenko, A.G. Kushner, N.V. Kuznetsov, A.A. Lazarev, A.I. Lyakhov, A.I. Matasov, S.M. Meerkov (USA), R.V. Mescheryakov, A.I. Mikhal'skii, B.M. Miller, O.V. Morzhin, R.A. Munasypov, A.V. Nazin, A.S. Nemirovskii (USA), D.A. Novikov, A.Ya. Oleinikov, P.V. Pakshin, D.E. Pal'chunov, A.E. Polyakov (France), V.Yu. Protasov, L.B. Rapoport, I.V. Rodionov, N.I. Selvesyuk, P.S. Shcherbakov, A.N. Sobolevski, O.A. Stepanov, A.B. Tsybakov (France), D.V. Vinogradov, V.M. Vishnevskii, K.V. Vorontsov, and L.Yu. Zhilyakova

Staff Editor E.A. Martekhina

SCOPE

Automation and Remote Control is one of the first journals on control theory. The scope of the journal is control theory problems and applications. The journal publishes reviews, original articles, and short communications (deterministic, stochastic, adaptive, and robust formulations) and its applications (computer control, components and instruments, process control, social and economy control, etc.).

Automation and Remote Control is abstracted and/or indexed in *ACM Digital Library, BFI List, CLOCKSS, CNKI, CNPIEC Current Contents/Engineering, Computing and Technology, DBLP, Dimensions, EBSCO Academic Search, EBSCO Advanced Placement Source, EBSCO Applied Science & Technology Source, EBSCO Computer Science Index, EBSCO Computers & Applied Sciences Complete, EBSCO Discovery Service, EBSCO Engineering Source, EBSCO STM Source, EI Compendex, Google Scholar, INSPEC, Japanese Science and Technology Agency (JST), Journal Citation Reports/Science Edition, Mathematical Reviews, Naver, OCLC WorldCat Discovery Service, Portico, ProQuest Advanced Technologies & Aerospace Database, ProQuest-ExLibris Primo, ProQuest-ExLibris Summon, SCImago, SCOPUS, Science Citation Index, Science Citation Index Expanded (Sci-Search), TD Net Discovery Service, UGC-CARE List (India), WTI Frankfurt eG, zbMATH.*

Journal website: http://ait.mtas.ru

© The Author(s), 2026 published by Trapeznikov Institute of Control Sciences, Russian Academy of Sciences.

Automation and Remote Control participates in the Copyright Clearance Center (CCC) Transactional Reporting Service.

Available via license: CC BY 4.0

0005-1179/26. *Automation and Remote Control* (ISSN: 0005-1179 print version, ISSN: 1608-3032 electronic version) is published monthly by Trapeznikov Institute of Control Sciences, Russian Academy of Sciences, 65 Profsoyuznaya street, Moscow 117997, Russia.

Volume 87 (12 issues) is published in 2026.

Publisher: Trapeznikov Institute of Control Sciences, Russian Academy of Sciences.

65 Profsoyuznaya street, Moscow 117997, Russia; e-mail: redacsia@ipu.rssi.ru; http://ait.mtas.ru, http://ait-arc.ru

Contents

Automation and Remote Control
Vol. 87, No. 9, 2026

Reviews

Data-Driven Control: Solving Some Classical Problems of Control Theory
Ya. I. Kvinto 763

Linear Systems

Closed-Loop State Predictor: Robustness and Safe Implementation
V. O. Nikiforov and D. N. Gerasimov 809

Block Coordinate Descent Method Applied to Stabilization and Time-Optimization Problems for Linear Discrete Systems
K. A. Tsarkov 831

Nonlinear Systems

Constructing Lyapunov Functions in Partial Stability Problems
V. I. Vorotnikov and Yu. G. Martyshenko 846

Notes, Chronicles, Information

The Problem of Optimal Iterative Learning
D. A. Novikov 859

ISSN 0005-1179 (print), ISSN 1608-3032 (online), Automation and Remote Control, 2026, Vol. 87, No. 9, pp. 763–808.
© The Author(s), 2026 published by Trapeznikov Institute of Control Sciences, Russian Academy of Sciences, 2026.
Russian Text © The Author(s), 2026, published in Avtomatika i Telemekhanika, 2026, No. 9, pp. 3–62.

REVIEWS

Data-Driven Control: Solving Some Classical Problems of Control Theory

Ya. I. Kvinto

Trapeznikov Institute of Control Sciences, Russian Academy of Sciences, Moscow, Russia
e-mail: yanakvinto@mail.ru

Received July 12, 2025
Revised May 20, 2026
Accepted May 29, 2026

Abstract—This paper is devoted to data-driven control methods for discrete linear time-invariant systems based on controller design from finite sets of input-output measurements of a system with unknown matrices. The discussed approach is also characterized by the use of linear matrix inequalities. We consider methods based on the Fundamental Lemma and the concept of Data Informativity for quadratic stabilization of a system (with exact data and in the case of bounded exogenous disturbances affecting the system), optimal linear quadratic regulator design, and $\mathcal{H}_2$- and $\mathcal{H}_\infty$-optimal control design. The application features of the methods are considered; their advantages and disadvantages are discussed; and numerical examples are provided to illustrate their performance.

Keywords: data-driven control, Fundamental Lemma, Data Informativity, discrete linear time-invariant system, linear matrix inequalities (LMIs), bounded exogenous disturbances, optimal control

DOI: 10.7868/S1608303226090019

1. INTRODUCTION

The past 10–15 years have seen a considerable expansion of a relatively young branch of control theory, known as data-driven control (DDC). Here, the main idea is to design control for a dynamic system with a partially or completely unknown model using measurements of the input and output signals of the system, without its identification.

At present, numerous different DDC methods are known for solving a great variety of control problems, and new ones are constantly appearing. The history and philosophy of DDC in general, as well as of its individual aspects, approaches, and methods, were discussed, e.g., in the surveys [1–7], books [8–12], and other publications.

Classical control problem formulations remain relevant. First of all, these include stabilization problems, particularly under exogenous disturbances, optimal linear quadratic regulator (LQR) design, and $\mathcal{H}_2$- and $\mathcal{H}_\infty$-optimal control design. Data-driven versions of control methods exist for all of them as well; and they will be considered in detail below.

This survey focuses on direct DDC methods for solving the above problems with various system parameterizations under incomplete information about its model that involve well-established conventional model-based control (MBC) methods, e.g., those with Lyapunov function construction. This approach unites DDC methods based on Jan Willems' behavioral systems theory [13, 14] and his Fundamental Lemma (FL) [15–17] with the methods involving the concept of Data Informativity [18–20].

According to the FL (also called Willems' lemma in the literature), for a controllable linear system, all input-output trajectories of finite length can be parameterized using a single finite-length trajectory generated by a persistently exciting input sequence, i.e., through finite-length vectors of signal measurements. To date, various modifications and extensions of the FL are known as follows: a constructive proof with an estimated lower bound for the persistency of excitation (PE) condition [22]; analysis and design of continuous-time systems [23, 24]; the FL for stochastic systems [25–27] and for controllability verification [28]; relaxation of controllability and PE conditions [29]; the FL for stability verification [30]; the robust FL [31, 32]; the FL for time-varying systems [33–35], systems with input saturation [36, 37], affine [38], switched [39], bilinear [40], and nonlinear [41, 42] systems, and autoregressive models [43]; in predictive control [44–49], optimal control with constraints under data with measurement errors [50], tracking [51, 52], and consensus design [53]; the extended FL in the case of multiple datasets [54]; the augmented FL for constructing alternative non-minimal state vectors [55] and universal input signals [56]; and many others.

In the alternative concept, the data must satisfy certain requirements for each particular problem that are generally milder compared to the FL, i.e., the so-called Data Informativity (DI) conditions. There are diverse ways to use the DI concept as well: analysis of the structural properties of linear systems based on noisy data [57]; observability investigation [58]; analysis and design of continuous-time systems [59]; distributed positive stabilization [60]; analysis of linear systems with conic constraints [61]; design of dissipative [62], affine [63], and parameter-varying [64] systems, and autoregressive models [65]; algebraic controller design in tracking problems [66]; deadbeat control [18]; Data Informativity in the case of multiple datasets and extensions of the FL [67]; applications in data-driven optimization problems [68]; and many other results.

The motivation to survey this particular group of DDC methods stems from the following considerations. First, according to the publications above, this group is quite universal, and the corresponding methods solve a wide variety of control problems. Second, these methods have a rigorous theoretical foundation, including minimal requirements on the amount and quality of the initial data. Third, they are very convenient from a practical standpoint: the controller design is performed using finite sets of experimental data; for theoretical derivations and computations, well-developed and efficient methods of linear algebra, convex optimization, and semidefinite programming, the technique of linear matrix inequalities (LMIs), etc., are used. Fourth, methods from this group can be combined with fundamentally different DDC approaches (e.g., with PID controllers under unknown system model [21]). Naturally, there exist some drawbacks: for example, the use of LMIs may limit the amount of data that can be manipulated without serious computational challenges.

The scope of a single survey is insufficient to cover all existing results. Here, we focus only on *discrete linear time-invariant systems* and the case of a finite set of measurements available for the input and output signals (of known dimensions) of the *open-loop system*. In addition, we restrict the presentation to linear static state-feedback control and use only experimental data for controller design, assuming the data to be exact. The cases of errors in the input and/or output measurements were addressed in [19, 39, 69–73], etc. Other classes of systems and controllers are not described below, nor are alternative methods, e.g., those based on various types of learning, online control, etc.

In some of the presented problems, by assumption, the system is subject to persistent *exogenous disturbances*, often unmeasurable directly but with known general bounds.

In all the problems under consideration, the systems have no other uncertainties apart from unknown matrices and exogenous disturbances. However, note significant results established recently for systems with structural and/or parametric uncertainty based on the combined use of experimental data and a priori information. For example, systems with structured uncertainty

were studied in [74]. In a number of publications by Russian researchers, the proposed methods for time-varying systems [75] and systems with parametric uncertainty [76–78] involve a Lyapunov function: the original uncertain system is "immersed" into an extended completely defined system with an additional artificial disturbance, without requiring the FL conditions or even, in some problems, the DI conditions.

This survey aims to present the diversity of DDC methods that are united by the common ideology of behavioral systems theory (the application of the FL and DI conditions) and the use of LMIs for solving a series of classical control problems, to investigate the features of their application in each problem, and to discuss their advantages and drawbacks.

The remainder of the paper is organized as follows. Section 2 is devoted to the quadratic stabilization problem and data-driven approaches to its solution. Section 3 deals with methods for solving the optimal LQR problem. In Section 4, we present DDC methods for systems with exogenous disturbances, namely, quadratic stabilization (Section 4.2), $\mathcal{H}_2$-optimal control design (Section 4.3), and $\mathcal{H}_\infty$-optimal control design (Section 4.4). For each of the problems, illustrative numerical examples are provided. The findings of this survey are summarized in Section 5.

We adopt the following notation: $\mathbb{R}^{m\times n}$ is the set of real matrices of dimensions $m\times n$; $\mathbb{S}^n$ is the set of symmetric matrices of dimensions $n\times n$; $\mathbb{S}^n_{++}$ is the set of symmetric positive definite matrices of dimensions $n\times n$; $\|\cdot\|$ stands for the spectral norm of a matrix; $^\top$ indicates transpose; † means the right pseudo-inverse; finally, I is an identity matrix of appropriate dimensions. All matrix inequalities are understood in the sense of sign definiteness of corresponding matrices; in LMIs, the symbol $*$ represents blocks that are symmetric to those explicitly specified.

For signals $x(k)\in\mathbb{R}^n$ and $u(k)\in\mathbb{R}^m$, $k=0,1,2,\dots$, we denote finite matrices of length $T\geqslant 0$ composed of their measurements as follows:

$$\begin{aligned}
U :=&[u(0) \quad u(1) \quad \dots \quad u(T-1)] \in \mathbb{R}^{m\times T},\\
X :=&[x(0) \quad x(1) \quad \dots \quad x(T)] \in \mathbb{R}^{n\times (T+1)},\\
X_- :=&[x(0) \quad x(1) \quad \dots \quad x(T-1)] \in \mathbb{R}^{n\times T},\\
X_+ :=&[x(1) \quad x(2) \quad \dots \quad x(T)] \in \mathbb{R}^{n\times T}.
\end{aligned}$$

2. QUADRATIC STABILIZATION

Consider a discrete linear time-invariant system of the form

$$x(k+1)=Ax(k)+Bu(k),\quad k=0,1,2,\dots, \tag{1}$$

where $x(k)\in\mathbb{R}^n$, $u(k)\in\mathbb{R}^m$, $A\in\mathbb{R}^{n\times n}$, and $B\in\mathbb{R}^{n\times m}$.

From this point onwards, let the pair (A,B) be controllable, and let the measured output coincide with the state of the system:

$$y(k)=x(k),\quad k=0,1,2,\dots.$$

The problem is to design a quadratically stabilizing static linear state-feedback controller of the form

$$u(k)=Kx(k),\quad K\in\mathbb{R}^{m\times n}.$$

When the matrices A and B are given (the conventional MBC approach), the problem reduces to finding matrices $P\in\mathbb{S}^n_{++}$ and $K\in\mathbb{R}^{m\times n}$ satisfying the well-known Lyapunov matrix inequality

$$(A+BK)P(A+BK)^\top - P\prec 0. \tag{2}$$

It can be equivalently written as [79]

$$\begin{bmatrix} P & AP + BY \\ * & P \end{bmatrix} \succ 0. \tag{3}$$

The solution $\hat{P}$, $\hat{Y}$ yields a controller with the gain matrix $\hat{K} = \hat{Y}\hat{P}^{-1}$ that quadratically stabilizes system (1).

Now we represent the system using measured data vectors: if (U, X) are given, then system (1) with exactly known matrices A_{tr} and B_{tr} can be written as

$$X_+ = \begin{bmatrix} B_{tr} & A_{tr} \end{bmatrix} \begin{bmatrix} U \\ X_- \end{bmatrix}. \tag{4}$$

The set of *all* systems with matrices A and B capable of generating such data (U, X) will be denoted by

$$\Sigma = \left\{ (A, B) : X_+ = \begin{bmatrix} B & A \end{bmatrix} \begin{bmatrix} U \\ X_- \end{bmatrix} \right\}. \tag{5}$$

Since the data are obtained from system (4), we have $(A_{tr}, B_{tr}) \in \Sigma$. However, based on these data, it is impossible to distinguish the system with matrices (A_{tr}, B_{tr}) from any other system in Σ. Consequently, in the control problem, it is required to design a controller that quadratically stabilizes *all* systems in Σ.

From the Lyapunov function perspective in the DDC approach, one needs to find a common matrix P satisfying (2) for all $(A, B) \in \Sigma$.

The following subsections present the most well-known methods for designing a quadratically stabilizing data-driven controller involving the FL (Section 2.1), the DI concept (Section 2.2), and quadratic matrix inequalities (QMIs, Section 2.3).

2.1. Parameterization Based on the FL

In [69], the conventional model-based formulation of the quadratic stabilization problem for system (1) was modified to the data-driven case using a nonparametric representation of the system from measured data and the FL (Willems et al.) [15, 16]. According to this lemma, the data (measurements) (U, X) must be sufficiently "*rich*" for system identifiability and, hence, for controller design.

Definition 1 [15]. An input signal $[u(0)\ \ u(1)\ \dots\ u(T-1)] \in \mathbb{R}^{m\times T}$ of a system is said to be persistently exciting of order $n + L$, $1 \leqslant L \leqslant T$, if the matrix composed of the shifts of input measurements and output (state) measurements

$$H_L = \begin{bmatrix} u(0) & u(1) & \dots & u(T-L) \\ \vdots & \vdots & & \vdots \\ u(L-1) & u(L) & \dots & u(T-1) \\ x(0) & x(1) & \dots & x(T-L) \end{bmatrix} \tag{6}$$

has full rank:

$$\operatorname{rank} H_L = n + mL.$$

Actually, this means that the set of input signals "covers" the entire space of possible system dynamics, thereby forcing all states and input-output interactions to manifest themselves. Hereinafter, we assume that $L = 1$.

Note that the PE conditions, which underlie the FL, originally appeared in the 1970s, i.e., long before the emergence of DDC theory, and were used in adaptive control. For example, in [80, Theorem 2.4.2] and [81], identifiability properties in adaptive systems were established under the

PE condition. In general, the ideology of DDC is internally closely related to adaptive and robust control [76] since it rests on the combined use of experimental and a priori data in control design for systems with model uncertainties and disturbances; for details, see [75–78].

Lemma 1 (fundamental Lemma, [15, Theorem 1], [16, Lemma 1]). *If in system* (1) *the input* U *of length* T *is persistently exciting of order* $n+1$, *then:*

1. *The rank condition of the FL holds:*

$$rank \begin{bmatrix} U \\ X_- \end{bmatrix} = n + m. \tag{7}$$

2. *Any trajectory* $\begin{bmatrix} \tilde{U} \\ \tilde{X}_- \end{bmatrix}$ *of length* T *can be represented as a linear combination of the measured data:*

$$\begin{bmatrix} \tilde{U} \\ \tilde{X}_- \end{bmatrix} = \begin{bmatrix} U \\ X_- \end{bmatrix} g \quad \forall g \in \mathbb{R}^T;$$

and conversely, any linear combination $\begin{bmatrix} U \\ X_- \end{bmatrix} g$, $g \in \mathbb{R}^T$, *is a trajectory of length* T *for system* (1).

In several methods, including the one presented here, condition (7) is crucial for data-driven controller design without prior model identification.

Let *the measurement vectors* U *and* X_- *be obtained by measuring signals of the open-loop system.* In the method under consideration, see [69, Theorem 2], the gain matrix K satisfies the relation

$$\begin{bmatrix} K \\ I \end{bmatrix} = \begin{bmatrix} U \\ X_- \end{bmatrix} G \tag{8}$$

with some (generally non-unique) unknown matrix $G \in \mathbb{R}^{T \times n}$, i.e., the controller has the form

$$K = UG, \tag{9}$$

and the closed-loop system matrix is expressed from the data as

$$A_K = A + BK = X_+ G \tag{10}$$

with the same matrix G.

Thus, for a stabilizing controller, the matrix $A_K = X_+ G$ must satisfy the classical Lyapunov condition (2). As a result, a stabilizing data-driven controller can be designed using a quadratic Lyapunov function and LMIs as follows.

Theorem 1 [69, Theorem 3]. *Let the rank condition* (7) *of the FL hold. Then any matrix* $Q \in \mathbb{R}^{T \times n}$ *satisfying the inequality*

$$\begin{bmatrix} X_- Q & X_+ Q \\ * & X_- Q \end{bmatrix} \succ 0 \tag{11}$$

yields the gain matrix

$$\hat{K} = U\hat{Q}(X_-\hat{Q})^{-1}. \tag{12}$$

The converse is also valid: if K *is a stabilizing controller, then it can be represented in the form* (12) *with a matrix* Q *satisfying* (11).

Remark 1. What is important, this theorem provides a characterization of the *entire set of stabilizing controllers* of the form $u = Kx$ for all systems with matrices $(A, B) \in \Sigma$ consistent with the original measurements.

Note that in this problem, the dimension of the matrix variable Q depends on the amount of experimental data T, and, in view of (7), $T \geqslant n + m$.

Remark 2. From this point onwards, we consider measurements obtained from the open-loop system. If this system is unstable, then the considered methods may lead to challenges: it may be physically impossible to obtain the required minimum amount of measurements due to instability. In such a situation, other controller design approaches are needed.

2.2. Controller Design Based on the Concept of Data Informativity

Willems' behavioral systems theory and the FL strongly stimulated the development of DDC and demonstrated broad possibilities for parameterizing the system and the control law using measured input and output signals. However, this lemma imposes significant restrictions on the minimum required amount and quality of the data used. Therefore, it was logical to develop this approach toward relaxing the measurement requirements, so that direct data-driven controller design no longer needs system identifiability (i.e., the fulfillment of the FL). This gave rise to the concept of Data Informativity [18, 19].

In [18], the PE condition was replaced by the milder Data Informativity conditions, under which the finite-length trajectory $\begin{bmatrix} U^\top & X_-^\top \end{bmatrix}^\top$ carries as much information as the state-space model (1) with known matrices. As a rule, necessary and sufficient DI conditions for the solvability of a particular problem are formulated individually as a linear matrix equality or inequalities (or systems thereof).

As applied to the stabilization problem for system (1) with a state-feedback control law $u = Kx$, the data (U, X) are informative if there exists a matrix $K \in \mathbb{R}^{m\times n}$ such that the closed-loop matrix $A + BK$ is Schur stable for all $(A, B) \in \Sigma$ consistent with the experimental data. As it turned out, a data-driven control law can be designed without the PE property of the data. More specifically, the following result was proved.

Theorem 2 [18, Theorem 16]. *The experimental data (U, X) are informative for stabilization by a state-feedback control law if and only if*

$$\operatorname{rank} X_- = n \tag{13}$$

and there exists a right pseudo-inverse $X_-^\dagger$ of X_- such that $X_+X_-^\dagger$ is stable.

Based on this theorem, a constructive method was formulated to design a stabilizing controller using the DI concept.

Theorem 3 [18, Theorem 17]. *The experimental data (U, X) are informative for stabilization by a state-feedback control law if and only if there exists a matrix $Q \in \mathbb{R}^{T\times n}$ such that*

$$\begin{bmatrix} X_-Q & X_+Q \\ * & X_-Q \end{bmatrix} \succ 0, \tag{14}$$
$$X_-Q = (X_-Q)^\top.$$

Moreover, K is a stabilizing controller if and only if for some $\hat{Q}$ satisfying (14),

$$\hat{K} = U\hat{Q}(X_-\hat{Q})^{-1}.$$

Obviously, here the LMI and the parameterization of the controller K have the same form (11)–(12) as in the FL-based design (Section 2.1). However, in the case of the DI conditions, the rank condition (7) is not required: the milder condition $X_-Q = (X_-Q)^\top$ appears instead.

Another important difference follows from (13): in the controller design based on Data Informativity, only $T \geqslant n$ measurements are needed, whereas the FL conditions involve at least $T \geqslant n+m$ measurements.

Thus, the DI concept expands the availability of experimental data suitable for controller design: for some (U, X) the FL conditions may fail (the system is not identifiable), but the method with the DI conditions successfully solves the control problem.

2.3. Controller Design via Quadratic Matrix Inequalities

Another parameterization method with Data Informativity conditions was proposed in [82–84]; it is based on Finsler's matrix lemma applied to constraints in the form of QMIs ([84, Theorem 1], see also Lemma 3 in Section 4.1). This result is a special case of a controller design theorem for systems with exogenous disturbances; the corresponding approach is presented in more detail in the second part of the survey, particularly in Section 4.2.2. Here, for the sake of brevity, we provide the final result, which is of interest from the computational point of view.

Theorem 4 [84, Theorem 2]. *The data (U, X) generated by system (1) are informative for stabilization by a state-feedback control law if and only if there exist matrices $P \in \mathbb{S}^n_{++}$, $Y \in \mathbb{R}^{m\times n}$ and a scalar $\beta > 0$ such that*

$$\begin{bmatrix} P-\beta I & * & * & * \\ 0 & -P & * & * \\ 0 & -Y & 0 & * \\ 0 & 0 & Y^\top & P \end{bmatrix} + \begin{bmatrix} X_+ \\ -X_- \\ -U \\ 0 \end{bmatrix} \begin{bmatrix} X_+ \\ -X_- \\ -U \\ 0 \end{bmatrix}^\top \succcurlyeq 0. \tag{15}$$

For such $\hat{P}$ and $\hat{Y}$ with some $\hat{\beta}$, the stabilizing controller has the form $\hat{K} = \hat{Y}\hat{P}^{-1}$ for all $(A, B) \in \Sigma$.

The principal distinction between this method and the previous two is the different structure of the variables in the LMI: in (11) and (14), the unknown matrix Q has dimensions $T \times n$, i.e., it directly depends on the experiment duration (measurement length) and the amount of data obtained, whereas the variables in the matrix inequality (15) are independent of T.

For this problem, the DI condition restricts the experiment duration, $T \geqslant n$, since the existence of a right pseudo-inverse $X_-^\dagger$ is required for a stabilizing controller design (see Theorem 2 in Section 2.2). In other words, X_- has full rank n [18].

Thus, one can choose the computationally most advantageous method depending on the system dimension and the available amount of measurements.

Example 1. In all examples, including this one, the plant is the discretized model of an unstable batch chemical reactor [69, 85] (with a sampling step of 0.1 s) with the matrices

$$A = \begin{bmatrix} 1.178 & 0.001 & 0.511 & -0.403 \\ -0.051 & 0.661 & -0.011 & 0.061 \\ 0.076 & 0.335 & 0.560 & 0.382 \\ 0 & 0.355 & 0.089 & 0.849 \end{bmatrix}, \quad B = \begin{bmatrix} 0.004 & -0.087 \\ 0.467 & 0.001 \\ 0.213 & -0.235 \\ 0.213 & -0.016 \end{bmatrix}. \tag{16}$$

In the data-driven methods, these matrices are supposed to be unknown; they are used only for simulating measurements and solving the problem by the MBC approach, for comparison purposes.

We designed quadratically stabilizing controllers by each of the above methods and compared the results with the MBC approach and with each other. Let the initial condition $x(0)$ be uniformly distributed on the unit sphere: $\|x(0)\| = 1$. We chose $T \geqslant n + m$ to satisfy the conditions of all the methods. By setting $T = 10$, we applied control inputs $u(k)$ uniformly distributed on $[-1, 1]$ to obtain the measurement vectors $U = \begin{bmatrix} u(0) & \dots & u(9) \end{bmatrix}$, $X_- = \begin{bmatrix} x(0) & \dots & x(9) \end{bmatrix}$, and $X_+ = \begin{bmatrix} x(1) & \dots & x(10) \end{bmatrix}$. All computations described in the paper were carried out in MATLAB using the CVX package [86].

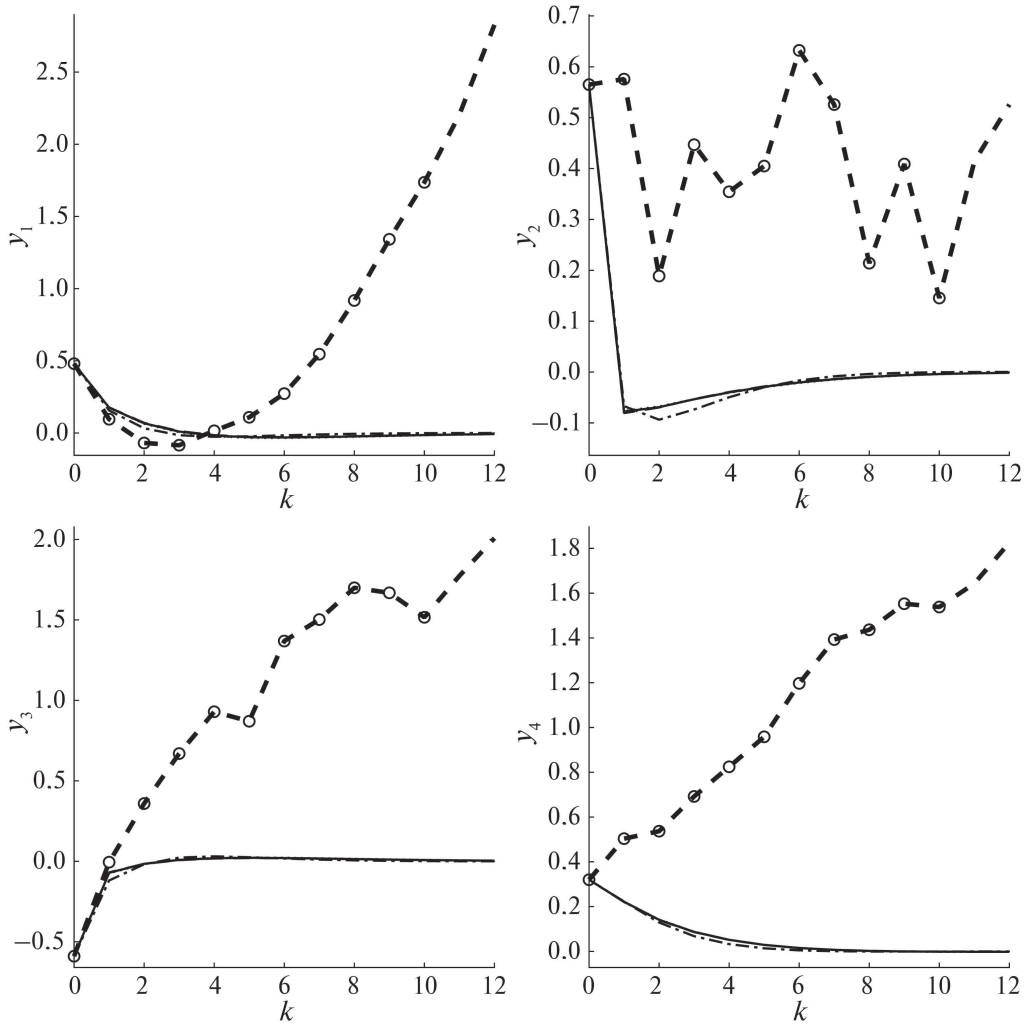


Fig. 1. System outputs in the quadratic stabilization problem.

The conventional MBC solution using formula (3) yielded the controller

$$\hat{K}_{mbc} = \begin{bmatrix} 0.6377 & -1.4696 & 0.0944 & -1.2149 \\ 3.0471 & 0.0622 & 2.9947 & -0.7239 \end{bmatrix}.$$

Before applying the FL-based method, the rank condition was checked again for the data. Computations using Theorem 1 of Section 2.1 gave the controller

$$\hat{K}_{fl} = \begin{bmatrix} 0.6320 & -1.4699 & 0.0940 & -1.2254 \\ 3.0371 & 0.0612 & 3.0009 & -0.7457 \end{bmatrix},$$

and the solution by the DI-based method (Theorem 3 of Section 2.2) was

$$\hat{K}_{di} = \begin{bmatrix} 0.6455 & -1.4713 & 0.1016 & -1.2018 \\ 3.0770 & 0.0563 & 3.0210 & -0.6770 \end{bmatrix}.$$

The controller

$$\hat{K}_{qmi} = \begin{bmatrix} 0.9537 - 1.45460.1305 - 1.5784 \\ 4.5344 - 0.15273.1986 - 1.5027 \end{bmatrix}$$

was obtained by solving the problem based on QMIs (Theorem 4 of Section 2.3) with $\hat{\beta} = 7.8 \cdot 10^{-6}$.

Figure 1 shows the following trajectories: the outputs $y(k) = x(k)$, $k = 0, 1, 2, \dots$, of the open-loop system (thick dashed line, with circles marking the measurements X used) and those of the closed-loop system, namely, the outputs $y_{mbc}(k)$ (the MBC solution, thin solid line), $y_{fl}(k)$ (the FL-based method, thin dashed line), $y_{di}(k)$ (the DI-based method, thin dotted line), and $y_{qmi}(k)$ (the DI-based method with QMIs, thin dash-dotted line).

The results were compared with the MBC solution by assessing the closeness of the corresponding trajectories:

$$\|y_{mbc}(k) - y_{fl}(k)\|_2 = 0.0073, \quad \|y_{mbc}(k) - y_{di}(k)\|_2 = 0.0088,$$
$$\|y_{mbc}(k) - y_{qmi}(k)\|_2 = 0.0696, \quad k = 0, 1, 2, \dots.$$

As the measurement length T increases, the trajectory in the method with QMIs approaches the MBC one more closely.

In fact, the problem formulation has no control requirements other than the stabilizing property. Therefore, the closeness of the matrices $\hat{K}$ and the trajectories does not characterize the quality of the solutions at all: the effectiveness of the methods is confirmed by the fact that they all successfully stabilized the system. However, an additional advantage of data-driven control is the observed degree of closeness of the MBC and DDC solutions.

3. THE OPTIMAL LINEAR QUADRATIC REGULATOR PROBLEM

Consider a system of the form

$$x(k+1) = Ax(k) + Bu(k) \tag{17}$$

with a fixed initial condition $x(0) = x_0$; $u(k) \in \mathbb{R}^m$, $x(k) \in \mathbb{R}^n$, $A \in \mathbb{R}^{n\times n}$, and $B \in \mathbb{R}^{n\times m}$.

Unless otherwise stated, the performance output is

$$z(k) = x(k), \quad k = 0, 1, 2, \dots.$$

The LQR problem is to design a state-feedback control law

$$u(k) = Kx(k), \quad K \in \mathbb{R}^{m\times n},$$

that minimizes the performance criterion (also termed the performance index or cost function in the literature)

$$J(x_0, u) = \sum_{k=1}^{\infty} (x(k)^\top Q x(k) + u(k)^\top R u(k)) \tag{18}$$

and makes the closed-loop matrix $A + BK$ stable. By assumption, the matrices of compatible dimensions $Q = Q^\top \succcurlyeq 0$ and $R = R^\top \succ 0$ are given, and the pair (Q, A) is observable.

As is well known (e.g., see [87]), the solution of this problem is provided by the controller

$$K = -(R + B^\top P^+ B)^{-1} B^\top P^* A, \tag{19}$$

where $P^* \in \mathbb{S}^n_{++}$ is the unique solution of the discrete Riccati equation

$$A^\top PA - P - A^\top PB(R + B^\top PB)^{-1} B^\top PA + Q = 0. \tag{20}$$

The minimum value of the performance criterion is

$$J^{LQR}_{MBC} = x_0^\top P^* x_0. \tag{21}$$

If the matrices A and B are unknown, the DDC framework currently suggests several solution methods. Most of them involve the Lyapunov function, the Riccati inequality, and various parameterizations of the system from the data. Below we discuss the approaches based on the Fundamental Lemma (Section 3.1) and the Data Informativity concept (Section 3.2). Also, the results for the suboptimal LQR problem with the DI condition (Section 3.3) and the finite-horizon LQR problem with the Fundamental Lemma (Section 3.4) will be presented.

3.1. Parameterization Based on the Fundamental Lemma

Let the rank condition (7) of the FL hold.

In [69, 88], the original LQR problem was written as

$$\min_{P,K} \operatorname{trace}(QP + K^\top RKP)$$

subject to the constraints

$$(A + BK)P(A + BK)^\top - P + I \preccurlyeq 0, \quad P \succcurlyeq I.$$

Using the parameterization (9)–(10), the cited authors passed directly to a data-driven formulation of the problem as follows:

$$\min_{P,K,G} \operatorname{trace}(QP + K^\top RKP)$$

subject to the constraints

$$(X_+G)P(X_+G)^\top - P + I \preccurlyeq 0,$$
$$\begin{bmatrix} K \\ I \end{bmatrix} = \begin{bmatrix} U \\ X_- \end{bmatrix} G, \quad P \succcurlyeq I.$$

After the change of variables $Y = KP$ and a series of standard transformations, one obtains a semidefinite programming problem with experimental data:

$$\min_{Y \in \mathbb{R}^{T\times n},\, Z \in \mathbb{R}^{m\times m}} \operatorname{trace}(QX_-Y) + \operatorname{trace}(Z) \tag{22}$$

subject to the constraints

$$\begin{bmatrix} X_-Y - I & X_+Y \\ * & X_-Y \end{bmatrix} \succcurlyeq 0, \quad \begin{bmatrix} Z & R^{1/2}UY \\ * & X_-Y \end{bmatrix} \succcurlyeq 0. \tag{23}$$

Its solution is provided by the optimal linear quadratic regulator

$$\hat{K} = U\hat{Y}(X_-\hat{Y})^{-1}. \tag{24}$$

To ensure the unique matrix G in the parameterization $K = UG$, $A + BK = X_+G$, it was proposed in [88] to augment the constraints (23) with the condition

$$\left(I - \begin{bmatrix} U \\ X_- \end{bmatrix}^\dagger \begin{bmatrix} U \\ X_- \end{bmatrix}\right) Y = 0;$$

in this case, the optimal regulator still has the form (24).

According to [69, Theorem 4], under the exact data (U, X) and the rank condition of the FL, the regulator (24) coincides with the exact MBC solution (19).

Obviously, the applicability of this method requires $T \geqslant n + m$ measurements.

3.2. Parameterization Based on the Data Informativity Concept

In the LQR case, the DI condition coincides with the FL. It was formulated in [18, Theorem 28]: an optimal linear quadratic regulator can be designed from the available measured data if and only if the rank condition (7) of the FL holds; consequently, one has the restriction $T \geqslant m + n$ on the measurement length. The following constructive algorithm was described therein to design the optimal feedback control law.

Algorithm 1 ([18], Theorem 29).

1. Solve the semidefinite programming problem

$$\max_{P \in \mathbb{S}^n_{++}} \operatorname{trace} P \tag{25}$$

subject to the constraint

$$L(P) \preccurlyeq 0, \tag{26}$$

where the linear operator $P \mapsto L(P)$ is

$$L(P) := {X_-}^\top P X_- - X_+^\top P X_+ - X_-^\top Q X_- - U^\top R U.$$

Denote by P^* the resulting unique solution.

2. Find a right pseudo-inverse $\hat{X}_-^\dagger$ such that

$$L(P^*)\hat{X}_-^\dagger = 0. \tag{27}$$

3. Compute the optimal linear quadratic regulator

$$\hat{K} = U\hat{X}_-^\dagger, \tag{28}$$

yielding the minimum value

$$J_{ID}^{LQR} = {x_0}^\top P^* x_0 \tag{29}$$

of the performance criterion (18).

When solving the LQR problem with this algorithm, the dimensions of the unknown matrix P are only $n \times n$ and do not depend on the measurement length, whereas the unknown matrices Y and Z in problem (22)–(23) have dimensions $T \times n$ and $m \times m$, respectively. An undoubted advantage of the DI-based solution is the ability to compute the minimum value of the criterion (29) directly from the measured data, which cannot be done using the FL-based method (see the previous subsection).

A fundamentally different approach to the infinite-horizon LQR problem was proposed in [89, 90]. According to this approach, finite-length measurement vectors are used to estimate the extended (one-step-ahead) observability matrix and an auxiliary matrix, and both matrices are employed to construct the regulator. As noted in [18, 89], the resulting control *tends to the optimal* linear quadratic solution only in the case of infinitely many measurements, and optimality cannot be ensured for experiments of insufficient length. This feature distinguishes the mentioned method from the data-driven design of the optimal LQR using the Fundamental Lemma (Section 3.1) and Data Informativity (Section 3.2), where the *exact optimal* solution is obtained from a finite dataset. Therefore, we will not dwell on this method further.

3.3. Suboptimal LQR Design with the Data Informativity Condition

The DI concept has also proved very useful for designing suboptimal linear quadratic control. For system (17) with a fixed initial condition $x(0) = x_0$ and the performance output

$$z(k) = Cx(k) + B_2u(k)$$

with *known* matrices $C \in \mathbb{R}^{l\times n}$ and $B_2 \in \mathbb{R}^{l\times m}$, a state-feedback control law $u = Kx$ such that the closed-loop matrix $A + BK$ is Schur stable and the quadratic performance criterion (18) takes values below a *prescribed level* $\gamma > 0$, $J(x_0, u) < \gamma$, was constructed from finite-length measured data in [91].

The authors' derivations stemmed from [91, Proposition 1]: if the matrices A and B are known, then K is a suboptimal linear quadratic regulator if and only if there exists a matrix $P \in \mathbb{S}^n_{++}$ such that

$$\begin{gathered}(A + BK)^\top P(A + BK) - P + Q + K^\top RK \prec 0,\\ {x_0}^\top P x_0 \prec \gamma.\end{gathered}$$

In the case of unknown system matrices, the suboptimal LQR design involves the DI condition for (U, X) to ensure stabilizability by a state-feedback control law $u = Kx$ (see Theorem 2 in Section 2.2).

As a consequence, the following result was established (see [91, Theorem 9]: for given x_0 and $\gamma > 0$, it is possible to construct a suboptimal linear quadratic regulator from measured data if and only if there exist a matrix $P \in \mathbb{S}^n_{++}$ and a right pseudo-inverse $X^\dagger_-$ of X_- such that

$$\begin{gathered}(X_+X^\dagger_-)^\top P X_+X^\dagger_- - P + Q + (UX^\dagger_-)^\top RUX^\dagger_- \prec 0,\\ {x_0}^\top P x_0 \prec \gamma.\end{gathered}$$

For the found $\hat{X}^\dagger_-$, the suboptimal linear quadratic regulator has the form $\hat{K} = U\hat{X}^\dagger_-$.

From a computational point of view, it is convenient to pass to an equivalent problem in an LMI form using standard transformations. To this end, in (18) let $Q = C^\top C$, $R = {B_2}^\top B_2$, $C^\top B_2 = 0$, $x_0 \in \mathbb{R}^n$, and $\gamma > 0$; after straightforward manipulations, one arrives at the following result.

Proposition 1 [91, Proposition 10]. *The data (U, X) are informative for suboptimal linear quadratic control if and only if there exist matrices $P \in \mathbb{S}^n_{++}$ and $Y \in \mathbb{R}^{T\times n}$ such that*

$$\begin{gathered}\begin{bmatrix} P & * & * \\ X_+Y & P & * \\ (CX_- + B_2U)Y & 0 & I \end{bmatrix} \succ 0, \quad \begin{bmatrix} \gamma & * \\ x_0^\top & P \end{bmatrix} \succ 0, \\ P = X_-Y.\end{gathered} \tag{30}$$

Moreover, K is a suboptimal linear quadratic regulator for all *systems with matrices $(A, B) \in \Sigma$ from* (5) *if and only if $\hat{K} = U\hat{Y}\hat{P}^{-1}$ for some matrices $\hat{P}$, $\hat{Y}$ found from* (30).

To satisfy these conditions, the matrix X_- must have full rank, i.e., only $T \geqslant n$ input-output measurements are required for designing a *suboptimal* linear quadratic regulator by the above method. Recall that $T \geqslant n + m$ measurements are needed for the unique identification of the system matrices A and B and the *optimal* LQR design (Sections 3.1 and 3.2).

Below we present one method for solving the finite-horizon optimal LQR problem from a finite set of measured data and compare its accuracy with the infinite-horizon methods in the numerical example.

3.4. Finite-Horizon LQR Design Based on the Fundamental Lemma

For system (17), the optimal LQR problem on a finite horizon $N \geqslant 0$ with the performance criterion

$$J = x(N)^\top Q_f x(N) + \sum_{k=1}^{N-1} (x^\top Q x + u^\top R u), \quad Q_f \succcurlyeq 0,$$

was considered in [92].

When the matrices A and B are known, the optimal solution in the form of a sequence $u^*(0), \dots, u^*(N-1)$ with $u^*(k) = K(k)x(k)$, $k = 1, \dots, N-1$, is given by

$$K(k) = -(R + B^\top P(k+1)B)^{-1} B^\top P(k+1)A, \tag{31}$$

where $P(k)$ is the solution of the discrete Riccati equation

$$P(k) = A^\top P(k+1)A - A^\top P(k+1)B(R + B^\top P(k+1)B)^{-1} B^\top P(k+1)A + Q,$$
$$P(N) = Q_f.$$

Obviously, the control law (31) depends on time and is defined on the time interval $[0, N]$. Assuming the observability of the pair (Q, A), the sequence of the matrices $P(k)$ converges to the matrix P^+ as $N \to \infty$, which is the solution of the discrete Riccati equation (20). Thus, one obtains the optimal infinite-horizon linear quadratic regulator (19).

If the matrices A, B are unknown, the data-driven parameterization (8) with the Fundamental Lemma (7) is employed.

Theorem 5 [92, Theorem 3]. *The optimal solution of the finite-horizon LQR problem is a sequence $K = (K_0, \dots, K_{N-1})$ such that*

$$\hat{K}_k = U\hat{Y}_k \hat{P}_k^{-1}, \tag{32}$$

where the matrices $\hat{P}_k \in \mathbb{S}^n_{++}$ and $\hat{Y}_k \in \mathbb{R}^{T\times n}$ are the solution of the semidefinite programming problem

$$\min_{P_k, Y_k, Z_k} \operatorname{trace}(Q_f P_N) + \sum_{k=1}^{N-1} \Big(\operatorname{trace} Q P_k + \operatorname{trace} Z_k \Big) \tag{33}$$

subject to the constraints

$$P_0 \succcurlyeq I_n, \quad P_k = X_- Y_k, \tag{34}$$
$$\begin{bmatrix} P_{k+1} - I_n & X_+ Y_k \\ * & P_k \end{bmatrix} \succcurlyeq 0, \quad \begin{bmatrix} Z_k & R^{1/2} U Y_k \\ * & P_k \end{bmatrix} \succcurlyeq 0, \quad k = 0, \dots, N-1,$$
$$P = (P_1, \dots, P_N), \quad Y = (Y_0, \dots, Y_{N-1}), \quad Z = (Z_0, \dots, Z_{N-1}).$$

For an observable pair (Q, A), as $N \to \infty$, this solution converges to the optimal infinite-horizon LQR solution (19), and the minimum required number of measurements is dictated by the Fundamental Lemma: $T \geqslant n + m$. Importantly, the dimensions of the matrix variables Y_k depend on T. Another distinguishing feature is inherent in the problem formulation and solution method: the solution is not a single matrix K but a set of matrices.

Let us mention another method for designing the optimal finite-horizon LQR from measured data [93], which combines stabilization conditions and the classical finite-horizon LQR. (For the sake of brevity, it is not presented in the survey.)

Example 2. To compare the above LQR design methods, we considered a more general reactor model from Example 1 with the matrices (16) and the performance output

$$z(k) = Cx(k) + B_2u(k),$$

setting $C = I$ and $B_2 = 0$ so that $C^\top B_2 = 0$. Let $Q = I$ and $R = I$. The input-output measurements were generated for $T = 10$ in the same way as in Example 1, so the applicability conditions of all the methods were satisfied.

We computed the model-based optimal LQR using formulas (19)–(20) (or in MATLAB using the *idare* function):

$$\hat{K}_{mbc}^{LQR} = \begin{bmatrix} 0.0811 & -0.7246 & -0.1444 & -0.6792 \\ 2.1509 & 0.0667 & 1.4934 & -0.9709 \end{bmatrix}, \quad \hat{J}_{mbc}^{LQR} = 3.9116.$$

The data-driven solutions using the Fundamental Lemma (22)–(24) (Section 3.1) and the Data Informativity conditions (25)–(28) (Section 3.2) were, respectively,

$$\hat{K}_{fl}^{LQR} = \begin{bmatrix} 0.0811 & -0.7246 & -0.1444 & -0.6791 \\ 2.1508 & 0.0667 & 1.4934 & -0.9709 \end{bmatrix},$$

$$\hat{K}_{di}^{LQR} = \begin{bmatrix} 0.0811 & -0.7246 & -0.1444 & -0.6792 \\ 2.1509 & 0.0667 & 1.4934 & -0.9709 \end{bmatrix}, \quad \hat{J}_{di}^{LQR} = 3.9116.$$

Obviously, these regulators almost coincide with the MBC solution.

Since this solution and the data-driven one with the DI conditions were very close: $\|\hat{K}_{mbc}^{LQR} - \hat{K}_{di}^{LQR}\| \approx 4.1350 \cdot 10^{-7}$, we also obtained very close values of the performance criteria (21) and (29).

In view of the strict inequality $J(x_0, u) < \gamma$ in the suboptimal control problem (30) from Section 3.3, we set a slightly larger value of $\gamma = 3.9117$. The formulation of this problem is somewhat different, so the result for the same parameters, where possible, was expected to vary from the MBC solution and the first two data-driven methods. For instance, for the above C and B_2, we obtained the suboptimal regulator

$$\hat{K}_{sub}^{LQR} = \begin{bmatrix} -0.6008 & -1.5819 & -0.2738 & -2.4524 \\ 3.0702 & -0.0725 & 3.0633 & -2.2513 \end{bmatrix}.$$

For designing the finite-horizon optimal linear quadratic regulator, we set $N = 20$ and $Q_f = I$ and solved problem (33)–(34) (Section 3.4). The solution was a sequence of control gains of the form (32): $\hat{K}_{fin}^{LQR} = (\hat{K}_{fin\,0}^{LQR}, \ldots, \hat{K}_{fin\,19}^{LQR})$. (The values are omitted here for brevity.) Applying them to the input yielded the system trajectory on the time interval $[0, 20]$.

Figure 2 shows the following trajectories: the performance outputs $z_i(k) = x_i(k)$, $i = 1, 2, 3, 4$, of the open-loop system (thick dashed line, with circles marking the initial measurements X) and those of the closed-loop system, namely, the outputs $z_{mbc}(k)$ (the MBC solution, thin solid line), $z_{fl}(k)$ (the FL-based method, thin dashed line), $z_{di}(k)$ (the DI-based method, thin dotted line), $z_{sub}(k)$ (the suboptimal control case, thin dash-dotted line), and $z_{fin}(k)$ (the finite-horizon LQR, thin solid line).

We numerically estimated the closeness of the trajectories:

$$\|z_{mbc}(k) - z_{fl}(k)\|_2 = 2.3563 \cdot 10^{-5}, \quad \|z_{mbc}(k) - z_{di}(k)\|_2 = 1.4229 \cdot 10^{-7},$$
$$\|z_{mbc}(k) - z_{sub}(k)\|_2 = 0.1030, \quad \|z_{mbc}(k) - z_{fin}(k)\|_2 = 0.0066, \quad k = 0, 1, 2, \ldots.$$

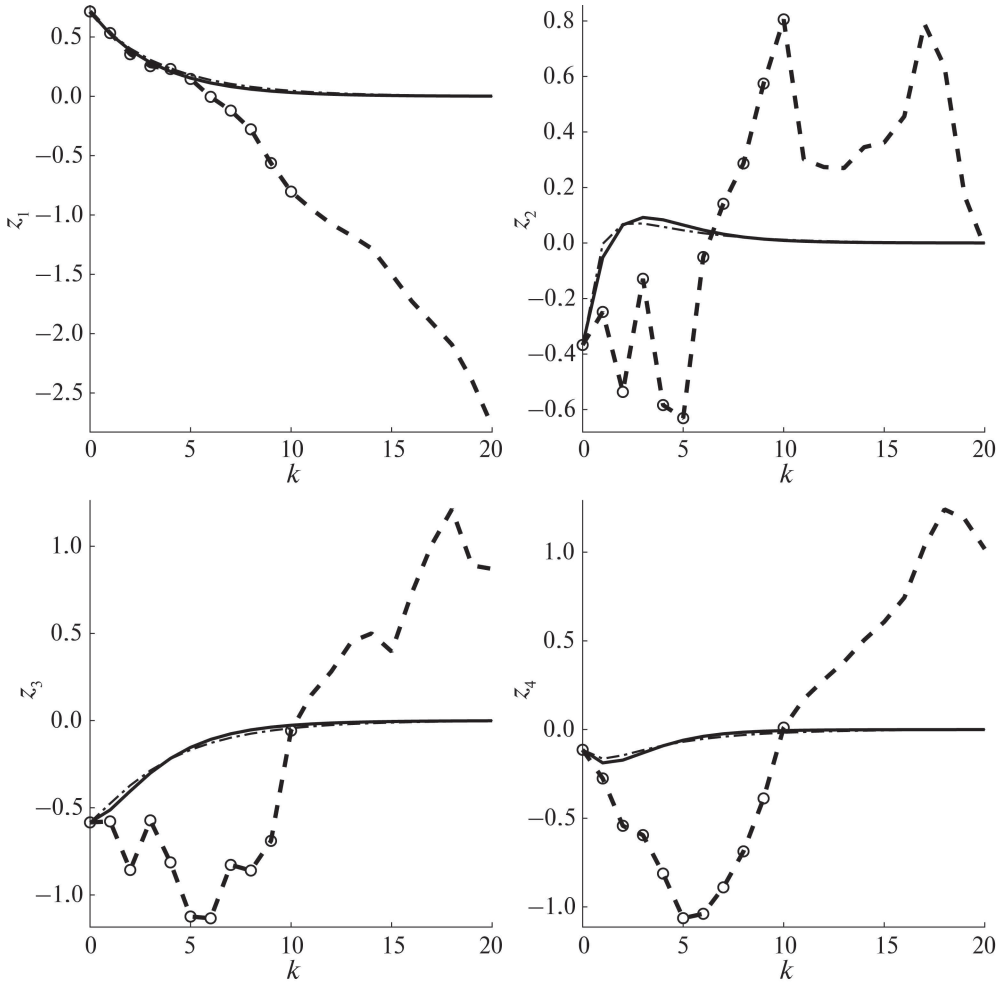


Fig. 2. System outputs in the LQR problem.

In this example, a performance criterion is specified and a unique optimal solution exists; therefore, the closeness of the solutions (the found regulators and the resulting output trajectories) to the MBC one directly characterizes the accuracy of the DDC methods under consideration. Clearly, the methods based on the Fundamental Lemma and the Data Informativity conditions provided high accuracy. The suboptimal LQR design gave a generally acceptable solution, taking into account the differences in problem formulations. For the finite-horizon LQR problem, already with 20 steps we obtained a solution very close to the exact one (in terms of the distance between the trajectory $x_{fin}(k)$ and the exact MBC counterpart $x_{mbc}(k)$); and the accuracy will improve when increasing the number of steps N.

4. CONTROL OF SYSTEMS WITH BOUNDED EXOGENOUS DISTURBANCES

In controller design for systems with exogenous disturbances, two alternative approaches are possible. According to the first approach, called Certainty Equivalence in the literature, all constructs are carried out under the assumption of no disturbances, but estimates of its level are provided under which the control law exists and stabilizes the system with the required performance. This approach seems less interesting from the data-driven perspective: in practice, from measured input and state/output values it is impossible to estimate the bounds within which this (generally speaking, unknown) disturbance can be neglected, owing to the frequent unavailability

of its measurements. As noted, such control methods are effective for small disturbances [88, 94]. From a computational viewpoint, methods of this group reduce to exact-data methods, e.g., those presented in Sections 2.1, 3.1, etc., and the main focus is on estimating the admissible disturbance level [69]. In this paper, we do not dwell on the Certainty Equivalence approach; interested readers are referred, e.g., to [18, 69, 88, 95–97].

The second approach directly accounts for the presence of disturbances, attempting to limit their effect. It is necessary to describe the set of admissible disturbances in some way, so as to design a control law (possibly optimal in terms of a certain criterion) that stabilizes the system for any possible disturbances from this set. This approach will be addressed in the following sections with regard to the quadratic stabilization problem for systems with exogenous disturbances and the problems of $\mathcal{H}_2$- and $\mathcal{H}_\infty$-optimal control design.

4.1. Description of the System and the Set of Disturbances

Consider a general discrete-time system with exogenous disturbances:

$$\begin{aligned} x(k+1) &= Ax(k) + B_1u(k) + Dw(k), \\ z(k) &= Cx(k) + B_2u(k), \end{aligned} \tag{35}$$

where $x(k) \in \mathbb{R}^n$, $u(k) \in \mathbb{R}^m$, and $z(k) \in \mathbb{R}^l$; the matrices $A \in \mathbb{R}^{n\times n}$ and $B_1 \in \mathbb{R}^{n\times m}$ are unknown in the DDC setting; the matrices $C \in \mathbb{R}^{l\times n}$, $B_2 \in \mathbb{R}^{l\times m}$, and $D \in \mathbb{R}^{n\times p}$ are given.

The values of a disturbance $w(k) \in \mathbb{R}^p$ are supposed to be unknown and unmeasurable. Their particular realization of length T will be denoted by

$$W = [w(0) \quad w(1) \quad \dots \quad w(T-1)] \in \mathbb{R}^{p\times T}.$$

A system of the form

$$x(k+1) = Ax(k) + B_1u(k) + Dw(k) \tag{36}$$

can be written in measurement terms:

$$X_+ = AX_- + B_1U + DW = \begin{bmatrix} B_1 & A & D \end{bmatrix} \begin{bmatrix} U \\ X_- \\ W \end{bmatrix}.$$

The complete set of systems with all such matrices (A, B_1, D) that can generate the data (U, X, W) will be denoted by

$$\Sigma_W = \left\{ (A, B_1, D) : X_+ = \begin{bmatrix} B_1 & A & D \end{bmatrix} \begin{bmatrix} U \\ X_- \\ W \end{bmatrix} \right\}; \tag{37}$$

in particular, system (36) with exactly known matrices $(A_{tr}, B_{1tr}, D_{tr})$ also belongs to this set: $(A_{tr}, B_{1tr}, D_{tr}) \in \Sigma_W$.

In many publications devoted to DDC of systems with exogenous disturbances (e.g., [19, 20, 82–84, 94]), the set of admissible exogenous disturbances is parameterized using QMIs. More specifically, the unknown disturbance matrix W is supposed to belong to the set

$$\mathbf{W} = \left\{ W \in \mathbb{R}^{p\times T} : \begin{bmatrix} I \\ W^\top \end{bmatrix}^\top \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{12}^\top & \Phi_{22} \end{bmatrix} \begin{bmatrix} I \\ W^\top \end{bmatrix} \succcurlyeq 0 \right\} \tag{38}$$

with given matrices $\Phi_{11} \in \mathbb{S}^p$, $\Phi_{12} \in \mathbb{R}^{p\times T}$, and $0 \succ \Phi_{22} \in \mathbb{S}^T$. Let us denote $\Phi = \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{12}^\top & \Phi_{22} \end{bmatrix} \in \mathbb{S}^{(n+T)\times(n+T)}$.

This inequality is quadratic in the disturbance matrix W, implying that all admissible disturbances lie *in a matrix ellipsoid*. A QMI of this form is convenient for describing the disturbance set because it can express various constraints on the disturbances.

For example, with $\Phi_{12} = 0$ and $\Phi_{22} = -I$, we obtain a disturbance W with bounded energy over the entire experiment duration $[0, \dots, T-1]$:

$$WW^\top = \sum_{k=0}^{T-1} w_k w_k^\top \preccurlyeq \Phi_{11}. \tag{39}$$

1. For W with an upper bound on the maximum eigenvalue $\sigma_{\max}(W) \leqslant \hat{\sigma}$, one should set $\Phi_{11} = \hat{\sigma} I$ in (39).

2. For W with bounded energy, i.e., $\|w_k\|_2 \leqslant \varepsilon$, $k = 0, \dots, T-1$, the constraints are written as (39) with $\Phi_{11} = \varepsilon^2 T I$. This constraint introduces conservatism, and its degree grows with T [19].

3. Upper bounds on the maximum deviations of each component of W, $\|w_{k,j}\|_2 \leqslant \hat{w}_j$, $k = 0, \dots, T-1$, $j = 1, \dots, p$, can be written as (39) with $\Phi_{11} = \sum_{j=1}^p \hat{w}_j{}^2 T I$.

4. The case of exact measurements ($w = 0$) is characterized by the matrix $\Phi = \begin{bmatrix} 0 & 0 \\ 0 & -I \end{bmatrix}$.

In the subsequent derivations, the description of the admissible disturbance set (38) allows using the matrix S-procedure [82–84] and Finsler's matrix lemma [84]. We present them in a form convenient for the problems under consideration.

Lemma 2 (strict matrix S-procedure, [84, Proposition 3]). *Consider given matrices $M, N \in \mathbb{S}^{k+l}$ of the form*

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{12}^\top & M_{22} \end{bmatrix} \quad \text{and} \quad N = \begin{bmatrix} N_{11} & N_{12} \\ N_{12}^\top & N_{22} \end{bmatrix}, \tag{40}$$

where $M_{22} \preccurlyeq 0$, $N_{22} \preccurlyeq 0$, and $\ker N_{22} \subseteq \ker N_{12}$.

Suppose the existence of a matrix $\tilde{Z} \in \mathbb{R}^{l\times k}$ such that

$$\begin{bmatrix} I \\ \tilde{Z} \end{bmatrix}^\top N \begin{bmatrix} I \\ \tilde{Z} \end{bmatrix} \succ 0.$$

Then the quadratic matrix inequality

$$\begin{bmatrix} I \\ Z \end{bmatrix}^\top M \begin{bmatrix} I \\ Z \end{bmatrix} \succ 0$$

holds for all $Z \in \mathbb{R}^{l\times k}$ such that

$$\begin{bmatrix} I \\ Z \end{bmatrix}^\top N \begin{bmatrix} I \\ Z \end{bmatrix} \succcurlyeq 0$$

if and only if there exist scalars $\alpha \geqslant 0$ and $\beta > 0$ satisfying

$$M - \alpha N \succcurlyeq \begin{bmatrix} \beta I & 0 \\ 0 & 0 \end{bmatrix}.$$

Lemma 3 (matrix Finsler's lemma, [84, Theorem 1]). *Let matrices $M, N \in \mathbb{S}^{k+l}$ be specified by (40), and let the following conditions be valid:*

1) $M_{11} = 0$ *and* $M_{22} \preccurlyeq 0$.

2) $N_{22} \preccurlyeq 0$ *and* $N_{11} - N_{12}N_{22}^{\dagger}N_{12}^{\top} = 0$.

3) *There exists a matrix* G *such that* $M_{11} + G^{\top}M_{22}G \succ 0$ *and* $N_{22}G = N_{12}^{\top}$.

Then the inequality

$$\begin{bmatrix} I \\ Z \end{bmatrix}^{\top} M \begin{bmatrix} I \\ Z \end{bmatrix} \succcurlyeq 0$$

holds for all $Z \in \mathbb{R}^{l\times k}$ *such that*

$$\begin{bmatrix} I \\ Z \end{bmatrix}^{\top} N \begin{bmatrix} I \\ Z \end{bmatrix} = 0$$

if and only if there exists a scalar $\alpha \in \mathbb{R}$ *satisfying*

$$M - \alpha N \succcurlyeq 0.$$

Now we proceed to the known data-driven methods for systems with bounded exogenous disturbances.

4.2. Quadratic Stabilization of Systems with Exogenous Disturbances

If the matrices A and B_1 in the system with exogenous disturbances

$$x(k+1) = Ax(k) + B_1u(k) + Dw(k), \quad \|w_k\|_2 \leqslant w^*, \quad k = 0, 1, 2, \ldots,$$

are known, then the quadratic stabilization problem of the closed-loop system with the controller $u = Kx$ reduces to finding matrices $P \in \mathbb{S}^n_{++}$ and $K \in \mathbb{R}^{m\times n}$ in the Lyapunov inequality (3) [79], by analogy with the disturbance-free case (Section 2).

In the case of unknown matrices A and B_1, one needs to construct a *common* quadratic Lyapunov matrix P for all systems with matrices $(A, B_1) \in \Sigma_W$ (37).

Below we will show the solution of this problem by data-driven methods using the FL and the linear fractional transformation (Section 4.2.1), the DI concept and QMIs (Section 4.2.2), the FL and Petersen's lemma (Section 4.2.3), and QMIs for each input-output measurement instant (Section 4.2.4). Then we will provide a numerical example to compare these methods.

4.2.1. Fundamental Lemma and Linear Fractional Transformation

A data-driven quadratically stabilizing controller design involving the FL-based parameterization (Section 2.1) and also a linear fractional transformation (LFT) was proposed in [98]. Let the original system be given by

$$x(k+1) = Ax(k) + B_1u(k) + Dw(k), \tag{41}$$

where the notation is the same as in (35).

The authors used an equivalent description of the disturbance set (38) in the form

$$\tilde{\mathbf{W}} = \left\{ W \in \mathbb{R}^{p\times T} : \begin{bmatrix} W \\ I \end{bmatrix}^{\top} \begin{bmatrix} \tilde{\Phi}_{11} & \tilde{\Phi}_{12} \\ \tilde{\Phi}_{12}^{\top} & \tilde{\Phi}_{22} \end{bmatrix} \begin{bmatrix} W \\ I \end{bmatrix} \succcurlyeq 0 \right\}, \tag{42}$$

where $\tilde{\Phi}_{11} \in \mathbb{S}^p$, $\tilde{\Phi}_{12} \in \mathbb{R}^{p\times T}$, and $\tilde{\Phi}_{22} \in \mathbb{S}^T_{++}$.

First, it is necessary to parameterize the set of all systems closed by the controller $u = Kx$. For the system written in measurement terms

$$X_+ = AX_- + B_1U + DW,$$

the set of quadratically stabilizing controllers can be parameterized using the measured data exactly as in (8):

$$\begin{bmatrix} K \\ I \end{bmatrix} = \begin{bmatrix} U \\ X_- \end{bmatrix} G, \quad G \in \mathbb{R}^{T\times n}.$$

Then the gain matrix is computed as $K = UG$ with the condition $X_-G = I$.

Therefore, we obtain the following parameterization of the set of system matrices (41) closed by a controller with the gain matrix $K = UG$ based on the measured data:

$$A_G = (X_+ - DW)G \tag{43}$$

for *some* $W \in \tilde{\mathbf{W}}$ such that $(X_+ - DW)\begin{bmatrix} U \\ X_- \end{bmatrix}^{\perp} = 0$. Here, $^{\perp}$ indicates the kernel basis matrix of an appropriate matrix.

Obviously, in contrast to (10), this parameterization of the closed-loop system matrices contains the unknown disturbance matrix W as a parameter, i.e., it is not unique.

Next, it is necessary to consider a broader set of all stable matrices of the form (43) with *any* admissible disturbances from the set $\tilde{\mathbf{W}}$:

$$A_G^{st} = \left\{ A_K : A_K = (X_+ - DW)G,\ W \in \tilde{\mathbf{W}} \right\}. \tag{44}$$

Although this transition leads to more conservative results, it avoids computing the kernel of the matrix $\begin{bmatrix} U^\top & X_-^\top \end{bmatrix}^\top$ and allows applying a more convenient parameterization and standard control methods. For instance, matrices from (44) are represented as a lower linear fractional transformation of the closed-loop original system (41) with the disturbance $W \in \tilde{\mathbf{W}}$:

$$\begin{bmatrix} x(k+1) \\ \tilde{z}(k) \end{bmatrix} = \begin{bmatrix} X_+G & D \\ -G & 0 \end{bmatrix} \begin{bmatrix} x(k) \\ \tilde{w}(k) \end{bmatrix},$$
$$\tilde{w}(k) = W\tilde{z}(k),$$
$$X_-G = I.$$

With the above representation, the following is the case.

Proposition 2 [98, Proposition 6]. *If there exist matrices* $\Psi \succ 0$ *and* $G \in \mathbb{R}^{T\times n}$ *satisfying*

$$\begin{bmatrix} * & * \\ * & * \\ * & * \\ * & * \end{bmatrix}^\top \begin{bmatrix} -\Psi & 0 & 0 & 0 \\ * & -\Psi & 0 & 0 \\ * & * & \tilde{\Phi}_{11} & \tilde{\Phi}_{12} \\ * & * & * & \tilde{\Phi}_{22} \end{bmatrix} \begin{bmatrix} I & 0 \\ X_+G & D \\ 0 & I \\ -G & 0 \end{bmatrix} \prec 0,$$
$$X_-G = I,$$

then the controller with the gain matrix $K = UG$ *is stabilizing for all* $(A, B_1) \in \Sigma_W$.

After a congruence transformation, application of the Schur complement lemma, and a change of variables, one arrives at the final result.

Proposition 3 [98, Remark 7]. *If the data (U, X) satisfy the rank condition of the FL, then the solution of the problem*

$$\begin{bmatrix} -P & * & * & * \\ -\tilde{\Phi}_{12}Y & \tilde{\Phi}_{11} & * & * \\ X_+Y & D & -P & * \\ Y & 0 & 0 & -\tilde{\Phi}_{22}^{-1} \end{bmatrix} \prec 0,$$

$$X_-Y = P$$

with respect to the matrix variables $P \in \mathbb{S}^n_{++}$ and $Y \in \mathbb{R}^{T\times n}$ yields the quadratically stabilizing controller $\hat{K} = U\hat{Y}\hat{P}^{-1}$ for all systems in Σ_W.

As noted above, this approach is conservative: the parameterization (43) contains the parameter W, and moreover, a common Lyapunov function is constructed for the entire set of systems $(A, B_1) \in \Sigma_W$ consistent with the measurements (U, X).

Here, the dimensions of the matrix variable Y depend on the amount of data T, and since the method is based on the Fundamental Lemma, the rank condition (7) must hold, i.e., $T \geqslant n + m$.

4.2.2. Data Informativity Condition Based on Quadratic Matrix Inequalities

The problem of designing a state-feedback control law $u = Kx$ that quadratically stabilizes the system with exogenous disturbances

$$x(k+1) = Ax(k) + B_1u(k) + w(k) \tag{45}$$

(in the same notation as (35)) was considered in [83, 84]. The authors addressed the case $D = I$; other cases require a simple modification of the method [19, 82, 83].

First, it is necessary to write the problem constraints as QMIs. For this purpose, using the system representation with unknown matrices A, B_1, and W and the measured data (U, X), one expresses

$$W = X_+ - AX_- - B_1U$$

and substitutes this formula into the QMI describing the set of admissible disturbances (38) to get

$$\begin{bmatrix} I \\ A^\top \\ B_1^\top \end{bmatrix}^\top \begin{bmatrix} I & X_+ \\ 0 & -X_- \\ 0 & -U \end{bmatrix} \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{12}^\top & \Phi_{22} \end{bmatrix} \begin{bmatrix} I & X_+ \\ 0 & -X_- \\ 0 & -U \end{bmatrix}^\top \begin{bmatrix} I \\ A^\top \\ B_1^\top \end{bmatrix} \succcurlyeq 0. \tag{46}$$

Next, the quadratic stabilization condition for all systems $(A, B_1) \in \Sigma$ using a common Lyapunov function is reduced to a similar form:

$$(A + B_1K)P(A + B_1K)^\top - P \prec 0, \quad P \in \mathbb{S}^n_{++}.$$

This yields the QMI with respect to the unknown matrices A and B_1 :

$$\begin{bmatrix} I \\ A^\top \\ B_1^\top \end{bmatrix}^\top \begin{bmatrix} P & * & * \\ 0 & -P & * \\ 0 & -KP & -KPK^\top \end{bmatrix} \begin{bmatrix} I \\ A^\top \\ B_1^\top \end{bmatrix} \succ 0. \tag{47}$$

Finally, the strict matrix S-procedure (Lemma 2, Section 4.1) with the matrices M from (47) and N from (46) is applied to the above QMIs. As a result, one arrives at the following necessary and sufficient condition for Data Informativity for the quadratic stabilization of the system with exogenous disturbances.

Proposition 4 [84, Proposition 4]. *If the Slater condition*

$$\begin{bmatrix} I \\ Z \end{bmatrix}^\top \begin{bmatrix} I & X_+ \\ 0 & -X_- \\ 0 & -U \end{bmatrix} \Phi \begin{bmatrix} I & X_+ \\ 0 & -X_- \\ 0 & -U \end{bmatrix}^\top \begin{bmatrix} I \\ Z \end{bmatrix} \succ 0$$

holds for some matrix $Z \in \mathbb{R}^{(n+m)\times n}$, then the data (U, X) generated by system (45) are informative for quadratic state-feedback stabilization if and only if there exist matrices $P \in \mathbb{S}^n_{++}$ and $Y \in \mathbb{R}^{m\times n}$ and a scalar $\beta > 0$ such that

$$\begin{bmatrix} P-\beta I & * & * & * \\ 0 & -P & * & * \\ 0 & -Y & 0 & * \\ 0 & 0 & Y^\top & P \end{bmatrix} - \begin{bmatrix} I & X_+ \\ 0 & -X_- \\ 0 & -U \\ 0 & 0 \end{bmatrix} \Phi \begin{bmatrix} I & X_+ \\ 0 & -X_- \\ 0 & -U \\ 0 & 0 \end{bmatrix}^\top \succcurlyeq 0. \tag{48}$$

For such $\hat{P}$ and $\hat{Y}$ with some $\hat{\beta}$, the stabilizing controller has the form $\hat{K} = \hat{Y}\hat{P}^{-1}$ for all $(A, B_1) \in \Sigma_W$.

In this approach, the measurement length is restricted by the full rank condition for X_-, which stems from the DI condition for the existence of a controller stabilizing the entire set of systems $(A, B_1) \in \Sigma_W$ [18]. Thus, $T \geqslant n$.

Moreover, if the hypotheses of Proposition 4 are supplemented by $\Phi_{22} \prec 0$ and $\operatorname{rank}[U^\top \ X_-^\top]^\top = n + m$ (i.e., the Fundamental Lemma holds), then condition (48) becomes a strict inequality [19]:

$$\begin{bmatrix} P & * & * & * \\ 0 & -P & * & * \\ 0 & -Y & 0 & * \\ 0 & 0 & Y^\top & P \end{bmatrix} - \begin{bmatrix} I & X_+ \\ 0 & -X_- \\ 0 & -U \\ 0 & 0 \end{bmatrix} \Phi \begin{bmatrix} I & X_+ \\ 0 & -X_- \\ 0 & -U \\ 0 & 0 \end{bmatrix}^\top \succ 0. \tag{49}$$

Note that its solution yields a controller of the same form: $\hat{K} = \hat{Y}\hat{P}^{-1}$.

Remark 3. In the above formulas, for the sake of simplicity, the original scalar α [83] has been included in the linear matrix variables P, Y, and β.

Remark 4. According to [83, 84], quadratic stabilization with exact data (see Theorem 4 in Section 2) is a special case of the above proposition (for $\Phi_{11} = 0$, $\Phi_{12} = 0$, and $\Phi_{22} = -I$). However, the Slater condition fails here because the matrix

$$N = \begin{bmatrix} I & X_+ \\ 0 & -X_- \\ 0 & -U \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} I & X_+ \\ 0 & -X_- \\ 0 & -U \end{bmatrix}^\top$$

is not positive definite. Nevertheless, one can establish a connection between these two cases using Finsler's matrix lemma (see Lemma 3 in Section 4.1), which does not require the Slater condition [84]. This result, discussed in detail in [84], has been presented in Section 2.3.

As noted by the authors, the method is non-conservative and convenient for large datasets since the dimension of the LMIs does not depend on T. Indeed, with the constraints $P \succ 0$ and $\beta > 0$, the linear matrix inequality (48) has dimension $4n + m + 1$ and contains $\frac{n(n+1)}{2} + nm + 1$ unknowns. This result can be improved in terms of computational complexity, so that the LMI dimension becomes $4n + 1$ with $\frac{n(n+1)}{2} + 1$ unknowns; see [20, Theorem 3].

4.2.3. Fundamental Lemma and Petersen's lemma

The system considered in [94] has the same form as in the previous subsection:

$$x(k+1) = Ax(k) + B_1u(k) + w(k), \tag{50}$$

where the set of bounded-energy disturbances $w(k) \in \mathbb{R}^n$ is described by the QMI

$$\bar{\mathbf{W}} = \left\{W \in \mathbb{R}^{n\times T} : \begin{bmatrix} I & W \end{bmatrix} \begin{bmatrix} -\Delta\Delta^\top & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} I \\ W^\top \end{bmatrix} \preccurlyeq 0\right\}, \quad \Delta > 0. \tag{51}$$

Obviously, this description is equivalent to (39) with $\Phi_{11} = \Delta\Delta^\top$, but a more general form (38) is also possible (omitted here).

In contrast to the results presented in Section 4.2.2, the authors imposed the FL condition (7) and proposed a *matrix ellipsoid* description for the set of all systems (A, B_1) consistent with the original data (U, X) under disturbances $W \in \bar{\mathbf{W}}$. For instance, using the QMI from (51), we obtain the set of systems

$$\begin{aligned}\Sigma_E &= \left\{(A, B_1) : \begin{bmatrix} I & A & B_1 \end{bmatrix} \begin{bmatrix} \mathbf{C} & * \\ \mathbf{B} & \mathbf{A} \end{bmatrix} \begin{bmatrix} I \\ A^\top \\ B_1^\top \end{bmatrix} \preccurlyeq 0\right\} \\ &= \{[A\ B_1] = Z^\top : \mathbf{C} + \mathbf{B}^\top Z + Z^\top\mathbf{B} + Z^\top\mathbf{A}Z \preccurlyeq 0\},\end{aligned} \tag{52}$$

where

$$\mathbf{A} = \begin{bmatrix} X_- \\ U \end{bmatrix} \begin{bmatrix} X_- \\ U \end{bmatrix}^\top, \quad \mathbf{B} = -X_+ \begin{bmatrix} X_- \\ U \end{bmatrix}^\top, \quad \text{and} \quad \mathbf{C} = X_+X_+^\top - \Delta\Delta^\top.$$

The FL condition ensures $\mathbf{A} \succ 0$, and this matrix ellipsoid can therefore first be written in a more convenient form:

$$\Sigma_E = \{[A\ B_1] = Z^\top : (Z - \zeta)^\top\mathbf{A}(Z - \zeta) \preccurlyeq Q\},$$

where $\zeta := -\mathbf{A}^{-1}\mathbf{B}$ and $\mathbf{Q} := \mathbf{B}^\top\mathbf{A}^{-1}\mathbf{B} - \mathbf{C}$. After proving that $\mathbf{Q} \succcurlyeq 0$ [94], one can pass to an equivalent parameterization of the system set Σ_E (52) as a matrix ellipsoid as well:

$$\mathcal{E} = \left\{(\zeta + \mathbf{A}^{-1/2}\Upsilon\mathbf{Q}^{1/2})^\top : \|\Upsilon\| \leqslant 0\right\}. \tag{53}$$

Next, it is necessary to use the Lyapunov function-based quadratic stabilization condition

$$P - (A + B_1K)P(A + B_1K)^\top \succ 0$$

with respect to the unknown matrices $P \in \mathbb{S}^n_{++}$ and Y, which is reduced to a QMI with respect to the unknown matrices A and B_1 :

$$P - \begin{bmatrix} A & B_1 \end{bmatrix} \begin{bmatrix} I \\ K \end{bmatrix} PP^{-1}P \begin{bmatrix} I \\ K \end{bmatrix}^\top \begin{bmatrix} A & B_1 \end{bmatrix}^\top \succ 0. \tag{54}$$

The considerations below employ Petersen's lemma [99, 100] in its strict form.

Lemma 4 (strict Petersen's lemma). *Let $G \in \mathbb{S}^n$, $M \neq 0$, $N \neq 0$ be matrices of appropriate dimensions, $\tilde{\Delta} \in \mathbb{S}^q$, $\tilde{\Delta} \succcurlyeq 0$, and let*

$$\mathcal{F} := \{\Delta \in \mathbb{R}^{p\times q} : \Delta^\top \Delta \preccurlyeq \tilde{\Delta}\}.$$

Then the inequality

$$G + M\Delta N + N^\top \Delta^\top M^\top \prec 0$$

holds for all $\Delta \in \mathcal{F}$ if and only if there exists $\varepsilon > 0$ such that

$$G + \varepsilon M M^\top + \varepsilon^{-1} N^\top \tilde{\Delta} N^\top \prec 0.$$

The parameterization (53), condition (54), and strict Petersen's lemma were jointly applied to establish the following necessary and sufficient condition for a data-driven design of a quadratically stabilizing controller. (Here, for the sake of convenience and brevity, we directly express this condition in terms of the original data (U, X).)

Theorem 6 [94, Theorem 1]. *For the measured data (U, X) of system (50) satisfying the rank condition of the FL, the solution of the LMI (if it exists)*

$$\begin{bmatrix} P + X_+X_+^\top - \Delta\Delta^\top & * & * \\ 0 & P & * \\ X_+ \begin{bmatrix} X_- \\ U \end{bmatrix}^\top & -\begin{bmatrix} P \\ Y \end{bmatrix} & \begin{bmatrix} X_- \\ U \end{bmatrix}\begin{bmatrix} X_- \\ U \end{bmatrix}^\top \end{bmatrix} \succ 0 \tag{55}$$

with respect to the unknown matrices $P \in \mathbb{S}^n_{++}$ and $Y \in \mathbb{R}^{m\times n}$ yields the quadratically stabilizing controller $\hat{K} = \hat{Y}\hat{P}^{-1}$ for all systems $(A, B_1) \in \Sigma_E$.

Upon applying the Schur complement lemma, condition (55) becomes very similar to condition (48) of the previous subsection. However, there are substantial differences. For instance, in the derivation of (48), the Slater condition was used without the rank condition of the FL; hence, the method from Section 4.2.2 can cover the case of an unbounded set Σ_E, but it does not directly include the exact-data case. (However, as already noted, this becomes possible by using some modifications and Finsler's matrix lemma [84, Theorem 2].)

Since the proposed method is based on the Fundamental Lemma, the rank condition requires $T \geqslant n + m$ measurements. Nevertheless, an undoubted advantage is that the sizes of the variables in (55) do not depend on the length of the data used.

4.2.4. Quadratic Matrix Inequalities for Each Input-Output Measurement Instant

In the paper [101], which appeared slightly earlier than [94] (see the previous subsection), the same authors also used a matrix-ellipsoid representation of the reachable set of system (50) and a quadratically stabilizing controller design very close to that considered in Section 4.2.2 [83, 84]. However, note essential differences of the two approaches that significantly reduce the degree of conservatism: the constraints in [101] apply to exogenous disturbances at each time instant, $\|w_k\|_2 \leqslant \varepsilon$, $k = 0, 1, \ldots, T-1$, whereas the previous methods typically assume bounded-energy disturbances over the entire sequence $W = [w(0) \quad w(1) \quad \ldots \quad w(T-1)] : WW^\top \preccurlyeq \varepsilon^2 TI$, which increases the degree of conservatism with T.

Due to such constraints, the set of systems (A, B_1) capable of generating the data (U, X) can be described by separate relations for each time instant:

$$\Sigma_i = \{(A, B_1) : x_{i+1} = Ax_i + B_1u_i + w_i, \quad w_i w_i^\top \preccurlyeq \varepsilon^2 I\} \quad i = 0, \ldots, T-1.$$

Therefore, we obtain a quadratic inequality of the form (46) for each measurement instant:

$$\begin{bmatrix} I \\ A^\top \\ B_1^\top \end{bmatrix}^\top \begin{bmatrix} I & x_{i+1} \\ 0 & -x_i \\ 0 & -u_i \end{bmatrix} \Phi \begin{bmatrix} I & x_{i+1} \\ 0 & -x_i \\ 0 & -u_i \end{bmatrix}^\top \begin{bmatrix} I \\ A^\top \\ B_1^\top \end{bmatrix} \succcurlyeq 0, \quad i = 0, \dots, T-1,$$

where the matrices (38) describe the set of admissible disturbances:

$$\Phi = \begin{bmatrix} \varepsilon^2 I & 0 \\ 0 & -1 \end{bmatrix} \in \mathbb{S}^{(n+1)\times(n+1)}.$$

In this case, the intersection

$$\Sigma_c = \bigcap_{i=0}^{T-1} \Sigma_i$$

yields the set of *all* systems (A, B_1) that can generate *all the obtained measurements* at time instants $i = 0, 1, \dots, T-1$.

According to [101, Proposition 3], the set given by the above intersection is a *subset* of all systems of the form

$$\begin{bmatrix} I \\ A^\top \\ B_1^\top \end{bmatrix}^\top \begin{bmatrix} I & X_+ \\ 0 & -X_- \\ 0 & -U \end{bmatrix} \begin{bmatrix} \varepsilon^2 TI & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} I & X_+ \\ 0 & -X_- \\ 0 & -U \end{bmatrix}^\top \begin{bmatrix} I \\ A^\top \\ B_1^\top \end{bmatrix} \succcurlyeq 0$$

(i.e., those using vectors composed of the measurements, assuming bounded-energy exogenous disturbances, see (46)) and, consequently, a subset of the system set Σ_E (52). Thus, the method considered in this subsection is less conservative than those in Sections 4.2.2 and 4.2.3 and gives more accurate results for a larger number of measurements T. (For comparisons and proofs, we refer to [101].)

The authors proposed a method for directly representing the reachable set from measured data, which ultimately leads to a quadratically stabilizing controller design. We omit the main derivations here: they are analogous to those presented in Section 4.2.2, taking into account the relations for each measurement instant. In fact, the following result was established.

Proposition 5 [101, Proposition 1]. *Let the rank condition* (7) *of the FL hold. Then the solution of the problem*

$$\begin{bmatrix} P - \beta I & * & * & * \\ 0 & -P & * & * \\ 0 & -Y & 0 & * \\ 0 & 0 & Y^\top & P \end{bmatrix} - \sum_{i=0}^{T-1} \tau_i \begin{bmatrix} I & x_{i+1} \\ 0 & -x_i \\ 0 & -u_i \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \varepsilon^2 I & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} I & x_{i+1} \\ 0 & -x_i \\ 0 & -u_i \\ 0 & 0 \end{bmatrix}^\top \succcurlyeq 0,$$
$$i = 0, \dots, T-1,$$

with respect to the unknown matrices $P \in \mathbb{S}^n_{++}$ and $Y \in \mathbb{R}^{m\times n}$ and scalar variables $\beta > 0$ and $\tau_i > 0$, $i = 0, \dots, T-1$, yields the controller $\hat{K} = \hat{Y}\hat{P}^{-1}$ that quadratically stabilizes all systems $(A, B_1) \in \Sigma_c$.

Obviously, the dimensions of the matrix variables in this problem do not depend on the length of the experimental data vectors, but the number of scalar variables grows with the number of measurements used. Generally, the FL conditions are not required initially, so the measurement

length restriction follows from the Data Informativity condition for stabilization: $T \geqslant n$. However, according to [101, Remark 1], $T \geqslant n + m$ ensures the closedness of the reachable set and facilitates the solution procedure of the LMIs. Moreover, increasing the number of measurements improves the solution accuracy but may also cause computational challenges. Therefore, when dealing with high-dimensional problems and large datasets, one should choose a trade-off between accuracy and computational complexity, or apply other methods.

Generally speaking, the use of individual measurements instead of their vectors distinguishes this method from most approaches covered by this survey.

Example 3. Consider the batch chemical reactor from Example 1. To model bounded exogenous disturbances $\|w_k\|_2 \leqslant \varepsilon$, $k = 0, \dots, T-1$, we set $D = I$ and $T = 10$.

Since all examples in this survey are illustrative, the results will be presented for a single realization of the exogenous disturbance W with a very small ε to obtain a solution by all methods. We chose $\varepsilon = 0.001$, generated a sequence of values $\check{w}(k) \in \mathbb{R}^n$ with the uniform distribution on $[-1, 1]$, and took

$$w(k) = \varepsilon \frac{\check{w}(k)}{\|\check{w}(k)\|_2}, \quad k = 0, 1, 2, \dots.$$

The input vector U and the initial condition $x(0)$ were the same as in Example 1.

In the conventional MBC approach, in the presence of bounded exogenous disturbances in system (36) with $\|w(k)\|_2 \leqslant 1$ (and with $D = I$, as in the current case), the controller is given by the solution of the same problem (3) as in the disturbance-free case:

$$\hat{K}^{st}_{mbc} = \begin{bmatrix} 0.6377 & -1.4696 & 0.0944 & -1.2149 \\ 3.0471 & 0.0622 & 2.9947 & -0.7239 \end{bmatrix}.$$

Solving the problem based on the linear fractional transformation (Proposition 3 of Section 4.2.1) yielded the controller

$$\hat{K}^{st}_{lft} = \begin{bmatrix} 1.0018 & -1.4901 & 0.1370 & -1.8788 \\ 4.1643 & -0.0818 & 3.2851 & -2.5132 \end{bmatrix}.$$

In this case, the experiment duration and the input signal were appropriately chosen to fulfill the FL; therefore, in the DI-based method with QMIs (Proposition 4 of Section 4.2.2), we used inequality (49). As a result,

$$\hat{K}^{st}_{qmi} = \begin{bmatrix} 0.9949 & -1.5262 & 0.1471 & -1.9208 \\ 4.1617 & -0.0413 & 3.2424 & -2.4442 \end{bmatrix}.$$

For stabilization using Petersen's lemma, we obtained the uncertainty $\Delta = \varepsilon\sqrt{T}I = 0.0032 I_4$, and the solution of the problem from Theorem 6 (Section 4.2.3) yielded the controller

$$\hat{K}^{st}_{pet} = \begin{bmatrix} 0.9894 & -1.5534 & 0.1013 & -1.9177 \\ 4.1600 & 0.0096 & 3.2913 & -2.4432 \end{bmatrix}.$$

The method with individual QMIs for each measurement instant (Proposition 5 of Section 4.2.4) gives a more accurate result for a larger number of measurements, so we set $T = 20$ to obtain twenty scalar variables $\hat{\tau}_i$ (omitted here for brevity) and $\hat{\beta} = 9.9971 \cdot 10^{-7}$. With these values, the controller had the gain matrix

$$\hat{K}^{st}_{ins} = \begin{bmatrix} 0.6576 & -1.4449 & 0.1234 & -1.1973 \\ 3.0850 & 0.0522 & 3.0250 & -0.7318 \end{bmatrix}.$$

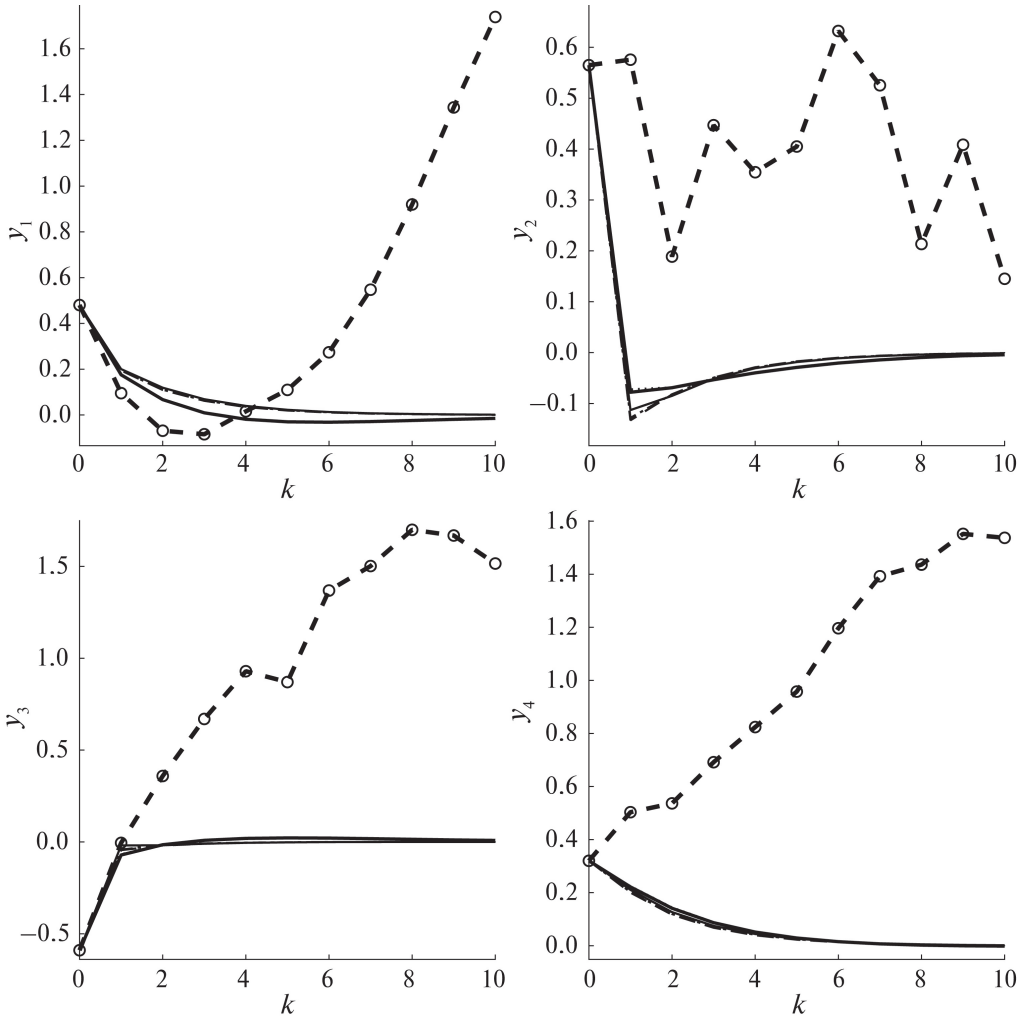


Fig. 3. System outputs in the quadratic stabilization problem under exogenous disturbances.

Figure 3 shows the following trajectories: the outputs $y_i(k)$, $i = 1, 2, 3, 4$, of the open-loop system (thick dashed line, with circles marking the measurements X used) and those of the closed-loop system, namely, the outputs $y_{mbc}(k)$ (the MBC solution, thick solid line), $y_{lft}(k)$ (the LFT-based method, thin solid line), $y_{qmi}(k)$ (the DI-based method with QMIs, thin dashed line), $y_{pet}(k)$ (Petersen's lemma, thin dash-dotted line), and $y_{ins}(k)$ (QMIs for individual measurements, thin dash-dotted line).

The closeness estimates of the trajectories were as follows:

$$\|y_{mbc}(k) - y_{lft}(k)\|_2 = 0.1409, \quad \|y_{mbc}(k) - y_{qmi}(k)\|_2 = 0.1328,$$
$$\|y_{mbc}(k) - y_{pet}(k)\|_2 = 0.1338, \quad \|y_{mbc}(k) - y_{ins}(k)\|_2 = 0.0079, \quad k = 0, 1, 2, \ldots.$$

Clearly, all the methods stabilized the system, i.e., the control objective was achieved in all cases: the problem formulation has no accuracy requirements (in terms of the closeness of controller matrices/trajectories to their MBC counterparts). However, in this example, the most accurate solution was provided by the method with individual measurements, which matches expectations owing to its conservatism-reducing structure.

The readers interested in a deeper performance analysis of the methods for systems with exogenous disturbances under larger uncertainty regions and various disturbance realizations (particularly, in the dependence of guaranteed performance bounds on disturbance levels and experiment duration) are referred to the original publications.

4.3. $\mathcal{H}_2$-Optimal Control

Consider a linear discrete time-invariant system with exogenous disturbances of the form

$$\begin{aligned} x(k+1) &= Ax(k) + B_1u(k) + Dw(k), \\ z(k) &= Cx(k) + B_2u(k), \end{aligned} \tag{56}$$

where $x(k) \in \mathbb{R}^n$, $u(k) \in \mathbb{R}^m$, $z(k) \in \mathbb{R}^l$, and $w(k) \in \mathbb{R}^p$; $A \in \mathbb{R}^{n\times n}$ and $B_1 \in \mathbb{R}^{n\times m}$; the matrices $C \in \mathbb{R}^{l\times n}$, $B_2 \in \mathbb{R}^{l\times m}$, and $D \in \mathbb{R}^{n\times p}$ are given.

The system is closed by a state-feedback controller $u = Kx$. The $\mathcal{H}_2$ norm of the transfer function from w to z is defined as

$$G(z) = (C + B_2K)(zI - (A + B_1K))^{-1}D.$$

The $\mathcal{H}_2$-suboptimal control problem is to find a control law $u = Kx$ such that $\|G(z)\|_{\mathcal{H}_2} \leqslant \gamma$ for some $\gamma > 0$. In the case of the smallest value $\gamma = \gamma^*$, we obtain $\mathcal{H}_2$-optimal control and $\|G(z)\|_{\mathcal{H}_2} = \gamma^*$.

When the matrices A and B_1 are known, the solution is given by the following statement: the matrix $A + B_1K$ is Schur stable and $\|G(z)\|_{\mathcal{H}_2} = \gamma$ if and only if there exists a matrix $P \in \mathbb{S}^n_{++}$ such that

$$\begin{gathered} P \succ (A + B_1K)^\top P(A + B_1K) + (C + B_2K)^\top (C + B_2K), \\ \operatorname{trace} D^\top PD < \gamma^2. \end{gathered} \tag{57}$$

In terms of linear matrix inequalities, the problem is formulated as follows [102].

For system (56), there exists an $\mathcal{H}_2$-optimal control law of the form $u = Kx$ if and only if the convex programming problem

$$\begin{gathered} \min_{P,Y,Z} \gamma, \\ \begin{bmatrix} P & AP + B_1Y & D \\ * & P & 0 \\ * & * & I \end{bmatrix} \succ 0, \quad \begin{bmatrix} Z & CP + B_2Y \\ * & P \end{bmatrix} \succ 0, \\ \operatorname{trace} Z < \gamma^2, \quad \gamma > 0 \end{gathered} \tag{58}$$

with respect to the matrix variables $P \in \mathbb{S}^n_{++}$, $Y \in \mathbb{R}^{m\times n}$, and $Z \in \mathbb{S}^p_{++}$ is solvable. In this case, $\hat{K} = \hat{Y}\hat{P}^{-1}$ and $\|G(z)\|_{\mathcal{H}_2} = \hat{\gamma}$.

If the matrices A, B_1 are unknown, we obtain an analogous formulation: from the measurements U and X of the system signals, it is required to design a static state-feedback control law so that the closed-loop system is stable and $\|G(z)\|_{\mathcal{H}_2} \leqslant \gamma$ for the entire set of systems consistent with these measurements.

In the following subsections, we present methods for solving this problem based on the DI condition and QMIs (Section 4.3.1), suboptimal control methods with the DI condition (Section 4.3.2), a method based on the FL with separation of system matrices and the Lyapunov matrix in the LMI (Section 4.3.3), and a method involving QMIs for individual input-output measurement instants (Section 4.3.4).

4.3.1. Data Informativity Condition Based on Quadratic Matrix Inequalities

A method for designing an $\mathcal{H}_2$-suboptimal controller using a parameterization of the closed-loop system matrices consistent with experimental data was proposed in [19, 83]. The main idea is to employ, within the DI framework, QMIs to describe both the set of corresponding systems and control objectives.

Consider system (56) with $D = I$ and exogenous disturbances w described by (38). The case $D \neq I$ requires minor modifications, see [19, 82, 83].

First, condition (57) is transformed to a QMI in the unknowns A and B_1, and then the strict matrix S-procedure (Lemma 2 of Section 4.1) is applied to the resulting expression. This approach leads to the following result.

Theorem 7 [83, Theorem 17]. *If the Slater condition*

$$\begin{bmatrix} I \\ \tilde{Z} \end{bmatrix}^\top \begin{bmatrix} I & X_+ \\ 0 & -X_- \\ 0 & -U \end{bmatrix} \Phi \begin{bmatrix} I & X_+ \\ 0 & -X_- \\ 0 & -U \end{bmatrix}^\top \begin{bmatrix} I \\ \tilde{Z} \end{bmatrix} \succ 0$$

holds for some matrix $\tilde{Z} \in \mathbb{R}^{(n+m)\times n}$, then the data (U, X) generated by system (56) are informative for $\mathcal{H}_2$-suboptimal control with $\gamma > 0$ if and only if there exist matrices $P \in \mathbb{S}^n_{++}$, $Y \in \mathbb{R}^{m\times n}$, and $Z \in \mathbb{S}^n_{++}$ and scalars $\alpha \geqslant 0$ and $\beta > 0$ such that

$$\begin{bmatrix} P-\beta I & * & * & * & * \\ 0 & 0 & * & * & * \\ 0 & 0 & 0 & * & * \\ 0 & P & Y^\top & P & * \\ 0 & 0 & 0 & CP+B_2Y & I \end{bmatrix} - \alpha \begin{bmatrix} I & X_+ \\ 0 & -X_- \\ 0 & -U \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \Phi \begin{bmatrix} I & X_+ \\ 0 & -X_- \\ 0 & -U \\ 0 & 0 \\ 0 & 0 \end{bmatrix}^\top \succcurlyeq 0,$$

$$\begin{bmatrix} P & * \\ CP+B_2Y & I \end{bmatrix} \succ 0, \quad \begin{bmatrix} Z & * \\ I & P \end{bmatrix} \succcurlyeq 0,$$

$$\operatorname{trace} Z < \gamma^2.$$

For such $\hat{P}$ and $\hat{Y}$ with some $\hat{\alpha}$ and $\hat{\beta}$, the $\mathcal{H}_2$-suboptimal controller has the form $\hat{K} = \hat{Y}\hat{P}^{-1}$ and $\|G(z)\|_{\mathcal{H}_2} < \hat{\gamma}$ for all $(A, B_1) \in \Sigma_W$.

Note that the dimensions of the matrix variables here do not depend on the data amount, which is limited only by the DI condition for stabilization of system (13), i.e., $T \geqslant n$.

4.3.2. $\mathcal{H}_2$-Suboptimal Control Based on the Data Informativity Concept

A necessary and sufficient condition for Data Informativity to design an $\mathcal{H}_2$-suboptimal control law $u = Kx$ for system (56) in the case where *it is possible to measure the values of the exogenous disturbances* (i.e., the vector $W = [w(0) \quad w(1) \quad \dots \quad w(T-1)] \in \mathbb{R}^{p\times T}$ is known) was provided in [91]. This condition fundamentally distinguishes the result discussed in this subsection from the others described by the survey.

Proposition 6 [91, Proposition 15]. *The data (U, X, W) are informative for $\mathcal{H}_2$-suboptimal control if and only if at least one of the following conditions holds for a given value $\gamma > 0$:*

1. *There exists a matrix $Q \in \mathbb{R}^{T\times n}$ such that $X_-Q = (X_-Q)^\top$,*

$$\begin{bmatrix} W_- \\ CX_- + B_2U \end{bmatrix} Q = 0, \quad \textit{and} \quad \begin{bmatrix} X_-Q & * \\ X_+Q & X_-Q \end{bmatrix} \succ 0.$$

2. *There exist a right pseudo-inverse* $W_-^\dagger$ *and matrices* $Y \in \mathbb{S}^n_{++}$ *and* $Q \in \mathbb{R}^{T\times n}$ *such that* $X_-Q = (X_-Q)^\top$, $W_-Q = X_-W_-^\dagger = UW_-^\dagger = 0$, *and*

$$\begin{bmatrix} X_-Q & * & * \\ X_+Q & X_-Q & * \\ (CX_- + B_2U)Q & 0 & I \end{bmatrix} \succ 0, \quad \begin{bmatrix} Y & * \\ X_+W_-^\dagger & X_-Q \end{bmatrix} \succ 0,$$

$$\operatorname{trace}(Y) < \gamma.$$

Moreover, K is an $\mathcal{H}_2$-suboptimal controller for all $(A, B_1) \in \Sigma_W$ and $\|G(z)\|_{\mathcal{H}_2} < \gamma$ if and only if the controller can be written as $\hat{K} = U\hat{Q}(X_-\hat{Q})^{-1}$ for any matrix $\hat{Q}$ satisfying conditions 1 or 2.

Obviously, with this solution method, the dimension of the matrix variable in the LMI depends on the experiment duration T and, hence, on the amount of data used. For this problem, the DI condition imposes the restriction $T \geqslant n$ (Theorem 2 of Section 2.2). Thus, the requirements on the minimum data amount are milder than the FL condition, but computational challenges may arise for large T.

Remark 5. The suboptimal LQR problem from Section 3.3 is a special case of the $\mathcal{H}_2$-suboptimal control problem presented here for $D = x_0$, $CC^\top = Q$, $B_2B_2^\top = R$, and $C^\top B_2 = 0$; however, the two methods slightly differ in the use of the original data in the solution.

The main drawback of this method lies in the assumption that the measurement vector of the exogenous disturbances W is known, which is not often realized in practice. However, when this condition holds, the method shows good results (see Example 4 below).

4.3.3. $\mathcal{H}_2$-Suboptimal Control Based on Quadratic Matrix Inequalities and an Auxiliary Matrix

In the case of a fully known system, the degree of conservatism can be reduced by the well-proved method [103, 104]: an LMI is constructed with separation of the system matrix A and the Lyapunov matrix by introducing an auxiliary matrix variable. For instance, this condition was applied to design a γ-suboptimal $\mathcal{H}_2$ controller, including a fixed-structure (sparse) controller (by adding extra constraints in the system) [105]. Here, the separation procedure is combined with the use of QMIs, and the method rests on the DI concept.

Consider system (56) closed by the controller $u = Kx$.

If the matrices A and B_1 are known, then the parameters of the γ-suboptimal $\mathcal{H}_2$ controller are obtained by solving the following problem [104].

Theorem 8 [104, Theorem 5]. *A controller K is a γ-suboptimal $\mathcal{H}_2$ controller for system* (56) *if and only if there exist matrices $P \in \mathbb{S}^n_{++}$, $Q \in \mathbb{S}^l$, $G \in \mathbb{R}^{n\times n}$, and $Y \in \mathbb{R}^{m\times n}$ such that*

$$\begin{bmatrix} P - DD^\top & AG + B_1Y \\ * & G + G^\top - P \end{bmatrix} \succ 0, \quad \begin{bmatrix} Q & CG + B_2Y \\ * & G + G^\top - P \end{bmatrix} \succ 0,$$

$$\operatorname{trace} Q < \gamma^2,$$

with $K = YG^{-1}$ and $\|G(z)\|_{\mathcal{H}_2} < \gamma$.

Passing to the design problem from measured data (U, X), we assume that the disturbance matrix W belongs to the set (38):

$$\mathbf{W} = \left\{ W \in \mathbb{R}^{n\times T} : \begin{bmatrix} I \\ W^\top \end{bmatrix}^\top \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{12}^\top & \Phi_{22} \end{bmatrix} \begin{bmatrix} I \\ W^\top \end{bmatrix} \succcurlyeq 0 \right\}.$$

Using [104, Theorem 1] and the matrix S-procedure [82], we arrive at the following γ-suboptimal $\mathcal{H}_2$ controller design.

Theorem 9 [105, Theorem 3.2]. *If for the measurements (U, X) of system (56) with exogenous disturbances (38) the problem*

$$\min_{P,Q,G,L,\alpha,\beta,\gamma} \gamma, \tag{59}$$
$$\begin{bmatrix} P - DD^\top - \beta I & 0 & 0 & 0 \\ * & 0 & 0 & G \\ * & * & 0 & Y \\ * & * & * & G + G^\top - P \end{bmatrix} - \alpha \begin{bmatrix} I & X_+ \\ 0 & -X_- \\ 0 & -U \\ 0 & 0 \end{bmatrix}^\top \Phi \begin{bmatrix} I & X_+ \\ 0 & -X_- \\ 0 & -U \\ 0 & 0 \end{bmatrix} \succcurlyeq 0,$$
$$\begin{bmatrix} Q & CG + B_2 Y \\ * & G + G^\top - P \end{bmatrix} \succcurlyeq 0,$$
$$\operatorname{trace} Q \leqslant \gamma^2$$

with respect to the matrices $P \in \mathbb{S}^n_{++}$, $Q \in \mathbb{S}^l$, $G \in \mathbb{R}^{n\times n}$, and $Y \in \mathbb{R}^{m\times n}$ and the scalars $\alpha \geqslant 0$, $\beta > 0$, and $\gamma > 0$ has a solution $\hat\gamma$ for some $\hat Y$ and $\hat G$, then the controller $\hat K = \hat Y \hat G^{-1}$ is a $\hat\gamma$-suboptimal $\mathcal{H}_2$ controller for all systems $(A, B_1) \in \Sigma_W$ and $\|G(z)\|_{\mathcal{H}_2} \leqslant \gamma$.

Note that the multiplier α in the first constraint cannot be eliminated here by scaling the other unknowns because the first term contains the constant matrix $DD^\top$.

Besides conservatism reduction (see above), an advantage of this γ-suboptimal $\mathcal{H}_2$ controller design method is that the computational complexity does not depend on the amount of data used; this amount must satisfy (13), i.e., $T \geqslant n$.

4.3.4. Quadratic Matrix Inequalities for Individual Input-Output Measurement Instants

Here, we outline the approach from [106] for system (56). Its form is largely similar to that of the one discussed in the previous subsection, although the authors do not separate the variable matrices. The main idea is to use constraints on the disturbances at each time instant and then apply QMIs for each measurement instant. Thus, all assumptions of Section 4.2.4 remain valid; in particular, we have

$$\Phi = \begin{bmatrix} \varepsilon^2 I & 0 \\ 0 & -1 \end{bmatrix} \in \mathbb{S}^{(n+1)\times(n+1)},$$
$$\begin{bmatrix} I \\ A^\top \\ B_1^\top \end{bmatrix}^\top \Psi_i \begin{bmatrix} I \\ A^\top \\ B_1^\top \end{bmatrix} \succcurlyeq 0,$$
$$\Psi_i = \begin{bmatrix} I & x_{i+1} \\ 0 & -x_i \\ 0 & -u_i \end{bmatrix} \Phi \begin{bmatrix} I & x_{i+1} \\ 0 & -x_i \\ 0 & -u_i \end{bmatrix}^\top, \quad i = 0, \dots, T-1.$$

Thus, the intersection

$$\Sigma_c = \bigcap_{i=0}^{T-1} \Sigma_i$$

yields the set of *all* systems of the form (56) with matrices (A, B_1) that can generate *all the measurements* at time instants $i = 0, 1, \dots, T-1$.

As claimed by the authors [106], the unique set obtained by this intersection is a *subset* of all systems from [105]. In other words, their method is less conservative and leads to more accurate results.

The subsequent derivations are standard, including the matrix S-procedure, the Schur complement, etc. The resulting problem with respect to the matrix variables $P \in \mathbb{S}^n_{++}$, $Q \in \mathbb{S}^n_{++}$, and $Y \in \mathbb{R}^{m\times n}$ and the scalar variables $\alpha_i \geqslant 0$, $i = 0, 1, \ldots, T-1$, $\beta > 0$, and $\gamma > 0$ has the following form:

$$\min_{P,Y,Q,\alpha_i,\beta,\gamma} \gamma \tag{60}$$

subject to the constraints

$$\begin{bmatrix} P - DD^\top - \beta I & 0 & 0 & 0 \\ * & 0 & 0 & P \\ * & * & 0 & Y \\ * & * & * & P \end{bmatrix} - \begin{bmatrix} \sum_{i=0}^{T-1} \alpha_i \Psi_i & 0 \\ 0 & 0 \end{bmatrix} \succcurlyeq 0, \tag{61}$$

$$\begin{bmatrix} Q & CP + B_2 Y \\ * & P \end{bmatrix} \succ 0,$$

$$\operatorname{trace} Q \leqslant \gamma^2.$$

Its solution gives the $\mathcal{H}_2$-suboptimal controller $\hat{K} = \hat{Y}\hat{P}^{-1}$, with $\|G(z)\|_{\mathcal{H}_2} \leqslant \hat{\gamma}$ for all $(A, B_1) \in \Sigma_c$.

Obviously, here the dimensions of the variables do not depend on the data amount T, but the latter directly determines the number of scalar variables. This may cause computational challenges for large measurement lengths.

The restriction on the experiment duration T follows from (13): $T \geqslant n$.

Remark 6. By [106, Theorem 3], the value γ^* representing the solution of problem (60)–(61) decreases monotonically with the time T.

Also, the authors of [106] proposed an extension of the method (an iterative algorithm) to the case of fixed-structure (sparse) controllers and demonstrated the advantages of the approach compared to the method discussed in Section 4.3.3 [105].

Among other $\mathcal{H}_2$-optimal control design techniques, we could mention, e.g., the linear fractional transformation [107]. Its advantage is that the variables in the semidefinite programming problem do not depend on the amount of data used.

Example 4. We continued modeling the unstable batch chemical reactor from Example 1. Let $C = I$ and $B_2 = 0$, i.e., the performance output of the system was given by

$$z(k) = x(k), \quad k = 0, 1, \ldots.$$

We set the experiment duration $T = 20$ and chose $\varepsilon = 0.001$. (For this value, all the methods under consideration yield a solution, for illustrative purposes.) For the matrix Φ in (38), we had $\Phi_{11} = 0.2 \cdot 10^{-4} I$, $\Phi_{12} = 0$, and $\Phi_{22} = -I$. The input and output measurements and exogenous disturbances were simulated as in Example 3.

The model-based computations using (58) (Section 4.3) yielded

$$\hat{K}^{H_2}_{mbc} = \begin{bmatrix} 0.9674 & -1.5118 & 0.1646 & -1.8382 \\ 4.7270 & -0.0118 & 3.4519 & -1.9159 \end{bmatrix},$$

with the optimal value

$$\hat{\gamma}_{mbc}^{H_2} = 2.7707.$$

Logically, data-driven methods cannot surpass the optimal value $\hat{\gamma}_{mbc}^{H_2}$ obtained with exactly known model parameters, so they naturally produced somewhat larger values of γ.

For instance, computations using the DI condition based on QMIs (Theorem 7 of Section 4.3.1) with the found values of $\hat{\alpha} = 319.54$ and $\hat{\beta} = 0.001$ led to the suboptimal controller

$$\hat{K}_{qmi}^{H_2} = \begin{bmatrix} 0.9745 & -1.5120 & 0.1616 & -1.8590 \\ 4.7494 & -0.0151 & 3.4368 & -1.9681 \end{bmatrix}, \quad \hat{\gamma}_{qmi}^{H_2} = 2.7951.$$

Due to the strict inequality criterion in the suboptimal control problem of the DI-based method (Proposition 6 of Section 4.3.2), we set γ slightly larger than the model-based optimum: $\gamma = 2.7708$. Also recall that in this method, the measurement vector of exogenous disturbances W is known and used directly in the computations. Both of these factors significantly improve the solution accuracy (see Fig. 4 and the closeness estimate below). For instance, we obtained

$$\hat{K}_{sub}^{H_2} = \begin{bmatrix} 0.9673 & -1.5118 & 0.1646 & -1.8378 \\ 4.7263 & -0.0118 & 3.4517 & -1.9151 \end{bmatrix}.$$

In the suboptimal controller design method based on QMIs and an auxiliary matrix in the LMI (59) (Section 4.3.3), the value of γ is not prescribed but computed as the minimum possible during the solution procedure. For instance, we found

$$\hat{K}_{do}^{H_2} = \begin{bmatrix} 0.9521 & -1.5119 & 0.1598 & -1.8413 \\ 4.7003 & -0.0124 & 3.4428 & -1.9355 \end{bmatrix}, \quad \hat{\gamma}_{do}^{H_2} = 2.8028.$$

Considering Remark 6 of Section 4.3.4, we used all 20 available measurements in the method with QMIs for individual input-output measurement instants of system (60)–(61). We obtained the scalar variables $\hat{\alpha}_i$, $i = 1, 2, \ldots, 20$ (omitted here for brevity), $\hat{\beta} = 2.2745 \cdot 10^{-5}$, and the controller

$$\hat{K}_{ins}^{H_2} = \begin{bmatrix} 0.9556 & -1.5116 & 0.1610 & -1.8425 \\ 4.7132 & -0.0132 & 3.4475 & -1.9450 \end{bmatrix}, \quad \hat{\gamma}_{ins}^{H_2} = 2.7957.$$

Figure 4 shows the following trajectories: the performance outputs $z_i(k) = x_i(k)$, $i = 1, 2, 3, 4$, of the open-loop system (thick dashed line, with circles marking the initial measurements X) and those of the closed-loop system, namely, the outputs $z_{mbc}(k)$ (the MBC solution, thin solid line), $z_{qmi}(k)$ (the DI-based method with QMIs, thin dashed line), $z_{sub}(k)$ (the suboptimal controller with the DI conditions, thin dotted line), $z_{do}(k)$ (the auxiliary-matrix method, dash-dotted line), and $z_{ins}(k)$ (the method with QMIs for individual measurements, thin solid line).

We compared the results with the MBC solution:

$$\begin{aligned}
&\|\hat{K}_{mbc}^{H_2} - \hat{K}_{qmi}^{H_2}\|_2 = 0.0629, \quad \|z_{mbc}(k) - z_{qmi}(k)\|_2 = 0.0021, \quad |\hat{\gamma}_{mbc}^{H_2} - \hat{\gamma}_{qmi}^{H_2}| = 0.0245, \\
&\|\hat{K}_{mbc}^{H_2} - \hat{K}_{sub}^{H_2}\|_2 = 0.0011, \quad \|z_{mbc}(k) - z_{sub}(k)\|_2 = 4.9881 \cdot 10^{-5}, \quad \hat{\gamma}_{sub}^{H_2} = 2.7708 (\text{given}), \\
&\|\hat{K}_{mbc}^{H_2} - \hat{K}_{do}^{H_2}\|_2 = 0.0376, \quad \|z_{mbc}(k) - z_{do}(k)\|_2 = 0.0045, \quad |\hat{\gamma}_{mbc}^{H_2} - \hat{\gamma}_{do}^{H_2}| = 0.0321, \\
&\|\hat{K}_{mbc}^{H_2} - \hat{K}_{ins}^{H_2}\|_2 = 0.0340, \quad \|z_{mbc}(k) - z_{ins}(k)\|_2 = 0.0044, \quad |\hat{\gamma}_{mbc}^{H_2} - \hat{\gamma}_{ins}^{H_2}| = 0.0250, \\
&\qquad\qquad k = 0, 1, 2, \ldots.
\end{aligned}$$

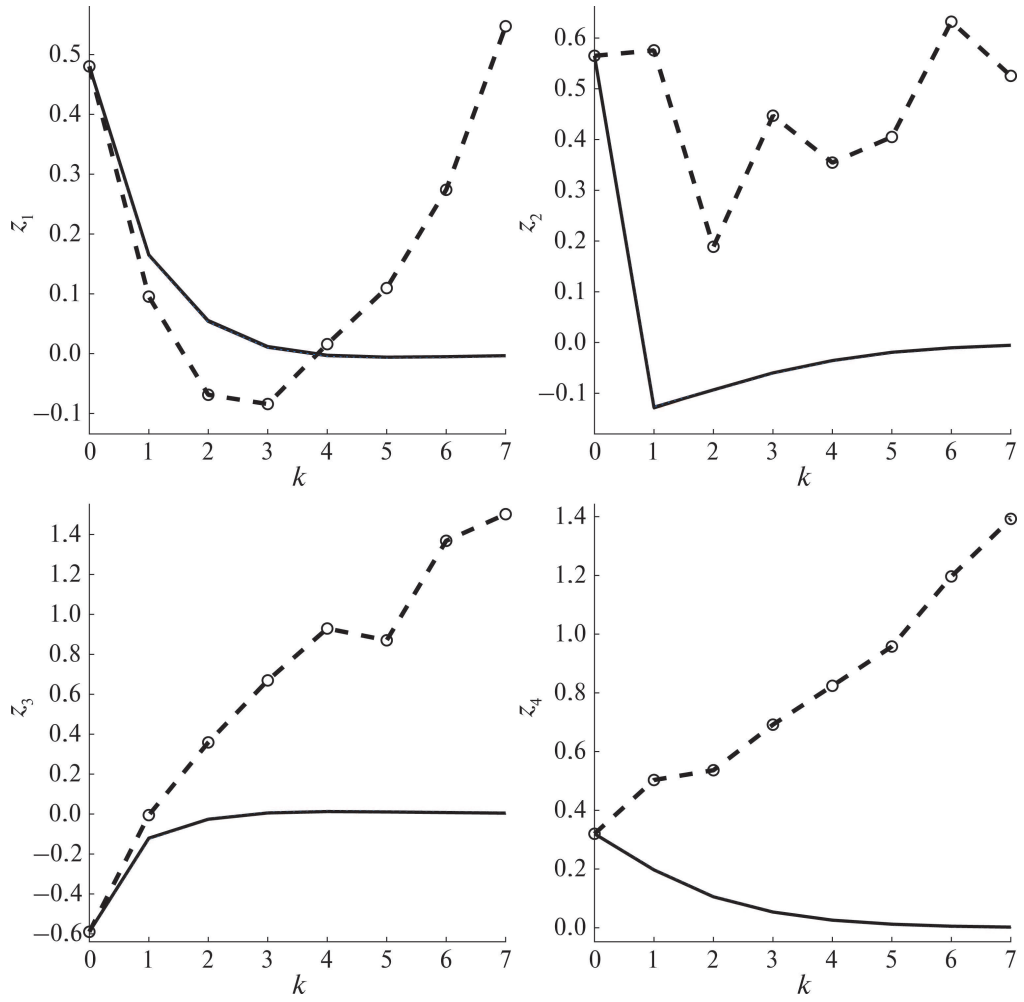


Fig. 4. System outputs in the $\mathcal{H}_2$-optimal control problem.

Evidently, all considered methods coped well with the problem: the controllers $\hat{K}$ and the values of $\hat{\gamma}$ were close to the optimal ones. In particular, in this example with unknown measurements of exogenous disturbances, the method with QMIs and the DI condition, as well as the method with individual QMIs for each time instant, gave more accurate solutions, and their accuracy will improve when increasing the initial experiment duration T. In general, one may use any of the presented methods, depending on the features of the available prior data.

4.4. $\mathcal{H}_\infty$-Optimal Control

Consider a discrete-time system of the form

$$x(k+1) = Ax(k) + B_1u(k) + Dw(k), \qquad (62)$$
$$z(k) = Cx(k) + B_2u(k),$$

where $x(k) \in \mathbb{R}^n$, $u(k) \in \mathbb{R}^m$, and $z(k) \in \mathbb{R}^l$, $w(k) \in \mathbb{R}^p$; $A \in \mathbb{R}^{n\times n}$ and $B_1 \in \mathbb{R}^{n\times m}$; the matrices $C \in \mathbb{R}^{l\times n}$, $B_2 \in \mathbb{R}^{l\times m}$, and $D \in \mathbb{R}^{n\times p}$ are given. It is required to design a control law $u = Kx$ that stabilizes the closed-loop system with the matrices

$$A_K = A + B_1K, \quad C_K = C + B_2K \qquad (63)$$

and simultaneously minimizes the $\mathcal{H}_\infty$ norm of the transfer function from the disturbance w to the output z :

$$G(z) = C_K(zI - A_K)^{-1}D.$$

If $\|G(z)\|_{\mathcal{H}_\infty} \leqslant \gamma$ for some $\gamma > 0$, we have an $\mathcal{H}_\infty$-suboptimal controller; in the case of the smallest value $\gamma = \gamma^*$, this statement leads to $\mathcal{H}_\infty$-optimal control and $\|G(z)\|_{\mathcal{H}_\infty} = \gamma^*$.

When the matrices (A, B_1) are known, the problem is written in LMI terms as follows [102].

An $\mathcal{H}_\infty$-optimal controller exists if and only if there exists a matrix $P \in \mathbb{S}^n_{++}$ such that

$$\min_{P\in\mathbb{S}^n_{++},\, Y\in\mathbb{R}^{m\times n},\, \gamma>0} \gamma, \tag{64}$$

$$\begin{bmatrix} P & AP + B_1Y & D & 0 \\ * & P & 0 & PC + Y^\top B_2^\top \\ * & * & \gamma I & 0 \\ * & * & * & \gamma I \end{bmatrix} \succ 0;$$

the solution $(\hat{P}, \hat{Y})$ yields the optimal controller $\hat{K} = \hat{Y}\hat{P}^{-1}$ and $\|G(z)\|_{\mathcal{H}_\infty} = \hat{\gamma}$.

Below we present DDC methods for solving this problem using the FL and LFT (Section 4.4.1), the FL and the discrete Bounded Real Lemma (Section 4.4.2), the DI condition and QMIs (Section 4.4.3), and QMIs for individual input-output measurement instants (Section 4.4.4).

4.4.1. Fundamental Lemma and Linear Fractional Transformation

System (62) with unknown matrices (A, B_1) was considered in [98].

Following the same methodology as in Section 4.2.1, under the rank condition (7) of the FL, the system closed by the controller $u = Kx$ can be represented as a linear fractional transformation with the channel from w to z :

$$\begin{bmatrix} x(k+1) \\ z(k) \\ \tilde{z}(k) \end{bmatrix} = \begin{bmatrix} X_+G & D & D \\ C + B_2UG & 0 & 0 \\ -G & 0 & 0 \end{bmatrix} \begin{bmatrix} x(k) \\ w(k) \\ \tilde{w}(k) \end{bmatrix},$$

$$\tilde{w}(k) = W\tilde{z}(k), \quad W \in \tilde{\mathbf{W}},$$

$$X_-G = I.$$

The authors formulated $\mathcal{H}_\infty$-suboptimal controller design conditions containing a matrix inequality and reduced them to a linear form by a change of variables and equivalent transformations. For brevity, let us present the final result.

Proposition 7 [98, Proposition 10]. *If the data (U, X) satisfy the rank condition of the FL and there exist matrices $P \in \mathbb{S}^n_{++}$ and $Y \in \mathbb{R}^{T\times n}$ and a scalar $\lambda > 0$ such that*

$$X_-Y = P,$$

then the solution of the problem

$$\begin{bmatrix} -P & * & * & * & * & * \\ 0 & -\gamma^2 I & * & * & * & * \\ -\lambda\Phi_{12}Y & 0 & \lambda\Phi_{11} & * & * & * \\ X_+Y & D & D & -P & * & * \\ CP + B_2U & 0 & 0 & 0 & -I & * \\ Y & 0 & 0 & 0 & 0 & -(\lambda\Phi_{22}^{-1}) \end{bmatrix} \prec 0 \tag{65}$$

with some $\hat{\lambda}$ yields the controller $\hat{K} = U\hat{Y}\hat{P}^{-1}$ for all $(A, B_1) \in \Sigma_W$ and $\|G(z)\|_{\mathcal{H}_\infty} \leqslant \hat{\gamma}$.

Thus, the LMI (65) can be solved by one-dimensional optimization over the parameter λ, and in this formulation, the experiment duration $T \geqslant n+m$ directly affects the problem dimension. Moreover, the solution accuracy improves with T [98, Example 1]; therefore, for large system dimensions and data amounts, one should choose a trade-off between satisfactory accuracy and the ability to solve LMIs of acceptable dimensions.

4.4.2. Fundamental Lemma and the Discrete Bounded Real Lemma

In contrast to the previous subsection and [98], the idea put forward in [108] is to design an $\mathcal{H}_\infty$-suboptimal controller for system (62) based on QMIs and the discrete Bounded Real Lemma (BRL) under the FL condition. For the closed-loop system with matrices (63), the author formulated and proved this lemma in the following form.

Lemma 5 [108, Lemma 6, Appendix A.1]. *The following statements are equivalent:*

1. *The matrix A_K is Schur stable and $\|G(z)\|_\infty < \gamma$.*
2. *The linear matrix inequality*

$$\begin{bmatrix} P & * & * & * \\ 0 & \gamma I & * & * \\ PA_K & PD & P & * \\ C_K & 0 & 0 & \gamma I \end{bmatrix} \succ 0 \tag{66}$$

holds for a Lyapunov matrix $P \in \mathbb{S}^n_{++}$ and $\gamma > 0$.

Then the known gain matrix parameterization (8) (see Sections 2.1 and 4.2.1) is applied to the LMI (66), which requires the rank condition (7) of the FL. In this case, $K = UG$ with $X_- G = I$, and the closed-loop system matrices can be expressed from the measured data as

$$A_K = (X_+ - DW)G, \quad C_K = C + B_2 UG \tag{67}$$

for an unknown (non-unique) matrix G and some disturbance matrix $W \in \mathbf{W}$ satisfying

$$(X_+ - DW)\begin{bmatrix} U \\ X_- \end{bmatrix}^\dagger = 0.$$

By substituting (67) into (66) and using the more general characterization of the disturbance set (42) (see Section 4.2.1), we obtain the data-driven $\mathcal{H}_\infty$-optimal control problem. After standard transformations and a change of the variables, it is reduced to the following LMI with the parameter $W \in \tilde{\mathbf{W}}$ and the variables $P \in \mathbb{S}^n_{++}$ and $Y \in \mathbb{R}^{T\times n}$:

$$\begin{bmatrix} P & * & * & * \\ 0 & \gamma I & * & * \\ (X_+ - DW)Y & D & P & * \\ CP + B_2 UY & 0 & 0 & \gamma I \end{bmatrix} \succ 0.$$

According to the approach of [108], the LFT, the matrix S-procedure (with the inequality in (42)) [19, 83], and the Schur complement lemma are successively applied to this constraint. And the data-driven $\mathcal{H}_\infty$-suboptimal controller design problem takes the final form

$$\min_{P,\,Y} \gamma \tag{68}$$

subject to the constraint

$$\begin{bmatrix} P & * & * & * & * & * \\ 0 & \gamma I & * & * & * & * \\ X_+Y & D & P & * & * & * \\ CP+B_2UY & 0 & 0 & \gamma I & * & * \\ Y\Phi_{12} & 0 & -D^\top & 0 & \Phi_{11} & * \\ Y & 0 & 0 & 0 & 0 & -\Phi_{22}^{-1} \end{bmatrix} \succ 0, \tag{69}$$

$$P = X_- Y.$$

Its solution yields the controller $\hat{K} = U\hat{Y}\hat{P}^{-1}$ and $\|G(z)\|_{\mathcal{H}_\infty} < \hat{\gamma}$.

Note that the parameterization of the matrix A_K in (67) is non-unique, and the control objective is achieved for all systems with the matrix $A_K = (X_+ - DW)G$ for any $W \in \tilde{\mathbf{W}}$. Therefore, the controller design in this case gives potentially conservative results.

The dimensions of the matrix variables in this problem directly depend on the amount of data used, which must satisfy $T \geqslant n + m$ by the FL.

4.4.3. Data Informativity Condition Based on Quadratic Matrix Inequalities

Following [19], consider a system of the form

$$\begin{aligned} x(k+1) &= Ax(k) + B_1u(k) + w(k), \\ z(k) &= Cx(k) + B_2u(k), \end{aligned} \tag{70}$$

where $x(k) \in \mathbb{R}^n$, $u(k) \in \mathbb{R}^m$, $z(k) \in \mathbb{R}^l$, and $w(k) \in \mathbb{R}^p$; $A \in \mathbb{R}^{n\times n}$ and $B_1 \in \mathbb{R}^{n\times m}$; the matrices $C \in \mathbb{R}^{l\times n}$, $B_2 \in \mathbb{R}^{l\times m}$, and $D \in \mathbb{R}^{n\times p}$ are given. The matrix W must satisfy (38):

$$\mathbf{W} = \left\{ W \in \mathbb{R}^{n\times T} : \begin{bmatrix} I \\ W^\top \end{bmatrix}^\top \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{12}^\top & \Phi_{22} \end{bmatrix} \begin{bmatrix} I \\ W^\top \end{bmatrix} \succcurlyeq 0 \right\}.$$

As in the case of $\mathcal{H}_2$-suboptimal control (Section 4.3.1), the authors used the DI concept and the technique of QMIs (the matrix S-procedure, the Slater condition, etc.) to establish control optimality conditions. As a result, the following result was formulated.

Theorem 10 [19, Theorem 13]. *The data (U, X) obtained from system (70) with disturbances described by the set $\mathbf{W}$ are informative for $\mathcal{H}_\infty$-suboptimal control design with $\gamma > 0$ if and only if there exist matrices $P \in \mathbb{S}^n_{++}$ and $Y \in \mathbb{R}^{m\times n}$ and scalars $\alpha \geqslant 0$ and $\beta > 0$ such that*

$$\begin{bmatrix} P - \frac{1}{\gamma^2}I - \beta I & 0 & 0 & 0 & 0 \\ * & 0 & 0 & P & 0 \\ * & * & 0 & Y & 0 \\ * & * & * & P & (CP+B_2Y)^\top \\ * & * & * & * & I \end{bmatrix} - \alpha \begin{bmatrix} I & X_+ \\ 0 & -X_- \\ 0 & -U \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \Phi \begin{bmatrix} I & X_+ \\ 0 & -X_- \\ 0 & -U \\ 0 & 0 \\ 0 & 0 \end{bmatrix}^\top \succcurlyeq 0,$$

$$\begin{bmatrix} P & * \\ CP+B_2Y & I \end{bmatrix} \succ 0.$$

Then the solution $\hat{P}$, $\hat{Y}$ of this problem yields the controller $\hat{K} = \hat{Y}\hat{P}^{-1}$ and $\|G(z)\|_{\mathcal{H}_\infty} < \hat{\gamma}$ for all $(A, B_1) \in \Sigma_W$.

In comparison with the methods from the previous two subsections, an undoubted advantage here is the independence of the dimensions of the matrix variables in the LMI from the length of the input-output measurements, which is limited only by the DI condition for stabilization of system (13): $T \geqslant n$.

An iterative method for designing an $\mathcal{H}_\infty$-optimal control law from multiple datasets was proposed in [109]. As claimed by the authors, their approach is less conservative than the one discussed in the current subsection; moreover, it constructs control for a wider set of systems because the matrices C and B_2 are supposed to be *unknown*. For brevity, we do not present the method of [109] here.

4.4.4. Quadratic Matrix Inequalities for Individual Input-Output Measurement Instants

By analogy with the solution of the $\mathcal{H}_2$-suboptimal control design from Section 4.3.4, the authors of [106] proposed to construct an $\mathcal{H}_\infty$-suboptimal control law by using individual measurements at each time instant. As proved in the paper, this approach reduces the degree of conservatism.

Since the initial data and subsequent constructs are the same as in Section 4.3.4, we directly provide the final result.

For system (62), the solution of the problem

$$\min_{P,Y,Q,\alpha_i,\beta,\gamma} \gamma \tag{71}$$

subject to the constraints

$$\begin{bmatrix} P-DD^\top-\beta I & * & * \\ 0 & -Q & * \\ 0 & \begin{bmatrix} C & B_2 \end{bmatrix} Q & \gamma^2 I - \begin{bmatrix} C & B_2 \end{bmatrix} Q \begin{bmatrix} C & B_2 \end{bmatrix}^\top \end{bmatrix} - \begin{bmatrix} \sum\limits_{i=0}^{T-1} \alpha_i \Psi_i & 0 \\ 0 & 0 \end{bmatrix} \succcurlyeq 0, \tag{72}$$

$$\Psi_i = \begin{bmatrix} I & x_{i+1} \\ 0 & -x_i \\ 0 & -u_i \end{bmatrix} \Phi \begin{bmatrix} I & x_{i+1} \\ 0 & -x_i \\ 0 & -u_i \end{bmatrix}^\top, \quad i = 0, \ldots, T-1,$$

$$\begin{bmatrix} Q - \begin{bmatrix} P & * \\ Y & 0 \end{bmatrix} & \begin{bmatrix} 0 \\ Y \end{bmatrix} \\ * & P \end{bmatrix} \succ 0$$

with respect to the matrix variables $P \in \mathbb{S}^n_{++}$, $Q \in \mathbb{S}^{n+m}_{++}$, and $Y \in \mathbb{R}^{m \times n}$ and the scalar variables $\alpha_i \geqslant 0$, $i = 0, 1, \ldots, T-1$, $\beta > 0$, and $\gamma > 0$ yields the H_∞-suboptimal controller $\hat{K} = \hat{Y}\hat{P}^{-1}$ with $\|G(z)\|_{\mathcal{H}_\infty} \leqslant \hat{\gamma}$ for all $(A, B_1) \in \Sigma_c$.

Remark 7. By [106, Theorem 5], the value γ^* representing the solution of problem (71)–(72) decreases monotonically with the time $T \geqslant n$. (This restriction follows from (13).)

In [106], the method was extended to the case of fixed-structure (sparse) controllers (an iterative algorithm), and its advantages were demonstrated theoretically and by an example.

Example 5. Under the conditions of the previous example, we designed $\mathcal{H}_\infty$-optimal control by the above methods.

First, we constructed a control law in the MBC approach by solving problem (64) (see the beginning of Section 4.4)

$$\hat{K}^{H_\infty}_{mbc} = \begin{bmatrix} 1.7141 & -1.5677 & 0.3249 & -3.2775 \\ 6.8823 & -0.1974 & 3.8621 & -4.6649 \end{bmatrix}, \quad \hat{\gamma}^\infty_{mbc} = 2.5607.$$

Next, with the LFT-based method (Proposition 7 of Section 4.4.1) and $\hat{\lambda} = 346.98$, we obtained

$$\hat{K}_{lft}^{H_\infty} = \begin{bmatrix} 1.6453 & -1.6010 & 0.2668 & -3.3096 \\ 6.8780 & -0.1381 & 3.9277 & -4.5610 \end{bmatrix}, \quad \hat{\gamma}_{lft}^{H_\infty} = 2.6102.$$

The solution of problem (68)–(69) based on the FL and the discrete BRL from Section 4.4.2 gave

$$\hat{K}_{brl}^{H_\infty} = \begin{bmatrix} 1.6642 & -1.6052 & 0.2650 & -3.3001 \\ 6.9270 & -0.1262 & 3.9583 & -4.5406 \end{bmatrix}, \quad \hat{\gamma}_{brl}^{H_\infty} = 2.7206.$$

Computations according to Theorem 10 (Section 4.4.3) using the DI conditions with QMIs led to $\hat{\alpha} = 149.81$, $\hat{\beta} = 0.001$, and

$$\hat{K}_{di}^{H_\infty} = \begin{bmatrix} 1.6645 & -1.5930 & 0.2825 & -3.2925 \\ 6.8529 & -0.1515 & 3.9036 & -4.5931 \end{bmatrix}, \quad \hat{\gamma}_{di}^{H_\infty} = 2.6132.$$

In view of Remark 7 (Section 4.4.4), the method with QMIs for individual input-output measurement instants (71)–(72) yielded $\hat{\alpha}_i$, $i = 1, 2, \ldots, 20$ (omitted for brevity), $\hat{\beta} = 2.2745 \cdot 10^{-5}$, and the controller

$$\hat{K}_{ins}^{H_\infty} = \begin{bmatrix} 1.6459 & -1.5946 & 0.2779 & -3.3306 \\ 6.9028 & -0.1491 & 3.9194 & -4.5696 \end{bmatrix}, \quad \hat{\gamma}_{ins}^{H_\infty} = 2.6031.$$

Figure 5 shows the following trajectories: the outputs $y_i(k)$, $i = 1, 2, 3, 4$, of the open-loop system (thick dashed line, with circles marking the measurements X used) and those of the closed-loop system, namely, the outputs $z_{mbc}(k)$ (the MBC solution, thick solid line), $z_{lft}(k)$ (the LFT-based method, thin dashed line), $z_{brl}(k)$ (the method based on the FL and discrete BRL, thin dotted line), $z_{di}(k)$ (the DI-based method with QMIs, thin dash-dotted line), and $z_{ins}(k)$ (QMIs for individual measurements, thin solid line).

We compared the results with the MPC solution:

$$\begin{gathered} \|\hat{K}_{mbc}^{H_\infty} - \hat{K}_{lft}^{H_\infty}\|_2 = 0.1555, \quad \|z_{mbc}(k) - z_{lft}(k)\|_2 = 0.0197, \quad |\hat{\gamma}_{mbc}^{H_\infty} - \hat{\gamma}_{lft}^{H_\infty}| = 0.0495, \\ \|\hat{K}_{mbc}^{H_\infty} - \hat{K}_{brl}^{H_\infty}\|_2 = 0.1946, \quad \|z_{mbc}(k) - z_{brl}(k)\|_2 = 0.0187, \quad |\hat{\gamma}_{mbc}^{H_\infty} - \hat{\gamma}_{brl}^{H_\infty}| = 0.1599, \\ \|\hat{K}_{mbc}^{H_\infty} - \hat{K}_{di}^{H_\infty}\|_2 = 0.1047, \quad \|z_{mbc}(k) - z_{di}(k)\|_2 = 0.0122, \quad |\hat{\gamma}_{mbc}^{H_\infty} - \hat{\gamma}_{di}^{H_\infty}| = 0.0524, \\ \|\hat{K}_{mbc}^{H_\infty} - \hat{K}_{ins}^{H_\infty}\|_2 = 0.1532, \quad \|z_{mbc}(k) - z_{ins}(k)\|_2 = 0.0254, \quad |\hat{\gamma}_{mbc}^{H_\infty} - \hat{\gamma}_{ins}^{H_\infty}| = 0.0423, \\ k = 0, 1, 2, \ldots. \end{gathered}$$

Clearly, all the methods successfully solved the stated problem. In terms of the accuracy of the estimate $\hat{\gamma}$ and the closeness of the trajectories, the method with individual measurements in the matrix inequality performed best in this example, and its accuracy increases with the number of initial measurements T. The method based on the joint application of the FL and the discrete BRL turned out to be the least accurate in the estimate $\hat{\gamma}$, although this difference is not critical. The other two methods are roughly similar.

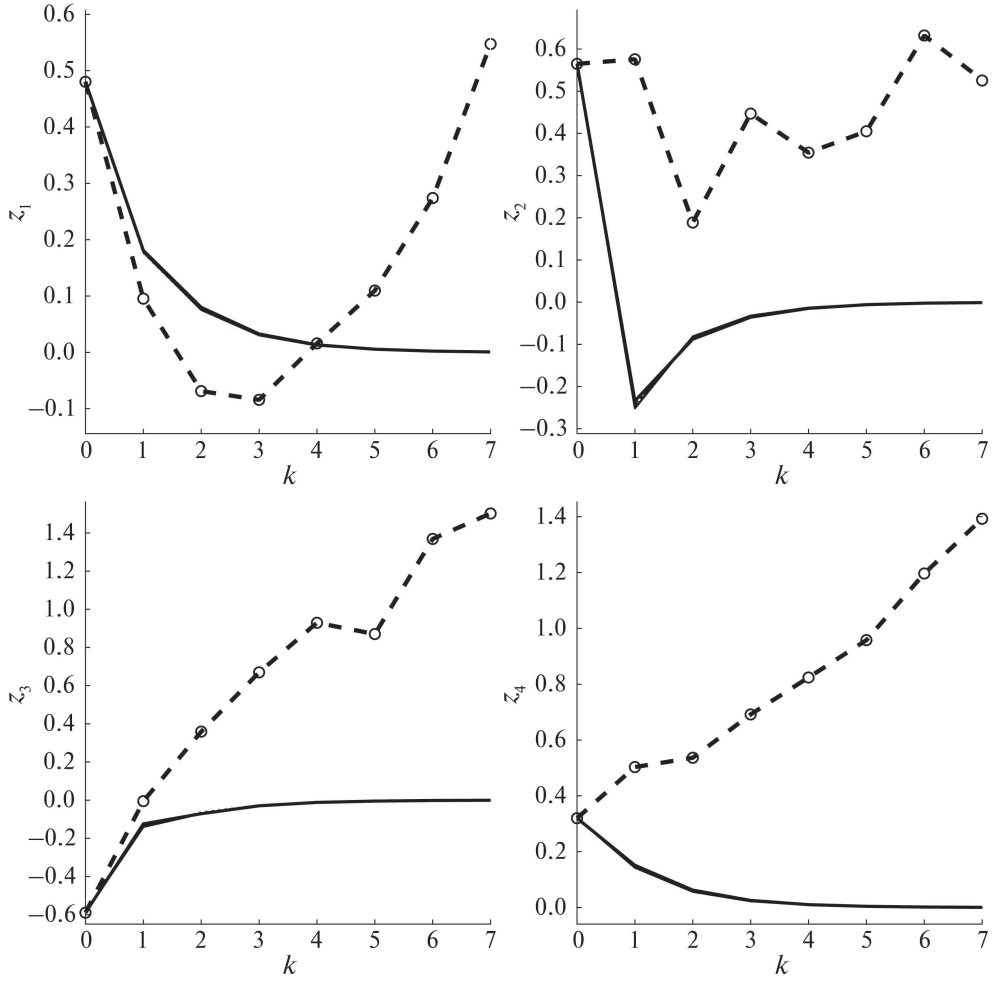


Fig. 5. System outputs in the $\mathcal{H}_\infty$-optimal control problem.

5. CONCLUSIONS

This survey has covered direct data-driven control methods based on finite sets of measurements, with the Fundamental Lemma or the Data Informativity concept at their core, and the use of linear matrix inequalities with application to quadratic stabilization (with exact measurements and under bounded exogenous disturbances), optimal linear quadratic regulator design, and $\mathcal{H}_2$- and $\mathcal{H}_\infty$-optimal control design for discrete linear time-invariant systems. Based on the above analysis, we draw the following conclusions:

— The methods involve well-known conventional MBC approaches and solution techniques, i.e., Lyapunov inequalities, the linear fractional transformation, Petersen's lemma, the S-procedure and Finsler's lemma (including their matrix versions), the discrete Bounded Real Lemma, and the techniques of linear and quadratic matrix inequalities are used.

— The only a priori knowledge is the linear time-invariant character of the system and the dimensions of its inputs and outputs (or some estimates of the dimensions).

— By assumption, all outputs are observable and measurements of *all states* are available, i.e., $y(k) = x(k)$, which is not always possible in real-world systems. In such cases, other DDC methods have to be applied, including those not included in this survey owing to differences in problem formulations.

— In all the presented methods, input-output measurements of the *open-loop system* are supposed to be available. First, this is not always feasible in practice; second, it may cause challenges in the case of unstable systems; in both cases mentioned, other DDC approaches are required.

— The Fundamental Lemma enables parameterizing the closed-loop system matrix directly, particularly in the presence of exogenous disturbances (at the cost of a higher degree of conservatism); however, its conditions require a larger amount of initial data.

— The Data Informativity conditions are milder and allow using less conservative methods, including those based on individual measurements, and smaller amounts of initial data.

— The dimensions of the matrix variables in different methods may/may not depend on the experiment duration, which significantly affects the applicability of a particular method: if the variable dimensions depend on the data amount, increasing this amount will improve the solution accuracy (an advantage of such methods), but computational challenges may arise when solving the LMIs.

— Persistent exogenous disturbances affecting the system (if any) are usually supposed to be unknown, and a description of the set of all admissible disturbances is used; bounded-energy disturbances or those with bounded norm values at individual time instants are considered (the latter approach reduces the degree of conservatism). However, there are methods with known disturbance measurements, and such measurements are directly employed in the solution (see Section 4.3.2).

— Methods have also been developed for partial problems, e.g., finite-horizon LQR design (Section 3.4).

Thus, the survey has presented a sufficiently diverse set of DDC methods for solving several classical control problems, discussed the features of their structure and application, and demonstrated their performance on numerical examples.

The above methods are part of one of the most actively developing directions within DDC. They are used, in the original form or numerous modifications, in the analysis, modeling, and control design for different real objects, systems, and processes: magnetic levitation systems [19]; single-link robotic arms [63]; simple distillation columns [70]; fighter aircraft [83]; pre-compensated surge subsystems of axial compressors [110]; stable active vehicle suspension systems and unstable rotational inverted pendulum systems [71]; inverted pendulums on fixed bases [69, 96] and carts [65]; two-mass spring systems [69]; decentralized control of DC microgrid systems with distributed generation units connected by transmission lines [111]; motion of robot groups [112]; motion of vehicles groups with a leader [60] and two vehicles at a safe distance [72]; other multi-agent systems [29, 53, 91]; population growth models [93], and other applications.

ACKNOWLEDGMENTS

The author is grateful to M.V. Khlebnikov, Dr. Sci. (Phys.–Math.), Prof. of the Russian Academy of Sciences, for careful reading of the manuscript and helpful remarks.

REFERENCES

1. Hou, Z.S. and Wang, Z., From Model-Based Control to Data-Driven Control: Survey, Classification and Perspective, *Information Sciences*, 2013, vol. 235, pp. 3–35. https://doi.org/10.1016/j.ins.2012.07.014

2. Xu, J.X. and Hou, Z.S., Notes on Data-Driven System Approaches, *Acta Automatica Sinica*, 2009, vol. 35, no. 6, pp. 668–675. https://doi.org/10.1016/S1874-1029(08)60090-7.

3. Dörfler, F., Data-Driven Control: Part Two of Two: Hot Take: Why not go with Models?, *IEEE Control Systems Magazine*, 2023, vol. 43, no. 6, pp. 27–31. https://doi.org/10.1109/MCS.2023.3310302

4. Soudbakhsh, D., Annaswamy, A.M., Wang, Y., et al., Data-Driven Control: Theory and Applications, *Proc. American Control Conf. (ACC)*, May 31, 2023–June 2, 2023, San Diego, CA, USA, pp. 1–18. https://doi.org/10.23919/ACC55779.2023.10156081.

5. Fernández, R., Data-Driven Control: A New Important Field in Control Theory, *Proc. 15th Latin American Control Conf.*, 2012, pp. 6.

6. Bazanella, A.S., Campestrini, L., and Eckhard, D., An Introduction to Data-Driven Control, From Kernels to Behaviors, *Proc. IEEE 61st Conf. Decision and Control (CDC)*, December 6–9, 2022, Cancún, Mexico. https://doi.org/10.1109/CDC51059.2022.9992589

7. Bazanella, A.S., Campestrini, L., and Eckhard, D., The Data-Driven Approach to Classical Control Theory, *Annual Reviews in Control*, 2023, vol. 56, art. no. 100906, pp. 1–20. https://doi.org/10.1016/j.arcontrol.2023.100906

8. Khaki-Sedigh, A., *An Introduction to Data-Driven Control Systems*, Hoboken: John Wiley & Sons, 2024.

9. Wang, J., Ramirez-Mendoza, R.A., and Morales-Menendez, R., *Data Driven Strategies: Theory and Applications*, CRC Press, 2023.

10. Bazanella, A.S., Campestrini, L., and Eckhard, D., *Data-Driven Controller Design: The H_2 Approach*, Dordrecht: Springer, 2012.

11. Brunton, S.L. and Kutz, J.N., *Data-Driven Science and Engineering. Machine Learning, Dynamical Systems, and Control*, Cambridge University Press, 2021.

12. Precup, R.-E., Roman, R.-C., and Safaei, A., *Data-Driven Model-Free Controllers*, Boca Raton: CRC Press, 2022.

13. Willems, J.C., From Time Series to Linear System: Part I. Finite Dimensional Linear Time Invariant Systems, *Automatica*, 1986, vol. 22, no. 5, pp. 561–580; Part II. Exact Modelling, *Automatica*, 1986, vol. 22, no. 6, pp. 675–694; Part III: Approximate Modelling, *Automatica*, 1987, vol. 23, no. 1, pp. 87–115.

14. Willems, J.C. and Polderman, J.W., *Introduction to Mathematical Systems Theory: A Behavioral Approach*, Texts in Applied Mathematics, New York: Springer, 1997.

15. Willems, J.C., Rapisarda, P., Markovsky, I., et al., A Note on Persistency of Excitation, *Systems & Control Letters*, 2005, vol. 54, no. 4, pp. 325–329. https://doi.org/10.1016/j.sysconle.2004.09.003

16. Markovsky, I., Willems, J., and De Moor, B., State Representations from Finite Time Series, *Proc. 44th IEEE Conf. on Decision and Control (CDC)*, Seville, Spain, 2005, pp. 832–835. https://doi.org/10.1109/CDC.2005.1582260

17. Markovsky, I. and Dörfler, F., Behavioral Systems Theory in Data-Driven Analysis, Signal Processing, and Control, *Annual Reviews in Control*, 2021, vol. 52, pp. 42–64. https://doi.org/10.1016/j.arcontrol.2021.09.005

18. Van Waarde, H.J., Eising, J., Trentelman, H.L., et al., Data Informativity: A New Perspective on Data-Driven Analysis and Control, *IEEE Trans. on Automatic Control*, 2020, vol. 65, no. 11, pp. 4753–4768. https://doi.org/10.1109/TAC.2020.2966717

19. Van Waarde, H.J., Eising, J., Camlibel, M.K., et al., The Informativity Approach: To Data-Driven Analysis and Control, *IEEE Control Systems*, 2023, vol. 43, no. 6, pp. 32–66. https://doi.org/10.1109/MCS.2023.3310305

20. Van Waarde, H.J., Eising, J., Camlibel, M.K., et al., A Tutorial on the Informativity Framework for Data-Driven Control, *Proc. 61st IEEE Conf. on Decision and Control (CDC)*, December 2022, pp. 1085–1090. https://doi.org/10.1109/CDC51059.2022.9992611

21. Rouis, N., N'Doye, I., and Laleg-Kirati, T.-M., Model-Free LQR Based PID Controller for Trajectory Tracking of 2-DoF Helicopter: Comparison and Experimental Results, *arXiv: 2103.10988v1.*

22. Berberich, J., Iannelli, A., Padoan, A., et al., A Quantitative and Constructive Proof of Willems' Fundamental Lemma and its Implications, *Proc. American Control Conf. (ACC)*, May 31, 2023–June 2, 2023, San Diego, CA, USA. https://doi.org/10.23919/ACC55779.2023.10156227

23. Bosso, A., Borghesi, M., Iannelli, A., et al., Derivative-Free Data-Driven Control of Continuous-Time Linear Time-Invariant Systems, *arXiv:2410.24167v1.*

24. Schmitz, P., Faulwasser, T., Rapisarda, P., et al., A Continuous-Time Fundamental Lemma and its Application in Data-Driven Optimal Control, *Systems & Control Letters*, 2024, vol. 194, art. no. 105950, pp. 1–11. https://doi.org/10.1016/j.sysconle.2024.105950

25. Pan, G., Ou, R., and Faulwasser, T., On a Stochastic Fundamental Lemma and Its Use for Data-Driven Optimal Control, *IEEE Trans. on Automatic Control*, 2023, vol. 68, no. 10, pp. 5922–5937. https://doi.org/10.1109/TAC.2022.3232442

26. Ou, R., Pan, G., and Faulwasser, T., A Stochastic Fundamental Lemma with Reduced Disturbance Data Requirements, *arXiv:2502.09131v1.*

27. Faulwasser, T., Ou, R., Pan, G., et al., Behavioral Theory for Stochastic Systems? A Data-driven Journey from Willems to Wiener and Back Again, *Annual Reviews in Control*, 2023, vol. 55, pp. 92–117. https://doi.org/10.1016/j.arcontrol.2023.03.005

28. Mishra, V.K., Markovsky, I., and Grossmann, B., Data-Driven Tests for Controllability, *IEEE Control Systems Letters*, 2021, vol. 5, no. 2, pp. 517–522. https://doi.org/10.1109/LCSYS.2020.3003770

29. Yu, Y., Talebi, Sh., Van Waarde, H.J., et al., On Controllability and Persistency of Excitation in Data-Driven Control: Extensions of Willems' Fundamental Lemma, *arXiv: 2102.02953v2.*

30. Yang, Y., Zheng, Z., and Li, S.E., On the Stability of Datatic Control Systems, *arXiv:2401.16793v1.*

31. Coulson, J., Van Waarde, H., and Dörfler, F., Robust Fundamental Lemma for Data-driven Control, *arXiv:2205.06636v1.*

32. Coulson, J., Van Waarde, H.J., Lygeros, J., et al., A Quantitative Notion of Persistency of Excitation and the Robust Fundamental Lemma, *IEEE Control Systems Letters*, 2023, vol. 7, pp. 1243–1248. https://doi.org/10.1109/LCSYS.2022.3232303

33. Verhoek, C., Tóth, R., Haesaert, S., et al., Fundamental Lemma for Data-Driven Analysis of Linear Parameter-Varying Systems, *Proc. 60th IEEE Conf. on Decision and Control (CDC)*, December 14–17, 2021, Austin, TX, USA. https://doi.org/10.1109/CDC45484.2021.9683151

34. Verhoek, C., Eising, J., Dörfler, F., and Tóth, R., Decoupling Parameter Variation from Noise: Bi-quadratic Lyapunov Forms in Data-Driven LPV Control, *arXiv:2403.16565v2.*

35. Nortmann, B. and Mylvaganam, T., Direct Data-Driven Control of LTV Systems, *IEEE Trans. on Automatic Control*, 2023, vol. 68, no. 8, pp. 4888–4895. https://doi.org/10.1109/TAC.2023.3276909

36. Breschi, V., Zaccarian, L., and Formentin, S., Data-Driven Stabilization of Input-Saturated Systems, *IEEE Control Systems Letters*, 2023, vol. 7, pp. 1640–1645. https://doi.org/10.1109/LCSYS.2023.3266254

37. Lu, X., Zhou, D., Chen, W.H., et al., Data-Driven Control for Linear Discrete-Time Systems with Input Saturation, *Circuits, Systems, and Signal Processing*, 2024, vol. 43, pp. 6211–6227. https://doi.org/10.1007/s00034-024-02751-w

38. Martinelli, A., Gargiani, M., Draskovic, M., et al., Data-Driven Optimal Control of Affine Systems: A Linear Programming Perspective, *IEEE Control Systems Letters*, 2022, vol. 6, pp. 3092–3097. https://doi.org/10.1109/LCSYS.2022.3180898

39. Chang, C.Y. and Bernstein, A., Robust Data-Driven Control with Noisy Data, *arXiv:2207.09587v4.*

40. Bisoffi, A., De Persis, C., and Tesi, P., Data-Based Stabilization of Unknown Bilinear Systems with Guaranteed Basin of Attraction, *Systems & Control Letters*, 2020, vol. 145, art. no. 104788, pp. 1–9. https://doi.org/10.1016/j.sysconle.2020.104788

41. De Persis, C. and Tesi, P., Designing Experiments for Data-Driven Control of Nonlinear Systems, *IFAC-PapersOnLine*, 2021, vol. 54, no. 9, pp. 285–290. https://doi.org/10.1016/j.ifacol.2021.06.085

42. Martin, T., Schönb, T.B., and Allgöwer, F., Guarantees for Data-Driven Control of Nonlinear Systems Using Semidefinite Programming: A Survey, *Annual Reviews in Control*, 2023, vol. 56, art. no. 100911. https://doi.org/10.1016/j.arcontrol.2023.100911

43. Padoan, A., Coulson, J., Van Waarde, H.J., et al., Behavioral Uncertainty Quantification for Data-Driven Control, *Proc. IEEE 61st Conf. on Decision and Control (CDC)*, December 6–9, 2022, Cancún, Mexico, pp. 4726–4731. https://doi.org/10.1109/CDC51059.2022.9993002

44. Alsalti, M., Barkey, M., Lopez, V.G., et al., Sample- and Computationally Efficient Data-Driven Predictive Control, *Proc. IEEE European Control Conf. (ECC)*, Stockholm, Sweden, 2024. https://doi.org/10.23919/ECC64448.2024.10591022

45. Coulson, J., Lygeros, J., and Dörfler, F., Data-enabled Predictive Control: In the Shallows of the DeePC, *Proc. 18th European Control Conf. (ECC)*, June 25–28, 2019, Naples, Italy, pp. 307–312. https://doi.org/10.23919/ECC.2019.8795639

46. Lazar, M. and Verheijen, P.C.N., Generalized Data-Driven Predictive Control: Merging Subspace and Hankel Predictors, *Mathematics*, 2023, vol. 11(9), art. no. 2216, 17 p. https://doi.org/10.3390/math11092216

47. Markovsky, I., Huang, L., and Dörfler, F., Data-Driven Control Based on the Behavioral Approach: From Theory to Applications in Power Systems, *IEEE Control Systems Magazine*, 2023, vol. 43, no. 5, pp. 28–68. https://doi.org/10.1109/MCS.2023.3291638

48. Mattsson, P., Bonassi, F., Breschi, V., et al., On the Equivalence of Direct and Indirect Data-Driven Predictive Control Approaches, *IEEE Control Systems Letters*, 2024, vol. 8, pp. 796–801. https://doi.org/10.1109/LCSYS.2024.3403473

49. Meijer, T.J., Scheres, K.J.A., Nouwens, S.A.N., et al., From a Frequency-Domain Willems' Lemma to Data-Driven Predictive Control, *arXiv:2501.19390v1*.

50. Dmitruk, N.M. and Manzhulina, E.A., Optimal Control of Linear Time-Invariant Discrete-Time Systems without Parametric Identification, *Autom. Remote Control*, 2022, vol. 83, no. 2, pp. 161–177.

51. Breschi, V., De Persis, C., Formentin, S., et al., Direct Data-Driven Model-Reference Control with Lyapunov Stability Guarantees, *Proc. 60th IEEE Conf. on Decision and Control (CDC)*, December 14–17, 2021, Austin, TX, USA, pp. 9212–9217. https://doi.org/10.1109/CDC45484.2021.9683437

52. Markovsky, I. and Rapisarda, P., On the Linear Quadratic Data-Driven Control, *Proc. European Control Conf. (ECC)*, Kos, Greece, July 2–5, 2007, pp. 5313–5318. https://doi.org/10.23919/ECC.2007.7068299

53. Chang, Z., Jiao, J., and Li, Z., Localized Data-driven Consensus Control, *arXiv:2401.12707v1*.

54. Van Waarde, H.J., De Persis, C., Camlibel, M.K., et al., Willems' Fundamental Lemma for State-space Systems and its Extension to Multiple Datasets, *IEEE Control Systems Letters*, 2020, vol. 4, no. 3, pp. 602–607. https://doi.org/10.1109/LCSYS.2020.2986991

55. Alsalti, M., Lopez, V.G., and Müller, M.A., Notes on Data-Driven Output-Feedback Control of Linear MIMO Systems, *IEEE Trans. on Automatic Control*, 2025, pp. 1–8. https://doi.org/10.1109/TAC.2025.3553073

56. Shakouri, A., Van Waarde, H.J., and Camlibel, M.K., A New Perspective on Willems' Fundamental Lemma: Universality of Persistently Exciting Inputs, *arXiv:2503.12489*.

57. Eising, J. and Trentelman, H.L., Informativity of Noisy Data for Structural Properties of Linear Systems, *Systems & Control Letters*, 2021, vol. 158, art. no. 105058, pp. 1–8. https://doi.org/10.1016/j.sysconle.2021.105058

58. Eising, J., Trentelman, H.L., and Camlibel, M.K., Data Informativity for Observability: An Invariance-based Approach, *Proc. European Control Conf. (ECC)*, May 12–15, 2020, St. Petersburg, Russia, pp. 1057–1059. https://doi.org/10.23919/ECC51009.2020.9143925

59. Berberich, J., Wildhagen, S., Hertneck, M., et al., Data-Driven Analysis and Control of Continuous-Time Systems under Aperiodic Sampling, *IFAC-PapersOnLine*, 2021, vol. 54, no. 7, pp. 210–215. https://doi.org/10.1016/j.ifacol.2021.08.360

60. Iwata, T., Azuma, S.-I., Ariizumi, R., et al., Data Informativity for Distributed Positive Stabilization, *IEEE Control Systems Letters*, 2024, vol. 8, pp. 826–831. https://doi.org/10.1109/LCSYS.2024.3404219

61. Eising, J. and Camlibel, M.K., Data Informativity for Analysis of Linear Systems with Convex Conic Constraints, *IEEE Control Systems Letters*, 2022, vol. 6, pp. 391–396. https://doi.org/10.1109/LCSYS.2021.3077367

62. Nguyen, E.T. and Van Waarde, H.J., Synthesis of Dissipative Systems Using Input-State Data, *Proc. IEEE European Control Conf. (ECC)*, pp. 2959–2964. https://doi.org/10.23919/ECC64448.2024.10590889

63. Hu, K. and Liu, T., Data-Driven H_∞ Control for Unknown Piecewise Affine Systems with Bounded Disturbances, *Proc. European Control Conf. (ECC)*, June 25–28, 2024, Stockholm, Sweden, pp. 2623–2628. https://doi.org/10.23919/ECC64448.2024.10590818

64. Miller, J. and Sznaier, M., Data-Driven Gain Scheduling Control of Linear Parameter-Varying Systems using Quadratic Matrix Inequalities, *IEEE Control Systems Letters*, 2022, vol. 7, pp. 835–840. https://doi.org/10.1109/LCSYS.2022.3227182

65. Van Waarde, H.J., Eising, J., Camlibel, M.K., et al., A Behavioral Approach to Data-Driven Control with Noisy Input-Output Data, *IEEE Trans. on Automatic Control*, 2024, vol. 69, no. 2, pp. 813–827. https://doi.org/10.1109/TAC.2023.3275014

66. Trentelman, H.L., Van Waarde, H.J., and Camlibel, M.K., An Informativity Approach to the Data-Driven Algebraic Regulator Problem, *IEEE Trans. on Automatic Control*, 2022, vol. 67, no. 11, pp. 6227–6233. https://doi.org/10.1109/TAC.2021.3129457

67. Cao, A. and Wang, F., Informativity Conditions for Multiple Signals: Properties, Experimental Design, and Applications, *arXiv:2501.10030v1.*

68. Eising, J. and Cortés, J., Cautious Optimization via Data Informativity, *arXiv:2307.10232v1.*

69. De Persis, C. and Tesi, P., Formulas for Data-Driven Control: Stabilization, Optimality, and Robustness, *IEEE Trans. on Automatic Control*, 2020, vol. 65, no. 3, pp. 909–924. https://doi.org/10.1109/TAC.2019.2959924

70. Bisoffi, A., Li, L., De Persis, C., et al., Controller Synthesis for Input-State Data With Measurement Errors, *IEEE Control Systems Letters*, 2024, vol. 8, pp. 1571–1576. https://doi.org/10.1109/LCSYS.2024.3402135

71. Esmzad, R., Sankar, G.S., Han, T., et al., Direct Data Driven Control Using Noisy Measurements, *arXiv:2505.06407.*

72. Bisoffi, A., De Persis, C., and Tesi, P., Controller Design for Robust Invariance From Noisy Data, *IEEE Trans. on Automatic Control*, 2023, vol. 68, no. 1, pp. 636–643. https://doi.org/10.1109/TAC.2022.3170373

73. Kaminaga, T. and Sasahara, H., Data Informativity under Data Perturbation, *arXiv:2505.01641v1.*

74. Berberich, J., Scherer, C.W., and Allgöwer, F., Combining Prior Knowledge and Data for Robust Controller Design, *IEEE Trans. Automat. Control*, 2023, vol. 68, no. 8, pp. 4618–4633. https://doi.org/10.1109/TAC.2022.3209342

75. Kogan, M.M. and Stepanov, A.V., How to Improve Robust Control of a Linear Time-Varying System by Using Experimental Data, *Autom. Remote Control*, 2024, vol. 85, no. 6, pp. 636–654.

76. Kogan, M.M., From Adaptive and Robust Control to Control Based on A Priori and Experimental Data, *Trudy XIV Vserossiiskogo soveshchaniya po problemam upravleniya (VSPU-2024)* (Proceedings of XIV All-Russian Meeting on Control Problems (AMCP-2024), June 17–20, 2024, Moscow, Novikov, D.A., Ed., Moscow: Trapeznikov Institute of Control Sciences RAS, 2024, pp. 450–454.

77. Kogan, M.M. and Stepanov, A.V., Design of Generalized H_∞-Suboptimal Controllers Based on Experimental and A Priori Data, *Autom. Remote Control*, 2024, vol. 85, no. 1, pp. 1–14.

78. Kogan, M.M. and Stepanov, A.V., Design of Suboptimal Robust Controllers Based on A Priori and Experimental Data, *Autom. Remote Control*, 2023, vol. 84, no. 8, pp. 918–932.

79. Polyak, B.T., Khlebnikov, M.V., and Shcherbakov, P.S., *Upravlenie lineinymi sistemami pri vneshnikh vozmushcheniyakh: Tekhnika lineinykh matrichnykh neravenstv* (Control of Linear Systems under Exogenous Disturbances: The Technique of Linear Matrix Inequalities), Moscow: LENAND, 2014.

80. Fomin, V.N., Fradkov, A.L., and Yakubovich, V.A., *Adaptivnoe upravlenie dinamicheskimi ob"ektami* (Adaptive Control of Dynamic Objects), Moscow: Nauka, 1981.

81. Fradkov, A.L., Miroshnik, I.V., and Nikiforov, V.O., *Nonlinear and Adaptive Control of Complex Systems*, Mathematics and Its Applications, vol. 491, Dordrecht: Kluwer Academic Publishers, 1999.

82. Van Waarde, H.J., Camlibel, M.K., Eising, J., et al., Quadratic Matrix Inequalities with Applications to Data-Based Control, *SIAM J. on Control and Optimization*, 2023, vol. 61(4), pp. 2251–2281. https://doi.org/10.1137/22M1486807

83. Van Waarde, H.J., Camlibel, M.K., and Mesbahi, M., From Noisy Data to Feedback Controllers: Nonconservative Design via a Matrix S-Lemma, *IEEE Trans. Autom. Control*, 2022, vol. 67, no. 1, pp. 162–175. https://doi.org/10.1109/TAC.2020.3047577

84. Van Waarde, H. and Camlibel, K., A Matrix Finsler's Lemma with Applications to Data-Driven Control, *Proc. 60th IEEE Conf. on Decision and Control*, 2022, pp. 5777–5782. https://doi.org/10.1109/CDC45484.2021.9683285

85. Walsh, G. and Ye, H., Scheduling of Networked Control Systems, *IEEE Control Syst. Mag.*, 2001, vol. 21, no. 1, pp. 57–65. https://doi.org/10.1109/37.898792

86. Grant, M. and Boyd, S., CVX: Matlab Software for Disciplined Convex Programming, ver. 2.1, Austin: CVX Research, Inc. URL: http://cvxr.com.

87. Chen, T. and Francis, B., *Optimal Sampled-Data Control Systems*, London: Springer, 1995.

88. Dörfler, F., Tesi, P., and De Persis, C., On the Certainty-Equivalence Approach to Direct Data-Driven LQR Design, *IEEE Trans. on Automatic Control*, 2023, vol. 68, no. 12, pp. 7989–7996. https://doi.org/10.1109/TAC.2023.3253787

89. Gonçalves Da Silva, G.R.G., Bazanella, A.S., et al., Data-Driven LQR Control Design, *IEEE Control Systems Letters*, 2018, vol. 3, no. 1, pp. 180–185. https://doi.org/10.1109/LCSYS.2018.2868183

90. Umar, A.A., Ryu, K., Back, J., et al., Stability Margin of Data-Driven LQR and Its Application to Consensus Problem, *Mathematics*, 2024, vol. 12, no. 2, art. no. 199. https://doi.org/10.3390/math12020199

91. Van Waarde, H.J. and Mesbahi, M., Data-Driven Parameterizations of Suboptimal LQR and H2 Controllers, *IFAC-PapersOnLine*, 2020, vol. 53, no. 2, pp. 4234–4239. https://doi.org/10.1016/j.ifacol.2020.12.2470

92. Rotulo, M., De Persis, C., and Tesi, P., Data-Driven Linear Quadratic Regulation via Semidefinite Programming, *IFAC-PapersOnLine*, 2020, vol. 53(2), pp. 3995–4000. https://doi.org/10.1016/j.ifacol.2020.12.2264

93. Scampicchio, A., Aravkin, A., and Pillonetto, G., LQR Design under Stability Constraints, *IFAC-PapersOnLine*, 2020, vol. 53, no. 2, pp. 5556–5560. https://doi.org/10.1016/j.ifacol.2020.12.1566

94. Bisoffi, A., De Persis, C., and Tesi, P., Data-driven Control via Petersen's Lemma, *Automatica*, 2022, vol. 145, art. no. 110537, pp. 1–14. https://doi.org/10.1016/j.automatica.2022.110537

95. Dörfler, F., Tesi, P., and De Persis, C., On the Role of Regularization in Direct Data-Driven LQR Control, *Proc. IEEE 61st Conf. on Decision and Control (CDC)*, 2022, pp. 1091–1098. https://doi.org/10.1109/CDC51059.2022.9992770

96. De Persis, C. and Tesi, P., Low-complexity Learning of Linear Quadratic Regulators from Noisy Data, *Automatica*, 2021, vol. 128, art. no. 109548. https://doi.org/10.1016/j.automatica.2021.109548

97. Zeng, X., Bako, L., and Ozay, N., Noise Sensitivity of the Semidefinite Programs for Direct Data-Driven LQR, *arXiv:2412.19705v1.*

98. Berberich, J., Koch, A., Scherer, C.W., et al., Robust Data-Driven State-Feedback Design, *Proc. American Control Conf. (ACC)*, Denver, CO, USA, 2020, pp. 1532–1538. https://doi.org/10.23919/ACC45564.2020.9147320

99. Petersen, I.R., A Stabilization Algorithm for a Class of Uncertain Linear Systems, *Systems & Control Letters*, 1987, vol. 8(4), pp. 351–357. https://doi.org/10.1016/0167-6911(87)90102-2

100. Khlebnikov, M.V. and Shcherbakov, P.S., Petersen's Lemma on Matrix Uncertainty and Its Generalizations, *Autom. Remote Control*, 2008, vol. 69, pp. 1932–1945. https://doi.org/10.1134/S000511790811009X

101. Bisoffi, A., De Persis, C., and Tesi, P., Trade-Offs in Learning Controllers from Noisy Data, *Systems & Control Letters*, 2021, vol. 154, art no. 104985. https://doi.org/10.1016/j.sysconle.2021.104985

102. Caverly, R.J. and Forbes, J.R., LMI Properties and Applications in Systems, Stability, and Control Theory, *arXiv:1903.08599.*

103. De Oliveira, M.C., Bernussou, J., and Geromel, J.C., A New Discrete-Time Robust Stability Condition, *Systems & Control Letters*, 1999, vol. 37, pp. 261–265. https://doi.org/10.1016/S0167-6911(99)00035-3

104. De Oliveira, M.C., Geromel, J.C., and Bernussou, J., Extended H_2 and H_∞ Norm Characterizations and Controller Parametrizations for Discrete-Time Systems, *Int. J. of Control*, 2002, vol. 75, no. 9, pp. 666–679. https://doi.org/10.1080/00207170210140212

105. Miller, J., Eising, J., Dörfler, F., et al., Data-Driven Structured Robust Control of Linear Systems, *arXiv: 2411.11542v1.*

106. Zhaohua, Y., Yuxing, Z., Nachuan, Y., et al., Data-Driven Structured Controller Design Using the Matrix S-Procedure, *arXiv:2503.14949v1.*

107. Brändle, F. and Allgöwer, F., A System Parametrization for Direct Data-Driven Analysis and Control with Error-in-Variables, *arXiv:2411.06787v2.*

108. De Boer, R., Robust Data-Driven State-Feedback Synthesis from Data Corrupted by Perturbations with Bounded Norms and Rates-of-Variation, *MSc Thesis*, University of Twente, 2023.

109. Hu, K. and Liu, T., A Robust Data-Driven Iterative Control Method for Linear Systems with Bounded Disturbances, *arXiv:2405.02537v1.*

110. Luppi, A., De Persis, C., and Tesi, P., On Data-Driven Stabilization of Systems with Nonlinearities Satisfying Quadratic Constraints, *Systems & Control Letters*, 2022, vol. 163, art. no. 105206. https://doi.org/10.1016/j.sysconle.2022.105206

111. Nakano, T., Aboudonia, A., Eising, J., et al., Dissipativity-Based Data-Driven Decentralized Control of Interconnected Systems, *arXiv:2509.14047.*

112. Liu, W., Li, Y., Sun, J., et al., Data-driven Internal Model Control for Output Regulation, *arXiv:2505.09255.*

This paper was recommended for publication by A.L. Fradkov, a member of the Editorial Board

ISSN 0005-1179 (print), ISSN 1608-3032 (online), Automation and Remote Control, 2026, Vol. 87, No. 9, pp. 809–830.
© The Author(s), 2026 published by Trapeznikov Institute of Control Sciences, Russian Academy of Sciences, 2026.
Russian Text © The Author(s), 2026, published in Avtomatika i Telemekhanika, 2026, No. 9, pp. 63–90.

LINEAR SYSTEMS

Closed-Loop State Predictor: Robustness and Safe Implementation

V. O. Nikiforov[*,a] **and D. N. Gerasimov**[*,b]

[*] *ITMO University, Saint-Petersburg, Russia*
e-mail: [a] *nikiforov_vo@itmo.ru,* [b] *dngerasimov@itmo.ru*

Received August 31, 2025
Revised January 29, 2026
Accepted February 29, 2026

Abstract—The paper addresses the problem of robust and safe implementation of a state predictor used for stabilization of the linear time-invariant system with known arbitrarily large yet bounded input delay. To this end, we design a new simple observer with resetting of initial conditions generating an estimate of the state prediction error. This estimate is used as the feedback for closed-loop correction of the state prediction. The robustness of the proposed closed-loop predictor with respect to parametric perturbations as well as to its numerical realization is proved and illustrated by simulation.

Keywords: linear systems, control delay, robustness

DOI: 10.7868/S1608303226090029

1. INTRODUCTION AND PROBLEM STATEMENT

It is considered a problem of practical implementation of the state-feedback control stabilizing linear time-invariant plant with the input delay[1]:

$$\dot{x}(t) = Ax(t) + bu(t-h), \tag{1}$$

where $x \in \mathbb{R}^n$ is the state with the initial condition x_0, $u \in \mathbb{R}^1$ is the control ($u(t) \equiv 0$ for $t < h$), $A \in \mathbb{R}^{n\times n}$ is not Hurwitz, $b \in \mathbb{R}^n$, the pair (A, b) is controllable, and h is the known arbitrarily large yet bounded time delay. We also assume that A and b are uncertain and their constant estimates $\hat{A}$ and $\hat{b}$ are used in the control design. In general, $\hat{A} \neq A$ and $\hat{b} \neq b$.

Remark 1. We will assume that the states of all considered dynamical systems are zero for $t < 0$, and under the initial condition x_0 we will assume that the state at the instance $t = 0$, i.e., $x_0 = x(0)$. For a dynamical subsystem with the state χ, by χ_ξ^+ we will denote the value of the state χ reset at the time instant ξ in accordance with some resetting rule.

If A and b are known exactly (i.e., $\hat{A} = A$ and $\hat{b} = b$), one can use the following *prediction-based* control [1–3]:

$$u(t) = -k^\top \left(e^{Ah}x(t) + \int\limits_{t-h}^{t} e^{A(t-\tau)}bu(\tau)d\tau \right), \tag{2}$$

where the vector of feedback gains $k \in \mathbb{R}^n$ is such that the matrix $A_k = A - bk^\top$ is Hurwitz. It is worth noting that the expression in the brackets is exactly the predicted value of the state, i.e.,

$$e^{Ah}x(t) + \int\limits_{t-h}^{t} e^{A(t-\tau)}bu(\tau)d\tau = x(t+h), \tag{3}$$

while the control (2) yields the closed-loop characteristic equation $\det\left(sI - A + bk^\top\right) = 0$.

[1] Sometimes the argument t will be omitted if it does not cause misunderstanding.

The control law (2) is widely used in such control problems as finite spectrum assignment [3, 4], adaptive regulation in systems with input delay [5, Chapter 6; 6–9], stabilization of multi-input multi-output systems with multiple delays [10] and so on (see also specialized books [2, 4, 11, 12]). However, this elegant theoretical result has essential drawback related to practical implementation of the integral term for unstable plants. In [3], it was proposed to approximate the integral with some numerical quadrature rule when the integral is replaced by a discrete sum with a finite number of terms. It was suggested that the closed-loop system should be stable for sufficiently high accuracy (i.e., for sufficiently large number of the terms).

However, in [13] it was demonstrated that for some conventional numerical quadrature rules the control law (2) may not stabilize the closed-loop system even for an arbitrary large number of the discrete terms in the sum. In other words, the predicted-based control is not robust with respect to numerical implementation of the predictor[2]. At the same time, as it was demonstrated in [3], implementation of (3) as a solution of the differential equation leads to cancellation of the unstable poles and zeros in the closed-loop system and results in instability for arbitrarily small parametric perturbations. In other words, in this case the predicted-based control is not robust with respect to parametric perturbations of the plant. It is worth noting that the control of asymptotically stable plants with input delays does not meet similar problems that gives opportunity to design a wide class of stabilizing control laws (for example, see [14, Section 2.2; 7, 15–17] and references therein).

To overcome this obstacle, in [18] (and in [19] for systems with both input and state delays) it was proposed to include a low-pass filter in the control loop and use *dynamic prediction-based* control of the form:

$$\dot{u} = -(\gamma + k^\top b)u - (k^\top A + \gamma k^\top)\left[e^{Ah}x(t) + \int\limits_{t-h}^{t} e^{A(t-\tau)}bu(\tau)d\tau\right], \tag{4}$$

where $\gamma > 0$. In this case, the closed-loop characteristic equation is $(s+\gamma)\det\left(sI - A + bk^\top\right) = 0$. It was demonstrated that the closed-loop system (1), (4) preserves the stability if the integral is approximated by a numerical quadrature rule with sufficiently high accuracy. Later, in [21–23] robustness of (4) with respect to small (in a certain sense) parametric (as well as delay) perturbations was proved.

Thus, the dynamic prediction-based controller (4) preserves the closed-loop stability even if the integral is approximated by a finite number of discrete terms and the real parameters A and b are slightly different form the nominal ones $\hat{A}$ and $\hat{b}$ used in the control law design. However, practical implementation of (4) requires the use of a special numerical quadrature rules (such as rectangular, trapezoidal, or Simpson's rules) of high precision (i.e., with a large number of the discrete terms).

There are two conceptually different approaches that completely avoid the problem of practical implementation of the infinite dimensional integral term. One of them assumes the use of different types of static finite dimension (SFD) feedbacks, while another one is based on the use of Luenberger-like observers for estimation of $x(t+\tau)$.

A simple SFD control law is represented by the *truncated predictor feedback* (TPF) [24–27]

$$u(t) = -k^\top(\sigma)e^{Ah}x(t), \tag{5}$$

where the feedback low gain $\hat{k} = \hat{k}(\sigma)$ is parameterized by a small parameter $\sigma \in (0, 1]$. As demonstrated in [26, 27], under a special choice of the feedback low gain $\hat{k}(\sigma)$ with sufficiently small σ, the

[2] In [20] with the use of linear matrix inequalities it was derives an estimate on admissible input delay allowing one to preserve stability of the closed-loop system under numerical implementation of the predictor for a given plant.

TPF provides global stabilization of a linear system with arbitrarily large input delay if the open-loop system is not exponentially unstable (i.e., eigenvalues of A are on the closed left-half plane). Further modification of a static finite dimension feedback known as *pseudo-predictor feedback* is applicable to a class of linear time-varying systems which are only polynomially unstable [28]. As mentioned in [28], since the static feedbacks are linear and finite dimensional, they are robust with respect to small parametric perturbations.

Thus, the static feedback laws provide robust stabilization of linear systems with arbitrarily large input delay and simultaneously completely remove the problem of practical implementation. However, the class of systems they can cope with is very restrictive.

Alternative method allowing one to completely remove the problem of the integral term implementation is based on the use of Luenberger-like observer [29]

$$\dot{\bar{x}}(t) = A\bar{x}(t) + bu(t) + L\left(\bar{x}(t-h) - x(t)\right), \tag{6}$$

where the feedback matrix L is chosen so that the model

$$\dot{e} = Ae + Le(t-h) \tag{7}$$

of the prediction error $e(t) = \bar{x}(t-h) - x(t)$ is asymptotically stable. If $e(t) \to 0$, then $\bar{x}(t) \to x(t+h)$ and one can use the simple state feedback $u(t) = -k^\top \bar{x}(t)$.

As well known [30], if $(A+L)$ is Hurwitz, then there exists sufficiently small $h_m > 0$ such that (7) is asymptotically stable for all $h \in [0, h_m]$. To cope with arbitrarily large input delays in [29], it was proposed to use *sequential sub-predictors* (SSP)—a sequence of coupled sub-predictors for small segments of the large delay (see also [31, 34, 36, 37], where SSP are used for different classes of linear and nonlinear systems with input, state, and output delays). In [29], it was demonstrated that the controller using the SSP is robust with respect to parametric perturbations, while in [37] it was proposed to couple infinite dimension prediction $x(t+r)$ with $r < h$ like (3) with the observer-based prediction $\bar{x}(t+h-r)$ like (6) to provide low-pass filtering needed for safe implementation of the integral term.

Thus, the observer-based predictors are robust with respect to parametric perturbations. However, for large input delays they require using a sequence of coupled sub-predictors or single observer-based predictor coupled with numerical implementation of infinite dimensional integral term.

In this paper, we propose a new closed-loop state predictor that is robust with respect to parametric perturbations and as well as numerical implementation. The proposed predictor consists of a single observer of simple structure and a resetting mechanism and has the following advantages over the known solutions:

—it is simple and does not require employment of special techniques of delayed systems stabilization;

—it is applicable for systems with arbitrarily large input delay without increasing complexity in the form of sequential sub-predictors;

—it is closed by actual value of the prediction error;

—it is robust with respect to parametric perturbations and numerical implementation;

—the predictor dynamic blocks can be practically implemented with the use of standard numerical solvers like `ode1 (Euler)`, `ode2 (Heun)`, `ode3 (Bogacki-Shampine)`, `ode4 (Runge-Kutta)`, `ode5`, or `ode8 (Dormand-Prince)` proposed in MatLab.

The present paper extends the results of the conference paper [38], where the closed-loop predictor was proposed but its robustness with respect to numerical realization was just conjected. In this paper, the properties of robustness with respect to both parametric perturbations and numerical

implementation are rigorously proved (Sections 3 and 4 respectively). The problem of robustness with respect to time-delay variations are not considered in this paper and perspective readers are referred to the corresponding works [2, 11].

The paper is organized as follows. In Section 2, the key idea of the proposed closed-loop predictor is introduced and its stability properties are analyzed for the ideal case, i.e., under assumption that the parameters A and b of the plant are known exactly, and there are no approximation errors in the integral implementation. Section 3 demonstrates the robustness of the proposed controller with respect to parametric perturbations, while Section 4 proves robustness with respect to numerical realization. Simulation results are presented in Section 5, while Appendices present the derivations of some models, proof of auxiliary lemmas.

2. CLOSED-LOOP PREDICTOR: KEY IDEA

Let us consider the integral term in (2) as the solution of the ordinary differential equation

$$\dot{\psi}(t) = A\psi(t) + bu(t) - e^{Ah}bu(t-h), \quad \psi(0) = 0. \tag{8}$$

In this case

$$x(t+h) = e^{Ah}x(t) + \psi(t), \tag{9}$$

and the control (2) takes the form

$$u = -k^\top \left(e^{Ah}x + \psi\right). \tag{10}$$

Hereafter, we will consider (8) and (9) as a virtual mathematical model of the *true state prediction* $x(t+h)$, while a solution $\psi(t)$ of (8) will be considered as the unmeasured *true integral term*.

We will assume that for implementation of a control law, the *integral term estimate* $\hat{\psi}(t)$ is generated as a solution (may be approximated by a numerical solver) of the equation

$$\dot{\hat{\psi}}(t) = \hat{A}\hat{\psi}(t) + \hat{b}u(t) - e^{\hat{A}h}\hat{b}u(t-h), \quad \hat{\psi}(0) = \hat{\psi}_0, \tag{11}$$

where $\hat{\psi} \in \mathbb{R}^n$. Then, the control law is defined as

$$u(t) = -\hat{k}^\top \left(e^{\hat{A}h}x(t) + \hat{\psi}(t)\right) \tag{12}$$

with the vector of feedback gains $\hat{k}$ selected so that the matrix $\hat{A}_k = \hat{A} - \hat{b}\hat{k}^\top$ is Hurwitz. In this case, *the state prediction estimate* is

$$\hat{x}(t+h) = e^{\hat{A}h}x(t) + \hat{\psi}(t). \tag{13}$$

As was mentioned above, the control (11), (12) is not robust due to cancellation of unstable (if A is not Hurwitz) poles and zeros in the closed-loop system [3]. It means that the closed-loop system (1), (11), (12) may lose the stability under *arbitrarily small* perturbations. Such perturbations can be caused either by parametric uncertainty of the plant (when some estimates $\hat{A}$ and $\hat{b}$ are used in the control law (11), (12) and $\hat{A} \neq A$, $\hat{b} \neq b$) or by the approximation errors of a numerical solver. To cope with this problem, we introduce a prediction error observer with resetting initial conditions.

2.1. Control Law Design: Ideal Case

To clarify the key idea of the design, we start with the ideal case when the control involves the true values of A and b, and the equation (11) is solved absolutely precisely. In this case, instead of the equations (11)–(13) we have

$$\dot{\hat{\psi}}(t) = A\hat{\psi}(t) + bu(t) - e^{Ah}bu(t-h),\ \hat{\psi}(0) = \hat{\psi}_0, \tag{14}$$

$$u(t) = -k^\top \left(e^{Ah}x(t) + \hat{\psi}(t)\right), \tag{15}$$

$$\hat{x}(t+h) = e^{Ah}x(t) + \hat{\psi}(t). \tag{16}$$

Here, we set $\hat{\psi}_0 \neq 0$ just to stress the fact that the designed closed-loop predictor is asymptotically stable.

Let us introduce *the state prediction error*

$$\varepsilon(t) := x(t+h) - \hat{x}(t+h). \tag{17}$$

In view of (9) and (16), we have

$$\varepsilon(t) = \psi(t) - \hat{\psi}(t), \tag{18}$$

and in view of (8) and (14) we obtain

$$\dot{\varepsilon}(t) = A\varepsilon(t), \quad \varepsilon_0 = -\hat{\psi}_0. \tag{19}$$

The state prediction error $\varepsilon(t)$ is not measured. However, the value[3]

$$y(t) := \varepsilon(t-h) = x(t) - \hat{x}(t) \tag{20}$$

or (see (16))

$$y(t) = x(t) - e^{Ah}x(t-h) - \hat{\psi}(t-h) \tag{21}$$

is measured and can be considered as the delayed output of the model (19). Then, in view of (19) and (20) we can write:

$$y(t) = e^{-Ah}\varepsilon(t). \tag{22}$$

The key idea of the closed-loop predictor design is to estimate the difference between the inaccessible true state prediction $x(t+h)$ and the state prediction estimate $\hat{x}(t+h)$ described by the model

$$\dot{\varepsilon}(t) = A\varepsilon(t), \quad y(t) = \varepsilon(t-h) \tag{23}$$

with the measured delayed output y, and then to use this estimate for online correction of $\hat{x}(t+h)$. To this end, we use the following observer (which is a particular case of the observer from Section 3.3 of [2]):

$$\dot{\hat{\varepsilon}}(t) = A\hat{\varepsilon}(t) + Le^{Ah}\left(y(t) - e^{-Ah}\hat{\varepsilon}(t)\right), \quad \hat{\varepsilon}_0 = 0, \tag{24}$$

where $\hat{\varepsilon} \in \mathbb{R}^n$ is *the estimate of the state prediction error* and the designer-chosen matrix $L \in \mathbb{R}^{n\times n}$ is such that $A_L = A - L$ is Hurwitz.

[3] Similar approach was employed in [39], where the delayed value of the state prediction error $\varepsilon(t-h) = x(t) - \hat{x}(t)$ was used to cope with constant external disturbances.

Introducing *the estimate error* $\tilde{\varepsilon} := \varepsilon - \hat{\varepsilon}$ and calculating its time derivative in view of (19), (22), and (24), we obtain

$$\dot{\tilde{\varepsilon}} = A_L \tilde{\varepsilon} \tag{25}$$

and, therefore, $\hat{\varepsilon}(t) \to \varepsilon(t) = x(t+h) - \hat{x}(t+h)$ as $t \to \infty$.

The expression (18) and the model (25) motivate us to correct the state prediction estimate according to the expression

$$\hat{x}^*(t+h) := e^{Ah}x(t) + \hat{\psi}(t) + \hat{\varepsilon}(t). \tag{26}$$

It is easy to see that the following equality holds:

$$\hat{x}^*(t+h) = x(t+h) - \tilde{\varepsilon}(t), \tag{27}$$

where $\tilde{\varepsilon}(t) \to 0$ as $t \to \infty$. Therefore, instead of (15) we use the control

$$u(t) = -k^\top \left(e^{Ah}x(t) + \hat{\psi}(t) + \hat{\varepsilon}(t) \right). \tag{28}$$

However, despite the fact that $\tilde{\varepsilon}(t) \to 0$ as $t \to \infty$, the estimates $\hat{\varepsilon}$ and $\hat{\psi}$ may grow unboundedly since the model (23) is not stable. To prevent unbounded growth of $\hat{\psi}$ and $\hat{\varepsilon}$, we propose to reset the states of the models (14) and (24) at each instance iT, where $i = 1, 2, 3, \ldots$ and T is a sufficiently large time interval to be defined later, according to the rule[4]

$$\hat{\psi}^+_{iT} = \hat{\psi}(iT) + \hat{\varepsilon}(iT), \qquad \hat{\varepsilon}^+_{iT} = 0. \tag{29}$$

Taking into account that $\hat{\varepsilon} = \varepsilon - \tilde{\varepsilon} = \psi - \hat{\psi} - \tilde{\varepsilon}$, we obtain

$$\hat{\psi}_{iT} = \psi(iT) - \tilde{\varepsilon}(iT). \tag{30}$$

Therefore, if the sequence $\tilde{\varepsilon}(iT)$ is asymptotically decaying, i.e., $\tilde{\varepsilon}(iT) \to 0$ for $i \to \infty$, then $\hat{\psi}_{iT} \to \psi(iT)$ for $i \to \infty$.

Remark 2. Variables $\hat{\varepsilon}(t)$ and $\hat{\psi}(t)$ are inner state variables of the controller and do not affect asymptotic behavior of the physical signals—state $x(t)$ and control $u(t)$. However, a rigorous poof of the system closed-loop stability as well as practical issue concerning large signal manipulations require to provide the boundedness of all the closed loop signal that is achieved by periodic resetting defined by (29).

It is worth noting that at the time instants $t = iT$ the signal $e^{-Ah}\hat{\varepsilon}(t)$ being the estimate of $y(t) = \varepsilon(t-h)$ is zeroed. Therefore, to provide correct work of the feedback $\left(y(t) - e^{-Ah}\hat{\varepsilon}(t)\right)$ in the observer (24) it is proposed to switch off this feedback on intervals $t \in [iT, iT+h]$ and instead of the observer (24) to use the observer with switching feedback

$$\dot{\hat{\varepsilon}}(t) = A\hat{\varepsilon}(t) + \gamma(t)Le^{Ah}\left(y(t) - e^{-Ah}\hat{\varepsilon}(t)\right), \quad \hat{\varepsilon}_0 = 0, \tag{31}$$

where

$$\gamma(t) = \begin{cases} 0 & \text{for} \quad iT \leqslant t < iT + h, \\ 1 & \text{for} \quad iT + h \leqslant t < (i+1)T. \end{cases} \tag{32}$$

[4] An alternative approach to the resetting of the model (14) was proposed in [40]. However, in this paper the predictor remains open loop and does not provide *asymptotic* behavior of the prediction.

Then, instead of the model (25) we have

$$\dot{\tilde{\varepsilon}} = \bar{A}_L \tilde{\varepsilon}, \tag{33}$$

where

$$\bar{A}_L = \begin{cases} A & \text{for} \quad iT \leqslant t < iT + h, \\ A_L & \text{for} \quad iT + h \leqslant t < (i+1)T. \end{cases}$$

Thus, the proposed control law consists of: the closed-loop state predictor (14), (26); the prediction error observer (31), (21); the controller (28); and the resetting rule (29).

The resetting interval T is chosen to provide the asymptotic stability of the model (33). To this end, we define a positive defined matrix P with minimum λ_1 and maximum λ_2 eigenvalues as the solution to the matrix equation $A_L^\top P + P A_L = -I$. Then, in view of (33) for the time interval $t \in [iT, (i+1)T]$, we can write

$$|\tilde{\varepsilon}((i+1)T)|^2 \leqslant \frac{\lambda_2}{\lambda_1} e^{-\frac{T-h}{\lambda_2}} |\tilde{\varepsilon}(iT+h)|^2 \leqslant \frac{\lambda_2}{\lambda_1} e^{-\frac{T-h}{\lambda_2}} \|e^{Ah}\|^2 |\tilde{\varepsilon}(iT)|^2.$$

To provide inequality $|\tilde{\varepsilon}((i+1)T)| < |\tilde{\varepsilon}(iT)|$, T is chosen from the condition[5]

$$T \geqslant h - \lambda_2 \ln\left(\frac{\lambda_1}{\lambda_2 \|e^{Ah}\|^2}\right). \tag{34}$$

Thus, we have proved the following lemma.

Lemma 1. *If the resetting time interval T obeys the inequality (34), then $\tilde{\varepsilon}(iT) \to 0$ for $i \to \infty$.*

Corollary 1. *Under the validity of (34), we have $\tilde{\varepsilon}(iT + h) \to 0$ for $i \to \infty$.*

Remark 3. The validity of (34) guarantees asymptotic convergence $\tilde{\varepsilon}(iT) \to 0$ for $i \to \infty$. At the same time, numerous simulations demonstrate that smooth transients without splashes after resetting in the closed-loop system are provided if T is chosen to be one and a half to two times greater than the value defined by inequality (34).

2.2. Stability under Ideal Conditions

To analyze the stability properties, we derive a model of the closed-loop system. It is convenient to derive separately two dynamic subsystems. The first one will be represented in the components $(\tilde{\varepsilon}, x, \psi)$, while the second one will be represented in $(\varepsilon, \hat{\varepsilon}, \hat{\psi})$.

Since the states $\hat{\psi}$ and $\hat{\varepsilon}$ are reset at the instances iT, and the feedback loop in the observer (31) is switched on with delay h, the derived dynamic models will be valid for the time intervals $iT + h \leq t < (i+1)T$. However, these models are used for analysis of the closed-loop stability for all time $t \in [0, \infty)$ taking into account asymptotic behavior of the discrete sequences $\tilde{\varepsilon}(iT)$ and $\tilde{\varepsilon}(iT + h)$.

Substituting (27) and (28) into (1), we obtain

$$\dot{x} = Ax - bk^\top x + bk^\top \tilde{\varepsilon}(t-h) = A_k x + bk^\top e^{-A_L h} \tilde{\varepsilon}.$$

Further, substituting (28) into (8), we have

$$\dot{\psi} = A\psi - bk^\top \left(e^{Ah} x + \hat{\psi} + \hat{\varepsilon} \pm \psi\right) + e^{Ah} bk^\top x - e^{Ah} bk^\top \tilde{\varepsilon}(t-h) = A_k \psi + Dx + C\tilde{\varepsilon},$$

where $C = bk^\top - e^{Ah} bk^\top e^{-A_L h}$ and $D = e^{Ah} bk^\top - bk^\top e^{Ah}$.

[5] Here we assume that A is not Hurwitz, hence $\|e^{Ah}\| \geqslant 1$.

Therefore, on the time intervals $iT + h \le t < (i+1)T$ we have

$$\begin{bmatrix} \dot{\tilde{\varepsilon}} \\ \dot{x} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} A_L & 0 & 0 \\ bk^\top e^{-A_L h} & A_k & 0 \\ C & D & A_k \end{bmatrix} \begin{bmatrix} \tilde{\varepsilon} \\ x \\ \psi \end{bmatrix},$$

where A_k and A_L are Hurwitz. Therefore, this subsystem is asymptotically stable due to its diagonal structure. If T is defined by (34), the resetting does not destroy asymptotic convergence $\tilde{\varepsilon}(iT) \to 0$ as $i \to \infty$, and $\tilde{\varepsilon}(t) \to 0$ as $t \to \infty$. Therefore, we conclude that $\tilde{\varepsilon}(t), x(t), \psi(t) \to 0$ as $t \to \infty$.

Now, we will derive a subsystem in the components $(\varepsilon, \hat{\varepsilon}, \hat{\psi})$. For $\hat{\varepsilon}$, we have

$$\dot{\hat{\varepsilon}} = A\hat{\varepsilon} + Le^{Ah}\left(e^{-Ah}\varepsilon - e^{-Ah}\hat{\varepsilon}\right) = A_L\hat{\varepsilon} + L\varepsilon.$$

Substituting (28) into (14), we have

$$\dot{\hat{\psi}} = A\hat{\psi} - bk^\top\left(e^{Ah}x + \hat{\psi} + \hat{\varepsilon}\right) + e^{Ah}bk^\top x - e^{Ah}bk^\top\left(\varepsilon(t-h) - \hat{\varepsilon}(t-h)\right) = A_k\hat{\psi} + Dx - K\varepsilon + \hat{K}\hat{\varepsilon},$$

where $K = e^{Ah}bk^\top e^{-Ah}$ and $\hat{K} = K - bk^\top$.

Therefore, on the time intervals $iT + h \le t < (i+1)T$ we have

$$\begin{bmatrix} \dot{\varepsilon} \\ \dot{\hat{\varepsilon}} \\ \dot{\hat{\psi}} \end{bmatrix} = \begin{bmatrix} A & 0 & 0 \\ L & A_L & 0 \\ -K & \hat{K} & A_k \end{bmatrix} \begin{bmatrix} \varepsilon \\ \hat{\varepsilon} \\ \hat{\psi} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ D \end{bmatrix} x. \tag{35}$$

This subsystem is unstable because of the matrix A. However, due to the states resetting (29), we have $\varepsilon(iT) = \tilde{\varepsilon}(iT)$, where $\tilde{\varepsilon}(iT) \to 0$ for $i \to \infty$. Therefore, $\varepsilon(iT)$ and $\varepsilon(iT + h)$ are asymptotically decreasing sequences. Now, taking into account that $x(t), \varepsilon(t) \to 0$ for $t \to \infty$, from (35) we conclude that $\hat{\varepsilon}(t), \hat{\psi}(t) \to 0$ for $t \to \infty$. Thus, we have proved the following statement.

Proposition 1. *For the ideal case, in the closed-loop system consisting of the plant* (1)*, the closed-loop state predictor* (14) *and* (26)*, the prediction error observer* (31) *and* (21)*, the controller* (28)*, and the resetting rule* (29) *all the signals are bounded and* $\tilde{\varepsilon}(t)$*,* $\hat{\varepsilon}(t)$*,* $\varepsilon(t)$*,* $x(t)$*,* $\psi(t)$*,* $\hat{\psi}(t) \to 0$ *as* $t \to \infty$ *for any initial conditions* x_0 *and* $\hat{\psi}_0$*.*

Thus, the closed-loop predictor provides the boundedness and asymptotic convergence of the signals for any initial conditions x_0 and $\hat{\psi}_0$, but not only for $\hat{\psi}_0 = 0$ as classical open-loop predictor does. In the next sections, we rigorously prove the robustness of the proposed controller with respect to parametric perturbations as well as to numerical solution of (14).

3. CLOSED-LOOP PREDICTOR: PARAMETRIC ROBUSTNESS

Now, let us assume that in the control law we use the estimate $\hat{A}$ instead of the true value A, and $\hat{A} \ne A$ and investigate the stability of the closed-loop system for small (in the sense of the norm $\|A - \hat{A}\|$) deviations of $\hat{A}$ from A.

Remark 4. All the results can be extended to the estimation of $\hat{b}$. However, we consider the case of $\hat{A}$ only to avoid tedious calculations.

Therefore, the control law consists of:

—the closed-loop state predictor

$$\dot{\hat{\psi}}(t) = \hat{A}\hat{\psi}(t) + bu(t) - e^{\hat{A}h}bu(t-h), \tag{36}$$

$$\hat{x}^*(t+h) := e^{\hat{A}h}x(t) + \hat{\psi}(t) + \hat{\varepsilon}(t); \tag{37}$$

—*the prediction error observer*

$$\dot{\hat{\varepsilon}}(t) = \hat{A}\hat{\varepsilon}(t) + \gamma(t)\hat{L}e^{\hat{A}h}\left(y(t) - e^{-\hat{A}h}\hat{\varepsilon}(t)\right), \tag{38}$$

$$y(t) = x(t) - e^{\hat{A}h}x(t-h) - \hat{\psi}(t-h), \tag{39}$$

where $\hat{L}$ is a designer-chosen matrix such that $\hat{A}_L = \hat{A} - \hat{L}$ is Hurwitz and $\gamma(t)$ is defined by (32);

—*the controller*

$$u(t) \ = -\hat{k}^\top \hat{x}^*(t+h), \tag{40}$$

where a vector $\hat{k}$ is such that the matrix $\hat{A}_k = \hat{A} - b\hat{k}^\top$ is Hurwitz;

—*the scheme of the states resetting* (29), where T is chosen as

$$T \geqslant h - \hat{\lambda}_2 \ln\left(\frac{\hat{\lambda}_1}{\hat{\lambda}_2\|e^{\hat{A}h}\|^2}\right), \tag{41}$$

$\hat{\lambda}_1$ and $\hat{\lambda}_2$ are the minimum and maximum eigenvalues of the matrix $\hat{P}$, respectively, while $\hat{P}$ is the solution of the matrix equation $\hat{A}_L^\top\hat{P} + \hat{P}\hat{A}_L = -I$.

As in Section 2.2, we first derive a model of the closed-loop system in the components $(\tilde{\varepsilon}, x, \psi)$.

In view of (13), for the prediction state error we have

$$\varepsilon(t) = x(t+h) - \hat{x}(t+h) = \tilde{E}x(t) + \psi(t) - \hat{\psi}(t), \tag{42}$$

where $\tilde{E} = e^{Ah} - e^{\hat{A}h}$. Therefore,

$$\hat{\varepsilon} = \tilde{E}x + \psi - \hat{\psi} - \tilde{\varepsilon}, \tag{43}$$

and (see (37))

$$\begin{aligned}\hat{x}^*(t+h) &= e^{\hat{A}h}x + \hat{\psi} + e^{Ah}x - e^{\hat{A}h}x + \psi \\ &= -\hat{\psi} - \tilde{\varepsilon}e^{Ah}x + \psi - \tilde{\varepsilon} = x(t+h) - \tilde{\varepsilon}.\end{aligned} \tag{44}$$

As demonstrated in Appendix A, on the time intervals $iT + h \leq t < (i+1)T$ we have

$$\begin{bmatrix}\dot{\tilde{\varepsilon}}\\ \dot{x}\\ \dot{\psi}\end{bmatrix} = \begin{bmatrix}\hat{A}_L & \tilde{E} & \tilde{A}\\ 0 & \bar{A}_k & 0\\ b\hat{k}^\top & \hat{D} & \bar{A}_k\end{bmatrix}\begin{bmatrix}\tilde{\varepsilon}\\ x\\ \psi\end{bmatrix} + \begin{bmatrix}0\\ b\hat{k}^\top\\ -e^{Ah}b\hat{k}^\top\end{bmatrix}\tilde{\varepsilon}(t-h) + \begin{bmatrix}\hat{L}e^{\hat{A}h}\\ 0\\ 0\end{bmatrix}\tilde{\eta}(x,\psi), \tag{45}$$

where $\tilde{\eta}(x,\psi)$ is defined in (A.3), and $\hat{D} = e^{Ah}b\hat{k}^\top - b\hat{k}^\top e^{Ah}$.

In view of Corollary 2, from Appendix B we conclude that the matrix $\bar{A}_k$ is Hurwitz for sufficiently small $\|\tilde{A}\|$. Since the matrices $\hat{A}_L$, $\bar{A}_k$ are Hurwitz, then the matrix

$$\begin{bmatrix}\hat{A}_L & 0 & 0\\ 0 & \bar{A}_k & 0\\ e^{Ah}b\hat{k}^\top & \hat{D} & \bar{A}_k\end{bmatrix}$$

is Hurwitz due to its diagonal structure. By virtue of Lemma 3 from Appendix B the matrix

$$\begin{bmatrix}\hat{A}_L & \tilde{E} & \tilde{A}\\ 0 & \bar{A}_k & 0\\ b\hat{k}^\top & \hat{D} & \bar{A}_k\end{bmatrix}$$

is Hurwitz for sufficiently small $\|\tilde{A}\|$ (and, as a result, for sufficiently small $\|\tilde{E}\|$). By Lemmas 4 and 5, from Appendix B we conclude that the origin of the system (45) is asymptotically stable on time intervals $iT \leqslant t < (i+1)T$. Further, we can prove that the sequence $\tilde{\varepsilon}(iT)$ is decreasing for sufficiently small $\|\tilde{A}\|$ if T is defined by (41). As a result, we have that $\tilde{\varepsilon}(t)$, $x(t)$, $\psi(t)$, $\eta(t) \to 0$ as $t \to \infty$ for sufficiently small $\|\tilde{A}\|$.

Now, we derive the subsystem in the coordinates $(\varepsilon, \hat{\varepsilon}, \hat{\psi})$. The model for ε is given by (A.7). As demonstrated in Appendix A (see (A.7) and (A.8)), on the time intervals $iT \leq t < (i+1)T$ we have

$$\begin{bmatrix} \dot{\varepsilon} \\ \dot{\hat{\varepsilon}} \\ \dot{\hat{\psi}} \end{bmatrix} = \begin{bmatrix} \hat{A} & 0 & 0 \\ \hat{L} & \hat{A}_L & 0 \\ 0 & -b\hat{k}^\top & \hat{A}_k \end{bmatrix} \begin{bmatrix} \varepsilon \\ \hat{\varepsilon} \\ \hat{\psi} \end{bmatrix} + \begin{bmatrix} \tilde{E}_A \\ 0 \\ \bar{D} \end{bmatrix} x + \begin{bmatrix} \tilde{A} \\ 0 \\ 0 \end{bmatrix} \psi + \begin{bmatrix} 0 \\ \hat{L}e^{\hat{A}h}\tilde{A} \\ 0 \end{bmatrix} \eta - \begin{bmatrix} 0 \\ 0 \\ e^{\hat{A}h}b\hat{k}^\top \end{bmatrix} \tilde{\varepsilon}(t-h).$$

As it was proved above, $x(t)$, $\tilde{\varepsilon}(t)$, $\psi(t)$, $\eta(t) \to 0$ as $t \to \infty$. However, this subsystem is unstable because of the matrix $\hat{A}$. At the same time, due to the states resetting (29) at the instances (41) we have $\varepsilon(iT) = \tilde{\varepsilon}(iT)$, where $\tilde{\varepsilon}(iT) \to 0$. As a result, $\varepsilon(iT)$ is the decreasing sequence. Taking into account that $\hat{A}_L$ and $\hat{A}_k$ are Hurwitz, we can conclude that $\varepsilon(t)$, $\hat{\varepsilon}(t)$, $\hat{\psi}(t) \to 0$ as $t \to \infty$. Thus, we have proved the following statement.

Proposition 2. *The control law* (36)–(40) *and* (29) *is robust with respect to parametric perturbations in the sense that the asymptotic stability declared by the Proposition 1 is preserved for small parametric variations* $\|\tilde{A}\|$*, i.e., all the closed-loop signals are bounded, and* $\tilde{\varepsilon}(t)$*,* $\hat{\varepsilon}(t)$*,* $\varepsilon(t)$*,* $x(t)$*,* $\psi(t)$*,* $\hat{\psi}(t) \to 0$ *for* $t \to \infty$ *and any initial conditions* x_0*,* $\hat{\psi}_0$*.*

4. CLOSED-LOOP PREDICTOR: ROBUSTNESS WITH RESPECT TO NUMERICAL REALIZATION

In this section, we consider the case when the parameters of the plant are known exactly (i.e., $\hat{A} = A$ and $\hat{b} = b$), while to implement the integral term ψ we solve the differential equation (14) with the use of a numerical solver like `ode1 (Euler)`, `ode2 (Heun)`, `ode3 (Bogacki-Shampine)`, `ode4 (Runge-Kutta)`, `ode5`, or `ode8 (Dormand-Prince)` proposed by MatLab. In other words, we investigate robustness of the open-loop predictor with respect to numerical solution of the differential equation (14).

To this end, let us assume that we use a discrete controller

$$u(m) = -k^\top \left(e^{Ah}x(m) + \bar{\psi}(m) \right) \tag{46}$$

with the sampling time δ, where $u(m)$ and $x(m)$ are the values of the signals u and x at the time instants $t = m\delta$, $m = 1, 2, 3 \ldots$, while $\bar{\psi}(m)$ is the *numerical solution* of the differential equation (14) obtained with the use of a numerical solver and the same sampling time δ. To interpolate the control signal within the sampling time δ, we use the zero-order hold.

It is reasonable to make the following assumption.

Assumption. The numerical solution $\bar{\psi}(m)$ is *convergent*, i.e.,

$$\lim_{\delta \to 0} \bar{\psi}(m) = \hat{\psi}(t)$$

for any fixed $t = m\delta$.

Since the equation (14) is linear in the state $\hat{\psi}$, we can construct such continuous-time model referred as *the model of the continuous majorant*

$$\dot{\bar{\psi}}^*(t) = \bar{A}\bar{\psi}^*(t) + \bar{b}u(t) - e^{\bar{A}h}\bar{b}u(t-h) \tag{47}$$

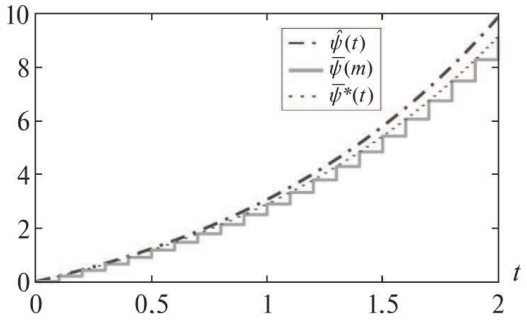


Fig. 1. The solution $\hat{\psi}(t)$ to the equation (14), numerical solution $\bar{\psi}(m)$, and the continuous-time majorant $\bar{\psi}^*(t)$.

with some $\bar{A} = \bar{A}(\delta)$ and $\bar{b} = \bar{b}(\delta)$ depending on δ that (see Fig.1)

$$\bar{\psi}^*(t) = \bar{\psi}(m) \quad \text{for any} \quad t = m\delta.$$

The properties of the model (47) are defined by the following statement proved in Appendix C.

Lemma 2. *The parameters $\bar{A}$ and $\bar{b}$ of the model* (47) *are such that*

$$\lim_{\delta\to 0} \bar{A}(\delta) = A, \quad \text{and} \quad \lim_{\delta\to 0} \bar{b}(\delta) = b.$$

Thus, at the time instants $t = m\delta$ the numerical solution $\bar{\psi}(m)$ can be replaced by the continuous majorant $\bar{\psi}^*(t)$ generated by the parametrically perturbed (since $\bar{A} \neq A$) model (47). In other words, we reduce the robustness analysis with respect to numerical solution to the robustness analysis with respect to parametric perturbations. Thus, in view of Proposition 2 we immediately obtain the following result.

Proposition 3. *The control law* (14), (26), (24), (21), (28), *and* (29) *is robust with respect to numerical solution of* (14) *in the sense that the property of the asymptotic convergence defined by Proposition 1 is preserved for a convergent numerical solution $\bar{\psi}(m)$ with sufficiently small sampling time δ, i.e., all the closed-loop signals are bounded and $\tilde{\varepsilon}(t)$, $\hat{\varepsilon}(t)$, $\varepsilon(t)$, $x(t)$, $\psi(t)$, $\hat{\psi}(t) \to 0$ as $t \to \infty$ for any initial conditions x_0, $\hat{\psi}_0$.*

Remark 5. It is obvious that Propositions 2 and 3 can be combined, and we can prove the robustness of the proposed closed-loop predictor simultaneously with respect to the parametric perturbations as well as to the numerical solution.

5. SIMULATION RESULTS

Example 1. Convergence under nonzero initial condition $\hat{\psi}_0 \neq 0$. For the methodological purposes we consider behavior of the open-loop predictor, closed-loop predictor without state resetting and closed-loop predictor with state resetting under nonzero initial condition $\hat{\psi}_0$.

Consider the second-order unstable plant (1) with $h = 1$ *s*,

$$\hat{A} = A = \begin{bmatrix} 0 & 1 \\ 0 & 0.1 \end{bmatrix}, \quad \hat{b} = b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad x_0 = \begin{bmatrix} 0 \\ 0.5 \end{bmatrix} \tag{48}$$

and set $k = [2 \;\; 3.1]^\top$. In this example, the simulations were carried out in MatLab/Simulink environment with the use of `ode8 (Dormand-Prince)` solver and fixed sampling time 0.00001. It is also assumed that A and b are known and used in the control law, i.e., $\hat{A} = A$ and $\hat{b} = b$.

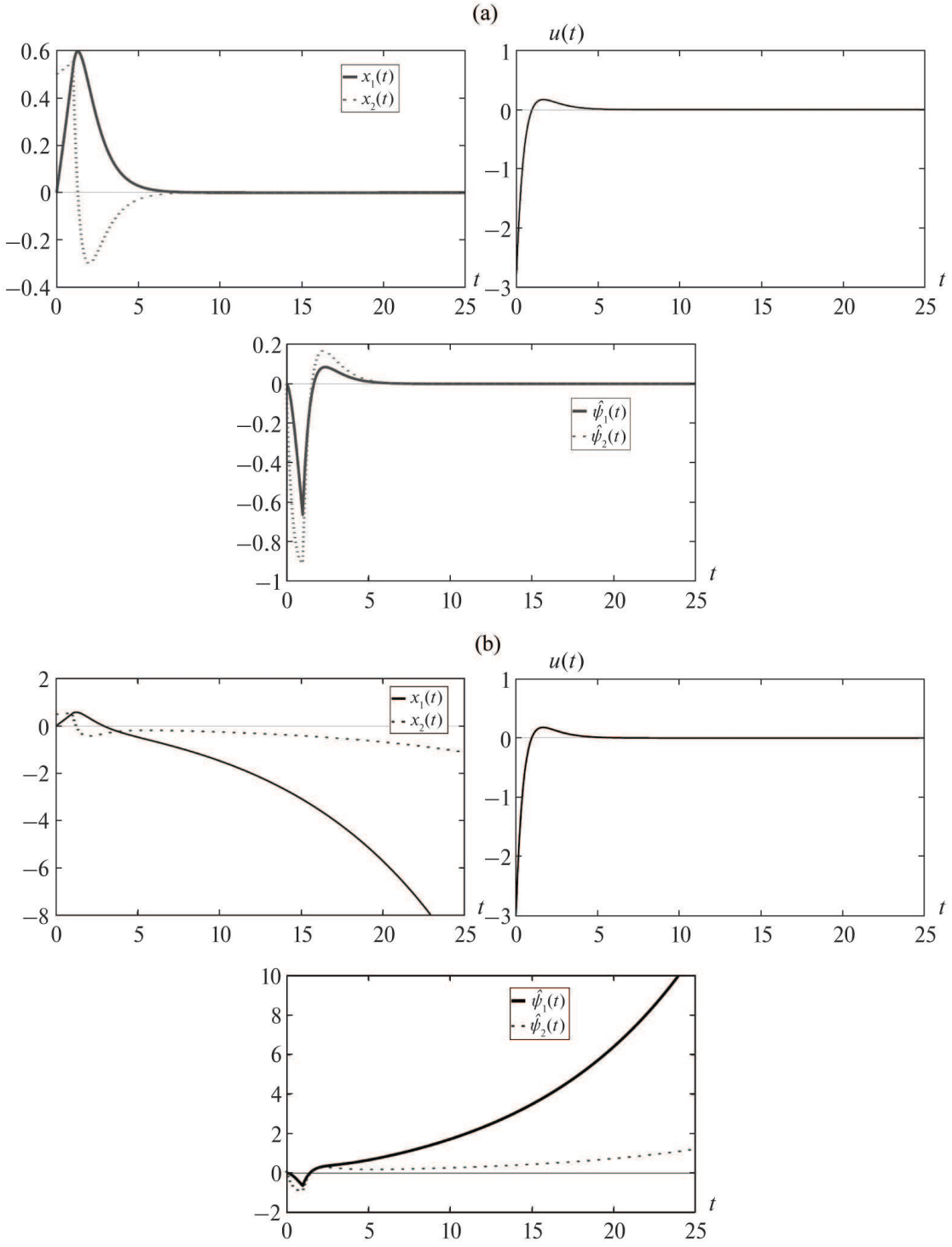


Fig. 2. (Example 1) Transients in the system obtained using the classical open-loop predictor with the parameters (48) and $h = 1$ s: (a) $\hat{\psi}_0 = 0$; (b) $\hat{\psi}_0 \neq 0$.

First, we consider controller with the open-loop state prediction estimate (14) and (15). Simulations results for $\hat{\psi}_0 = 0$ are presented in Fig. 2a and demonstrate asymptotic convergence. However, for $\hat{\psi}_0 = [0, 0.1]^\top$ we obtain unbounded growth of x and ψ (see Fig. 2b).

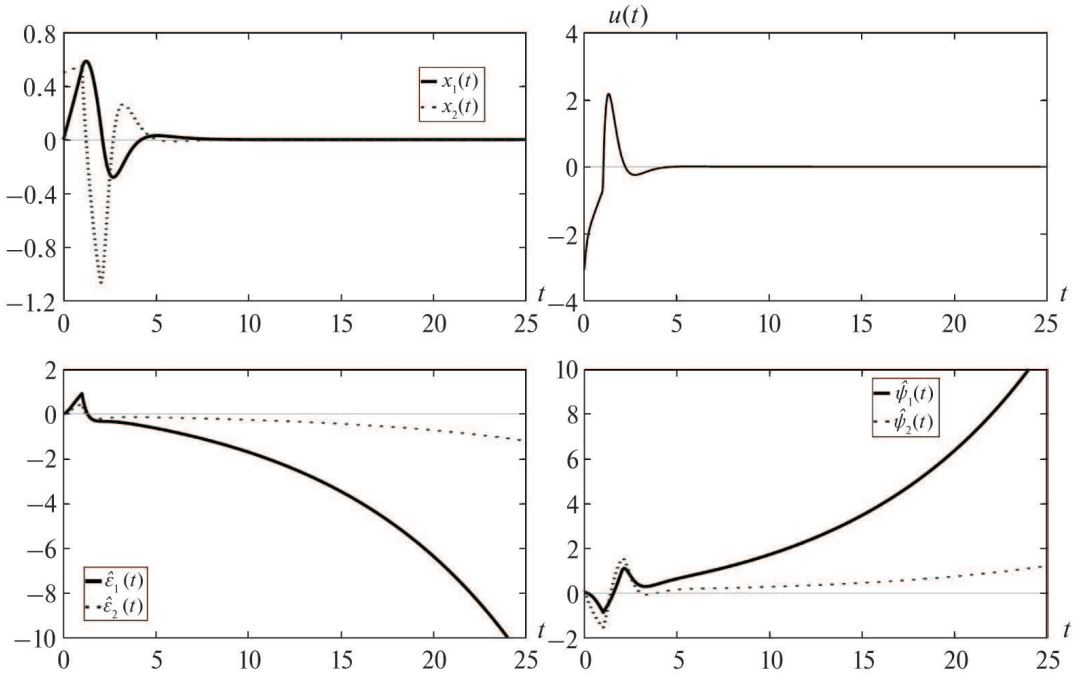


Fig. 3. (Example 1) Transients in the system obtained for $\hat{\psi}_0 \neq 0$ using the closed-loop predictor without state resetting with the parameters (48) and $h = 1$ s.

Now, consider controller with the closed-loop predictor but without state resetting. In this case the controller is described by the following equations: (14), (20), (24), and (28). In simulation, we set:

$$L = \begin{bmatrix} 5 & 0 \\ 6 & 0.1 \end{bmatrix}, \quad T = 10 \text{ s}. \tag{49}$$

Simulation results for $\hat{\psi}_0 = [0, 0.1]^\top$ are presented in Fig. 3 and demonstrate asymptotic convergence on x (in contrast to the open-loop predictor), but unbounded growth on $\hat{\psi}$ and $\hat{\varepsilon}$.

Finally, Fig. 4 presents simulation results obtained for $\hat{\psi}_0 = [0, 0.1]^\top$ with the use of the proposed closed-loop predictor with the state resetting. In this case, the controller is described by the following equations: (14), (26), (31), (21), (28), and (29). As seen from the presented plots, the proposed controller provides asymptotic convergence of x and boundedness and zeroing of $\hat{\psi}$ and $\hat{\varepsilon}$.

Example 2. Robustness with respect to parametric disturbances. Consider the plant (1) with $h = 1$ s,

$$A = \begin{bmatrix} 0 & 1 \\ 0.45 & 0.2 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad x_0 = \begin{bmatrix} 0 \\ 0.5 \end{bmatrix}, \tag{50}$$

and

$$\hat{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0.1 \end{bmatrix}, \quad \hat{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{51}$$

We set $\hat{k} = [2 \;\; 3.1]^\top$. Simulations are carried out in MatLab/ Simulink with the use of `ode8 (Dormand -Prince)` solver and fixed sampling time 0.00001.

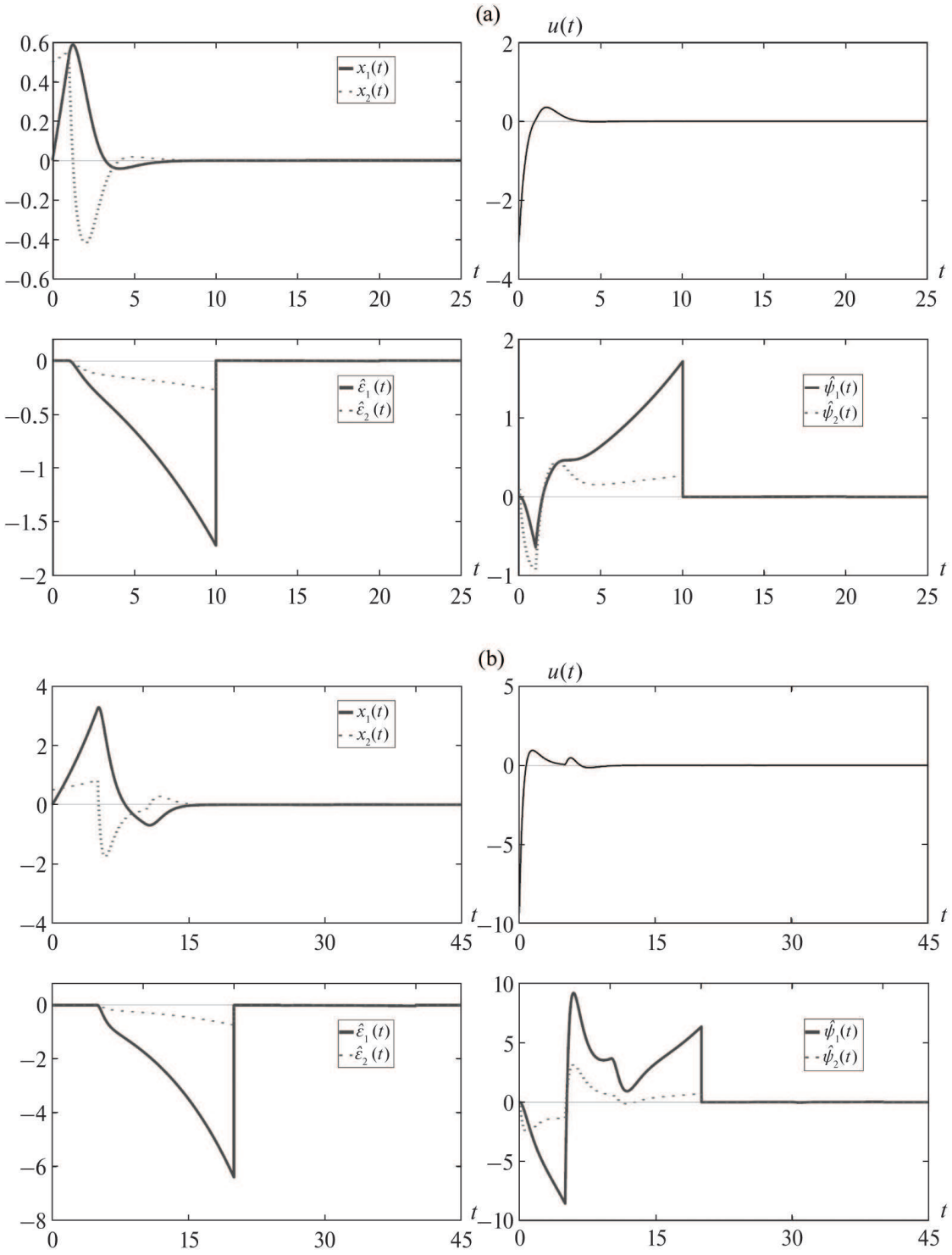


Fig. 4. (Example 1) Transients in the system obtained for $\hat{\psi}_0 \neq 0$ using the closed-loop predictor with state resetting with the parameters (48): (a) $h = 1$ s and $T = 10$ s; (b) $h = 5$ s and $T = 20$ s.

Simulation results obtained for the controller with the open-loop state prediction estimate with $\hat{\psi}_0 = 0$ are presented in Fig. 5 and demonstrate unbounded growth of $x(t)$. Figure 6 presents simulation results obtained for the proposed controller with the closed-loop state prediction and the states resetting. In this simulation, we set L and T as in (49). As seen from the plots, in spite

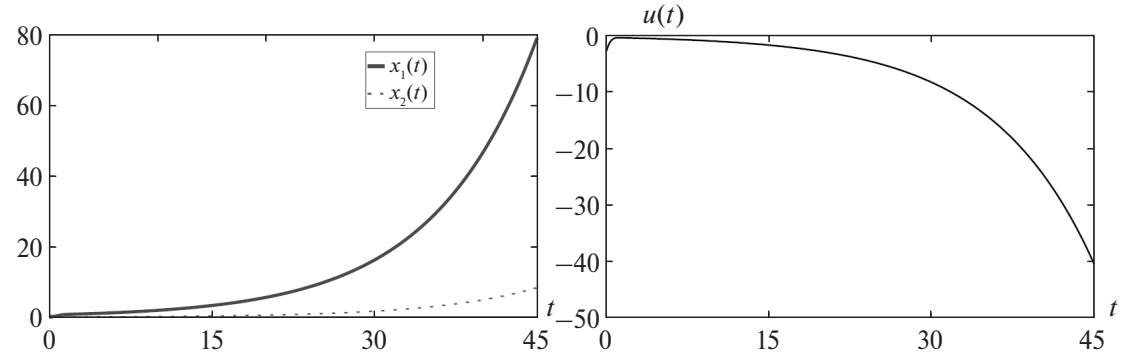


Fig. 5. (Example 2) Transients in the system obtained using the classical open-loop predictor with the parameters (50), (51), and $h = 1$ s.

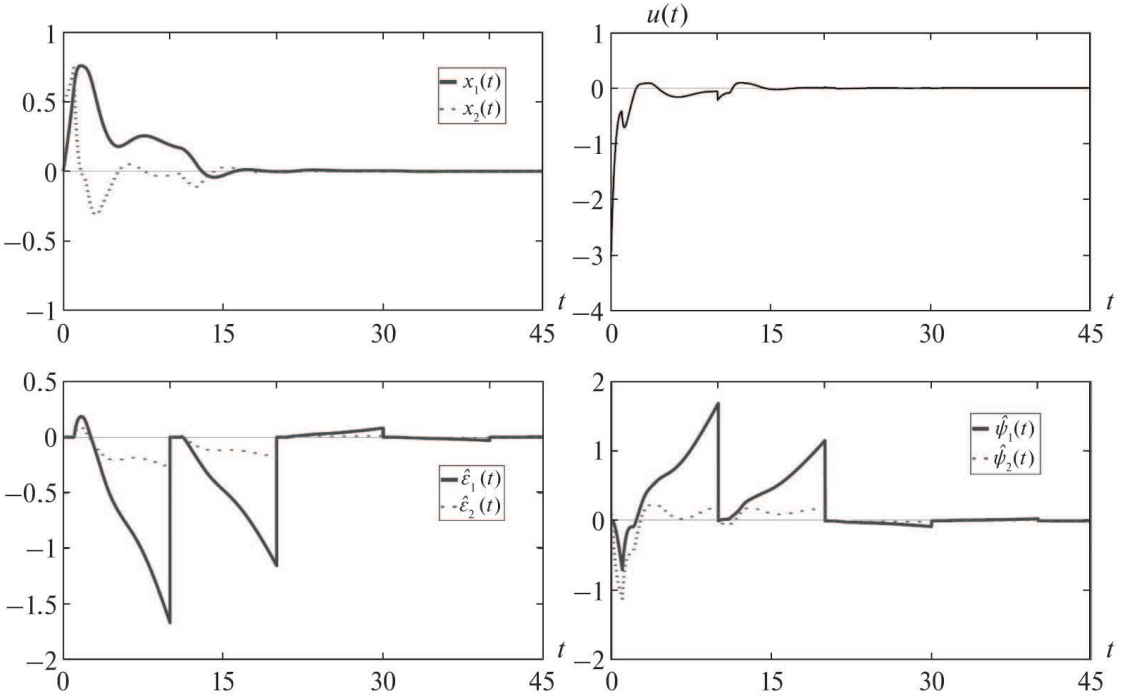


Fig. 6. (Example 2) Transients in the system obtained using the closed-loop predictor with state resetting with the parameters (50), (51), and $h = 1$ s.

of the parametric perturbations the proposed controller provides asymptotic zeroing of $x(t)$, $\hat{\psi}(t)$, and $\hat{\varepsilon}(t)$ with all the closed-loop signals bounded.

Example 3. Robustness with respect to numerical solution. Consider again the plant (1) with the parameters (48), $h = 1$ s, and $k = [2\ \ 3.1]^\top$. Simulation results for the controller with the open-loop state prediction obtained with the use of `ode8 (Dormand-Prince)` solver and fixed sampling time 0.00001 are presented in Fig. 2a and demonstrate zeroing of the state $x(t)$ visible at least for the given zoom and $t \leqslant 25$ s. However, for the less accurate `ode1 (Euler)` solver with fixed sampling time 0.0001 we obtain fast diverging of $x(t)$ (see Fig. 7). The proposed controller with the closed-loop state predictor and L, T defined by (49) provides for `ode1 (Euler)` solver with fixed sampling time 0.0001 the boundedness of all the closed-loop signals and asymptotic zeroing of $x(t)$, $\hat{\psi}(t)$, and $\hat{\varepsilon}(t)$ with small transformations of transient processes presented in Fig. 4a.

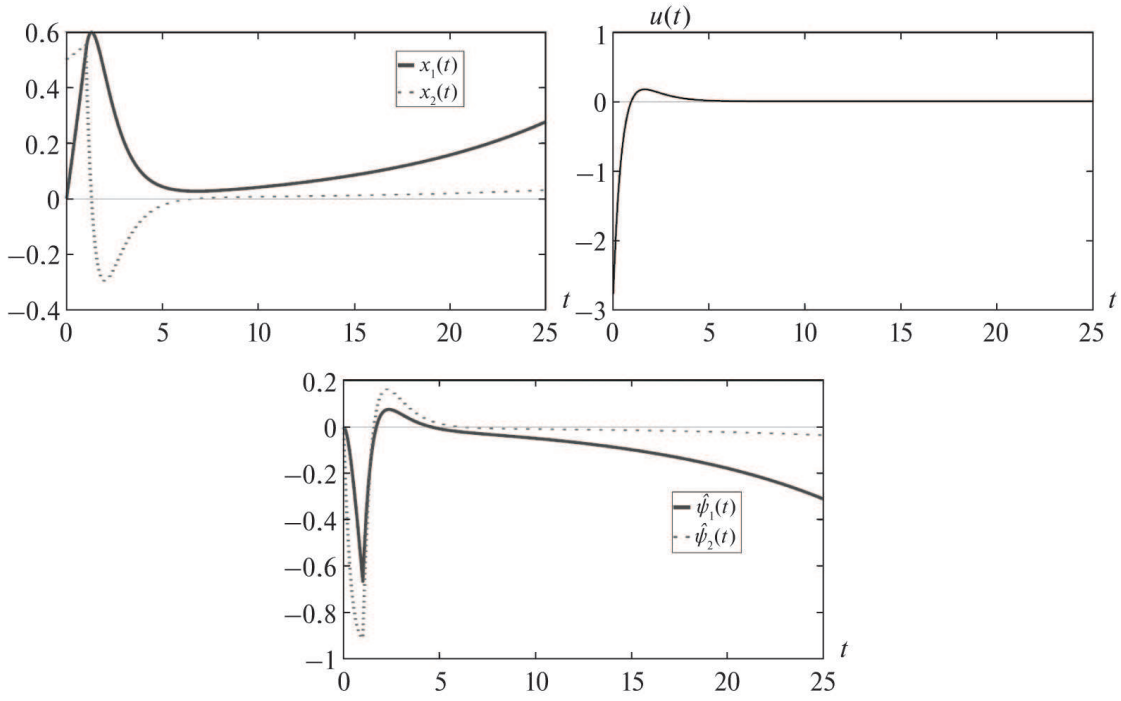


Fig. 7. (Example 3) Transients in the system obtained using the classical open-loop predictor with the parameters (48) and $h = 1$ s and the method of numerical integration `ode1 (Euler)` with the fixed sampling time 0.0001 s.

6. CONCLUSION

A new simple closed-loop state predictor is designed and its robustness with respect to parametric perturbation as well as numerical realization is proved. The key idea of the closed-loop predictor is to estimate the difference between the inaccessible true state prediction $x(t+h)$ and the state prediction estimate $\hat{x}(t+h)$ generated by the predictor and use this estimate for online correction of $\hat{x}(t+h)$. The estimation problem is solved with the use of the observer designed for autonomous systems with output delay. In order to prevent the unbounded growth of some internal signals, a special resetting rule is introduced.

FUNDING

The work is supported by the Ministry of Science and Higher Education of the Russian Federation (project no. FSER-2025-0002).

APPENDIX A

Derivation of the Model (45). Substituting (44) into (40) and (1), we receive

$$\dot{x} = \bar{A}_k x + b\hat{k}^\top \tilde{\varepsilon}(t-h), \tag{A.1}$$

where $\bar{A}_k = A - b\hat{k}^\top$.

Now, let us consider (39):

$$\begin{aligned} y(t) &= e^{Ah}x(t-h) + \psi(t-h) - e^{\hat{A}h}x(t-h) - \hat{\psi}^*(t-h) \\ &= \tilde{E}x(t-h) + \psi(t-h) - \hat{\psi}^*(t-h) = \varepsilon(t-h). \end{aligned}$$

In view of (42), (8) and (36), we obtain

$$\begin{aligned}\dot{\varepsilon} = \tilde{E}\dot{x} + \dot{\psi} - \dot{\hat{\psi}} = \tilde{E}Ax + \tilde{E}bu(t-h) + A\psi + bu - e^{Ah}bu(t-h)\\ - \hat{A}\hat{\psi} - bu + e^{\hat{A}h}bu(t-h) = \hat{A}\varepsilon + \tilde{E}_A x + \tilde{A}\psi,\end{aligned} \tag{A.2}$$

where $\tilde{A} = A - \hat{A}$ and $\tilde{E}_A = \tilde{A}e^{Ah} - e^{\hat{A}h}\tilde{A}$.

In view (A.2) we have (see Section 3.3 in [2])

$$\varepsilon(t-h) \;=\; e^{-\hat{A}h}\varepsilon(t) - \int\limits_{t-h}^{t} e^{\hat{A}(t-h-\tau)}\left(\tilde{E}_A x(\tau) + \tilde{A}\psi(\tau)\right)d\tau.$$

Let us define

$$\tilde{\eta}(x,\psi) = -\int\limits_{t-h}^{t} e^{\hat{A}(t-h-\tau)}\left(\tilde{E}_A x(\tau) + \tilde{A}\psi(\tau)\right)d\tau. \tag{A.3}$$

Then

$$y(t) = \varepsilon(t-h) = e^{-\hat{A}h}\varepsilon(t) + \tilde{\eta}(x,\psi), \tag{A.4}$$

and (in view of (A.2) and (38)) we have

$$\begin{aligned}\dot{\tilde{\varepsilon}} = \dot{\varepsilon} - \dot{\hat{\varepsilon}} = \hat{A}\varepsilon + \tilde{E}_A x + \tilde{A}\psi - \hat{A}\hat{\varepsilon} + \hat{L}e^{\hat{A}h}\left(e^{-\hat{A}h}\varepsilon + \tilde{\eta}(x,\psi) - e^{-\hat{A}h}\hat{\varepsilon}\right)\\ = \hat{A}_L\tilde{\varepsilon} + \tilde{E}x + \tilde{A}\psi + \hat{L}e^{\hat{A}h}\tilde{\eta}(x,\psi).\end{aligned} \tag{A.5}$$

In view of (8), (40) and (44), we have

$$\begin{aligned}\dot{\psi} = A\psi - b\hat{k}^\top\left(e^{\hat{A}h}x + \hat{\psi} + \hat{\varepsilon} \pm \psi\right) + e^{Ah}b\hat{k}^\top x - e^{Ah}b\hat{k}^\top\tilde{\varepsilon}(t-h)\\ = \bar{A}_k\psi + \hat{D}x + b\hat{k}^\top\tilde{\varepsilon} - e^{Ah}b\hat{k}^\top\tilde{\varepsilon}(t-h),\end{aligned} \tag{A.6}$$

where $\hat{D} = e^{Ah}b\hat{k}^\top - b\hat{k}^\top e^{Ah}$.

Taking into account (A.5), (A.1) and (A.6), we have (45).

For $\hat{\varepsilon}$ in view of (38) and (A.4), we have

$$\dot{\hat{\varepsilon}} \;=\; \hat{A}\hat{\varepsilon} + \hat{L}e^{\hat{A}h}\left(e^{-\hat{A}h}\varepsilon + \tilde{A}\eta - e^{-\hat{A}h}\hat{\varepsilon}\right) = \hat{A}_L\hat{\varepsilon} + \hat{L}\varepsilon + \hat{L}e^{\hat{A}h}\tilde{A}\eta. \tag{A.7}$$

Substituting (40) into (36) instead of $u(t)$, and (40), (44) instead of $u(t-h)$, we finally have

$$\begin{aligned}\dot{\hat{\psi}} = \hat{A}\hat{\psi} - b\hat{k}^\top\left(e^{\hat{A}h}x + \hat{\psi} + \hat{\varepsilon}\right) + e^{\hat{A}h}b\hat{k}^\top x - e^{\hat{A}h}b\hat{k}^\top\tilde{\varepsilon}(t-h)\\ = \hat{A}_k\hat{\psi} - b\hat{k}^\top\hat{\varepsilon} + \bar{D}x - e^{\hat{A}h}b\hat{k}^\top\tilde{\varepsilon}(t-h),\end{aligned} \tag{A.8}$$

where $\bar{D} = e^{\hat{A}h}b\hat{k}^\top - b\hat{k}^\top e^{\hat{A}h}$.

APPENDIX B

Useful mathematical background.

In this appendix, we summarize some mathematical tools used in the proof of the main result in Section 3.

Lemma 3. *If a matrix $A \in \mathbb{R}^{n\times n}$ is Hurwitz, then a matrix $\hat{A} \in \mathbb{R}^{n\times n}$ is also Hurwitz for sufficiently small $\|\tilde{A}\|$, where $\tilde{A} = A - \hat{A}$ and $\|A\|$ is the spectral norm of the matrix.*

The validity of this lemma directly follows from the robustness of the exponential stability with respect to multiplicative (i.e., parametric) perturbations (see Section 2.1.4 in [32]).

Corollary 2. *If a matrix $\hat{A}_k = \hat{A} - b\hat{k}^\top$ is Hurwitz, where $\hat{A} \in \mathbb{R}^{n\times n}$ and $b, \hat{k} \in \mathbb{R}^n$, then a matrix $\bar{A}_k = A - b\hat{k}^\top$ is also Hurwitz for sufficiently small $\|\tilde{A}\|$.*

Lemma 4. *If the linear system*

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}A_1 & 0\\ 0 & A_2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix},$$

where $A, A_1, A_2, B \in \mathbb{R}^{n\times n}$ and $x_1, x_2 \in \mathbb{R}^n$, is exponentially stable, then the system

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}A_1 & \tilde{A}\\ 0 & A_2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\\ B\end{bmatrix} x_1(t-h)$$

is also exponentially stable for sufficiently small $\|\tilde{A}\|$.

Proof. For x_1 we have

$$\dot{x}_1 = A_1 x_1 + \tilde{A}W(s)[x_1(t-h)], \tag{B.1}$$

where $W(s) = (sI_2 - A_2)^{-1}B$. It can be shown that the characteristic polynomial $p(s)$ of (B.1) can be represented in the form [12]:

$$p(s) = P(s) + \sum_{i=1}^{n} Q_i(\tilde{A}, s)e^{-ihs},$$

where $P(s) = \det(sI_1 - A_1)\det(sI_2 - A_2)$,

$$Q_i(\tilde{A}, s) = \sum_{k=0}^{2(n-1)} q_{ik}(\tilde{A})s^k,$$

and the coefficients $q_{ik}(\tilde{A})$ are such that

$$\lim_{\|\tilde{A}\|\to 0} q_{ik}(\tilde{A}) = 0.$$

Therefore, for sufficiently small $\|\tilde{A}\|$ and all $\omega \in \mathbb{R}_+$ we have

$$\frac{\sum_{i=1}^{n}\left|Q_i(\tilde{A}, j\omega)\right|}{P(j\omega)} < 1,$$

where j is the imaginary unit. Then, in accordance with Lemma 1 of [12, page 45] we conclude that the system (B.1) is asymptotically stable independently from delay (see also [11, Proposition 2.1]).

Lemma 5. *If a matrix $A_1 \in \mathbb{R}^{n\times n}$ is Hurwitz, then the origin of the system*

$$\dot{x} = A_1 x + B_1\eta(x), \tag{B.2}$$

where $x \in \mathbb{R}^n$, $B_1 \in \mathbb{R}^{n\times m}$,

$$\eta(x) = \int_{t-h}^{t} e^{A_2(t-h-\tau)}B_2 x(\tau)d\tau, \qquad \eta \in \mathbb{R}^n,$$

$A_2 \in \mathbb{R}^{m\times m}$, $B_2 \in \mathbb{R}^{m\times m}$, and $h > 0$ is a constant, is asymptotically stable for sufficiently small $\|B_2\|$.

Proof. Let us consider the Lyapunov function candidate $V(x) = \frac{1}{2}x^\top Px$, where the matrix $P = P^\top > 0$ is the solution of the matrix equation $A_1^\top P + PA_1 = -I_n$. Let λ_1 and λ_2 be the minimum and maximum eigenvalues of P, respectively. Therefore,

$$\frac{\lambda_1}{2}|x(t)|^2 \leqslant V(t) \leqslant \frac{\lambda_2}{2}|x(t)|^2. \tag{B.3}$$

Then, the time derivative of V along solutions of (B.2) obeys

$$\dot V = -|x|^2 + x^\top PB_1\eta(x) \leqslant -\frac{2}{\lambda_2}V + \|PB_1\||x||\eta(x)|.$$

Via the properties of defined integrals, we obtain [41]

$$|\eta(x)| = \left|\int\limits_{t-h}^{t} e^{A_2(t-h-\tau)}B_2x(\tau)d\tau\right| \leqslant \int\limits_{t-h}^{t}\left|e^{A_2(t-h-\tau)}B_2x(\tau)\right|d\tau \leqslant \|B_2\|\int\limits_{t-h}^{t}\left\|e^{A_2(t-h-\tau)}\right\||x(\tau)|d\tau.$$

Since for any $\tau \in [t-h, t]$ the function $\left\|e^{A_2(t-h-\tau)}\right\|$ is bounded, e.g. $r \leqslant \left\|e^{A_2(t-h-\tau)}\right\| \leqslant R < \infty$, and the constants r and R do not depend on t, we can write [41]

$$|\eta(x)| \leqslant \mu\|B_2\|\int\limits_{t-h}^{t}|x(\tau)|d\tau,$$

where $\mu > 0$ is some constant depending on A_2 and h. Therefore, we have

$$\dot V \leqslant -\frac{2}{\lambda_2}V + \mu\|PB_1\|\|B_2\||x|\int\limits_{t-h}^{t}|x(\tau)|d\tau. \tag{B.4}$$

Now let us check if it is possible to find such a value of $\|B_2\|$ that (B.2) be asymptotically stable and $\dot V = -\alpha V$, where $\alpha > 0$ is a positive constant. In this case, $V(t) = e^{-\alpha t}V(0)$ and in view of (B.3) $|x(t)| \leqslant \sqrt{\frac{\lambda_2}{\lambda_1}}e^{-\frac{\alpha}{2}t}|x(0)|$. Therefore,

$$\dot V \leqslant -\frac{2}{\lambda_2}V + \mu\sqrt{\frac{\lambda_2}{\lambda_1}}e^{-\frac{\alpha}{2}t}\|PB_1\|\|B_2\||x(0)|\int\limits_{t-h}^{t}|x(\tau)|d\tau,$$

where

$$\int\limits_{t-h}^{t}|x(\tau)|d\tau \leqslant \sqrt{\frac{\lambda_2}{\lambda_1}}\int\limits_{t-h}^{t}e^{-\frac{\alpha}{2}\tau}|x(0)|\,d\tau = \frac{2}{\alpha}\sqrt{\frac{\lambda_2}{\lambda_1}}\left(e^{\frac{\alpha}{2}h}-1\right)e^{-\frac{\alpha}{2}t}|x(0)|.$$

Then, we can write

$$\begin{aligned} r\dot V &\leqslant -\frac{2}{\lambda_2}V + \frac{2\mu}{\alpha}\frac{\lambda_2}{\lambda_1}\left(e^{\frac{\alpha}{2}h}-1\right)\|PB_1\|\|B_2\|e^{-\alpha t}|x(0)|^2 \\ &\leqslant -\frac{2}{\lambda_2}V + \frac{4\mu}{\alpha}\frac{\lambda_2}{\lambda_1^2}\left(e^{\frac{\alpha}{2}h}-1\right)\|PB_1\|\|B_2\|e^{-\alpha t}V(0). \end{aligned}$$

Consider the worst case, when

$$\dot{V} = -\frac{2}{\lambda_2}V + \frac{4\mu}{\alpha}\frac{\lambda_2}{\lambda_1^2}\left(e^{\frac{\alpha}{2}h} - 1\right)\|PB_1\|\|B_2\|e^{-\alpha t}V(0)$$

and, therefore,

$$\dot{V} = -\left(\frac{2}{\lambda_2} - \frac{4\mu\lambda_2}{\alpha\lambda_1^2}(e^{\frac{\alpha}{2}h} - 1)\|PB_1\|\|B_2\|\right)V.$$

From the inequality

$$\alpha = \frac{2}{\lambda_2} - \frac{4\mu\lambda_2}{\alpha\lambda_1^2}\left(e^{\frac{\alpha}{2}h} - 1\right)\|PB_1\|\|B_2\|$$

we can conclude that for any preselected $0 < \alpha < \frac{2}{\lambda_2}$ we can calculate corresponding $\|B_2\| > 0$:

$$\|B_2(\alpha)\| = \frac{\alpha\lambda_1^2}{4\mu\lambda_2}\left(e^{\frac{\alpha}{2}h} - 1\right)^{-1}\|PB_1\|^{-1}\left(\frac{2}{\lambda_2} - \alpha\right).$$

Moreover, if $\alpha \to \frac{2}{\lambda_2}$, then $\|B_2(\alpha)\| \to 0$, and if $\alpha \to 0$, then $\|B_2(\alpha)\| \to \frac{\lambda_1^2}{\mu\lambda_2^2 h}\|PB_1\|^{-1}$. Therefore, for all $\|B_2\| < \frac{\lambda_1^2}{\mu\lambda_2^2 h}\|PB_1\|^{-1}$ we have $\alpha > 0$, and the system (B.2) is exponentially stable.

APPENDIX C

Proof of Lemma 2. Let us consider a linear time-invariant system

$$\dot{\chi}(t) = A\chi(t) + b\nu(m), \tag{C.1}$$

where $\chi \in \mathbb{R}^n$, $\nu(m)$ is the discrete input , δ is the sampling period and $m = 1, 2, 3, \ldots$. We prove the lemma considering the particular case of the Euler numerical solver of the equation (C.1):

$$\bar{\chi}(m+1) = (I + \delta A)\bar{\chi}(m) + \delta b\nu(m). \tag{C.2}$$

The continuous majorant model is given by

$$\dot{\bar{\chi}}^*(t) = \bar{A}\bar{\chi}^*(t) + \bar{b}\nu(m). \tag{C.3}$$

To define the matrix $\bar{A}$, we equate the zero-input responses of the models (C.2) and (C.3) at the time instants $t = m\delta$:

$$e^{\bar{A}m\delta}x_0 = (I + \delta A)^m x_0.$$

From the latter equality we immediately obtain that

$$\bar{A}(\delta) = \frac{1}{\delta}\ln(I + \delta A),$$

where $\ln(\cdot)$ is the matrix natural logarithm. Then, with the use of the L'Hopital's rule, we can show that

$$\lim_{\delta\to 0}\bar{A}(\delta) = \lim_{\delta\to 0}\frac{1}{\delta}\log(I + \delta A) = A.$$

For invertible matrix A, we can show that $\bar{b}(\delta) = \bar{A}(\delta)A^{-1}b$, and, therefore, $\lim_{\delta\to 0}\bar{b}(\delta) = b$.

REFERENCES

1. Artstein, Z., Linear systems with delayed control: a reduction, *IEEE Trans. Automatic Control*, 1982, vol. 27, no. 8, pp. 869–879.
2. Krstić, M., Delay Compensation for Nonlinear, Adaptive and PDE Systems, Birkhauser, Boston, 2009.
3. Manitius, A. and Olbrot, A., Finite spectrum assignment problem for systems with delay, *IEEE Trans. Automatic Control*, 1979, vol. 24, no. 4, pp. 541–553.
4. Wang, Q.-G., Lee, T.H., and Tan, K.K., Finite Spectrum Assignment for Time-Delay Systems, London: Springer-Verlag, 1998.
5. Nikiforov, V.O. and Gerasimov, D.N., Adaptive Regulation: Reference Tracking and Disturbance Rejection, Springer Nature, 2022.
6. Pyrkin, A.A. and Bobtsov, A.A., Adaptive controller for linear system with input delay and output disturbance, *IEEE Trans. Automatic Control*, 2016, vol. 61, pp. 4229–4234.
7. Bui, V.H. and Margun, A.A., Compensation of output external disturbances for a class of linear systems with control delay, *Scientific and Technical J. of Information Technologies, Mechanics and Optics*, 2022, vol. 22, no. 6, pp. 1072–1077 (in Russian).
8. Nikiforov, V.O., Paramonov, A.V., and Gerasimov, D.N., Adaptive control algorithms in MIMO linear systems with control delay, *Automation and Remote Control*, 2020, vol. 81, no. 6, pp. 1091–1106.
9. Pyrkin, A.A., Bobtsov, A.A., Nikiforov, V.O., Kolyubin, S.A., Vedyakov, A.A., Borisov, O.I., and Gromov, V.S., Compensation of polyharmonic disturbance of state and output of a linear plant with delay in the control channel, *Automation and Remote Control*, 2015, vol. 76, no. 12, pp. 2124–2142.
10. Tsubakino, D., Krstić, M., and Olivera, T.R., Exact predictor feedbacks for multi-input LTI systems with distinct input delays, *Automatica*, 2016, vol. 71, pp. 143–150.
11. Fridman, E., Introduction to Time Delay Systems: Analysis and Control, Birkhauser: Boston–New York–Dordrecht–London, 2010.
12. Niculescu, S-I. and Gu, K., (Eds.) Advances in Time-Delay Systems, Springer Nature, 2004.
13. Engelborghs, K., Dambrine, M., and Roose, D., Limitations of a class of stabilization methods for delay systems, *IEEE Trans. Automatic Control*, 2001, vol. 46, no. 2, pp. 336–339.
14. Tsykunov, A.M., Adaptive and Robust Output-Feedback Control of Dynamic Systems, Moscow: Fizmatlit, 2009 (in Russian).
15. Smith, J., Closer control of loops with delay time, *Chemical Engineering Progress*, 1957, vol. 53, no. 5, pp. 217–219.
16. Furtat, I.B., Adaptive predictor-free control of a plant with delayed input signal, *Automation and Remote Control*, 2014, vol. 75, no. 1, pp. 139–151.
17. Nikiforova, L.V. and Shelenok, E.A., A system with adaptive filters for dynamic plant with state and input time delay, *Bulletin of PNU*, 2025, vol. 77, pp. 77–86 (in Russian).
18. Mondie, S. and Michiels, W., Finite spectrum assignment of unstable time-delay systems with a safe implementation, *IEEE Trans. Automatic Control*, 2003, vol. 48, no. 12, pp. 2207–2212.
19. Kharitonov, V.L., Predictor-based controls: the implementation problem, *Differential Equations*, vol. 51, no. 13, pp. 1675–1682.
20. Furtat, I., Fridman, E., and Fradkov, A., Disturbance compensation with finite spectrum assignment for plants with input delay, *IEEE Trans. Automatic Control*, 2018, vol. 63, no. 1, pp. 298–305.
21. Juarez, L., Mondie, S., and Kharitonov, V.L., Dynamic predictor for systems with state and input delay: a time-domain robust stability analysis, *Int. J. on Robust and Nonlinear Control*, 2020, vol. 30, no. 6, pp. 2204–2218.

22. Rodriguez-Guerrero, L., Kharitonov, V.L., and Mondiea, S., Robust stability of dynamic predictor based control laws for input and state delay systems, *Systems and Control Letters*, 2016, vol. 96, pp. 95–102.

23. Yoon, S.Y. and Lin, Z.I., Robust output regulation of linear time-delay systems: A state predictor approach, *Int. J. on Robust and Nonlinear Control*, 2016, vol. 26, no. 8, pp. 1686–1704.

24. Lin, Z. and Fang, H., On asymptotic stabilizability of linear systems with delayed input, *IEEE Trans. Automatic Control*, 2007, vol. 52, no. 6, pp. 998–1013.

25. Zhou, B., Lin, Z.I., and Duan, G., Stabilization of linear systems with input delay and saturation—A parametric Lyapunov equation approach, *Int. J. on Robust and Nonlinear Control*, 2010, vol. 20, pp. 1502–1519.

26. Zhou, B., Lin, Z., and Duan, G., Truncated predictor feedback for linear systems with long time-varying input delays, *Automatica*, 2012, vol. 48, pp. 2387–2399.

27. Zhou, B., Truncated Predictor Feedback for Time-Delay Systems, Berlin, Heidelberg: Springer-Verlag, 2014.

28. Zhou, B., Pseudo-predictor feedback stabilization of linear systems with time-varying input delays, *Automatica*, 2014, vol. 50, no. 11, pp. 2861–2871.

29. Najafi, M., Hosseinnia, S., Sheikholeslam, F., and Karimadini, M., Closed-loop control of dead time systems via sequential sub-predictors, *Int. J. of Control*, 2013, vol. 86, no. 4, pp. 599–609.

30. Dugard, L. and Verriet, E., Stability and Control of Time-delay Systems, London: Springer, 1997.

31. Cacace, F., Conte, F., Germani, A., and Pepe, P., Stabilization of strict-feedback nonlinear systems with input delay using closed-loop predictors, *Int. J. on Robust and Nonlinear Control*, 2016, vol. 26, pp. 3524–3540.

32. Fradkov, A.L., Miroshnik, I.V., and Nikiforov, V.O., Nonlinear and Adaptive Control of Complex Systems. Ser. Mathematics and Its Applications, vol. 491, Dordrecht: Kluwer Academic Publishers, 1999.

33. Furtat, I.B. and Gushchin, P.A., A control algorithm for an object with delayed input signal based on subpredictors of the controlled variable and disturbance, *Automation and Remote Control*, 2019, vol. 80, no. 2, pp. 201–216.

34. Cacace, F. and Germani, A., Output feedback control of linear systems with input, state and output delays by chains of predictors, *Automatica*, 2017, vol. 85, pp. 455–461.

35. Furtat, I. and Gushchin, P., Tracking control algorithms for plants with input time-delays based on state and disturbance predictors and sub-predictors, *J. of The Franklin Institute*, 2019, vol. 356, no. 8, pp. 4496–4512.

36. Liu, Q. and Zhou, B., Extended observer based feedback control of linear systems with both state and input delays, *J. of The Franklin Institute*, 2017, vol. 354, no. 18, pp. 8232–8255.

37. Zhou, B., Liu, Q., and Mazenc, F., Stabilization of linear systems with both input and state delays by observer-predictors, *Automatica*, 2017, vol. 83, pp. 368–377.

38. Nikiforov, V.O. and Gerasimov, D.N., Robust closed-loop state predictor for unstable systems with input delay, *62nd IEEE Conf. On Decision And Control, Singapore, Dec. 13–15*, 2023, pp. 5708–5713.

39. Léchappé, V., Moulay, E., Plestan, F., Glumineau, A., and Chriette, A., New predictive scheme for the control of LTI systems with input delay and unknown disturbances, *Automatica*, 2015, vol. 52, pp. 179–184.

40. Tam, P.K.S. and Moore, J.B., Stable realization of fixed-lag smoothing equations for continuous-time signals, *IEEE Trans. Automatic Control*, 1974, vol. 19, no. 1, pp. 84–87.

41. Korn, G. and Korn, T., Mathematical Handbook for Scientists and Engineers, Moscow: Nauka, 1970 (in Russian).

This paper was recommended for publication by M.V. Khlebnikov, a member of the Editorial Board

ISSN 0005-1179 (print), ISSN 1608-3032 (online), Automation and Remote Control, 2026, Vol. 87, No. 9, pp. 831–845.
© The Author(s), 2026 published by Trapeznikov Institute of Control Sciences, Russian Academy of Sciences, 2026.
Russian Text © The Author(s), 2026, published in Avtomatika i Telemekhanika, 2026, No. 9, pp. 91–109.

LINEAR SYSTEMS

Block Coordinate Descent Method Applied to Stabilization and Time-Optimization Problems for Linear Discrete Systems

K. A. Tsarkov

Trapeznikov Institute of Control Sciences, Russian Academy of Sciences, Moscow, Russia
e-mail: k6472@mail.ru

Received January 12, 2026
Revised March 13, 2026
Accepted March 23, 2026

Abstract—In the paper, we propose algorithms for solving terminal stabilization and time-optimization problems for discrete-time linear dynamical systems with convex sets of geometric constraints on the control. The algorithms are based on the well-known block coordinate descent method. We present some corollaries to classical results and give new assertions on the convergence of the constructed algorithms to a solution in the problems.

Keywords: terminal stabilization, optimal time, discrete time, linear systems, sequentional global improvement

DOI: 10.7868/S1608303226090039

1. INTRODUCTION

The present study is motivated by the results obtained in [1, 2] on applications of V.F. Krotov's method of sequentional global improvements to problems of terminal stabilization and time-optimization for discrete-time linear systems. An analysis of these results shows that one of the possible implementations of Krotov method in the case under consideration leads to an algorithm widely known as block coordinate descent. With that, a natural desire arises to present an approach to solving the mentioned problems, based on the use of results on the convergence of block coordinate descent. In this regard, we intend to formulate the most significant known results and derive some new meaningful consequences from them. Ultimately, we plan to obtain a transparent justification for the convergence of the algorithms from [1, 2] and propose potentially more efficient numerical methods for solving these problems.

It should be noted that Krotov method, in general, differs significantly from the block coordinate descent method in its design, applicability, and results, although they share many similarities. For example, the formulation of Krotov method involves an unspecified function (the Krotov function), which can be selected from a fairly wide class of admissible functions. Obtaining identical algorithmic implementations for these methods is largely due to the fact that Krotov method is used here with a very specific choice of the Krotov function. The stages of constructing such a function are described in detail in [1, 2].

One of the advantages of the block coordinate descent method is its widespread popularity. There is an extensive list of works devoted to various theoretical and computational aspects of its application to various problems (see, for example, a list of more than two dozen relevant applications of the method in [5]). It is also known that the block coordinate descent can be a very effective computational tool for the problems with explicit block-composition structure. At the same time, it is practically never used to solve terminal stabilization and time-optimization problems for discrete-time linear systems, although it is precisely in the theory of optimal control of discrete systems

that the domain of admissible values of variables is decomposed into a product of a finite number of sets of geometric constraints (blocks) by the very problem statement. In fact, specialized methods are rarely used in studying these problems at all.

Among recent works related to the study of original methods for solving time-optimization problem for a linear discrete system, we should specially mention [6]. In [6], a numerical approach to time-optimization problem is developed, based on support functions of sets of geometric constraints on the control. A distinctive feature is that it presupposes the solution of a series of non-convex and non-smooth optimization problems, while the original problem, generally speaking, decomposes into a sequence of convex problems with a smooth objective function. Nevertheless, almost all of these problems can be replaced by equivalent convex problems. However, a difficulty remains associated with the need to use support functions of sets of geometric constraints in the structure of the objective function to be optimized. Moreover, after determining the optimal time, the problem of constructing an optimal process remains. In [6], it is shown that an optimal process can be constructed using the solution of the last of the mentioned optimization problems. At the same time, such a construction is not always trivial and, in the non-strictly convex case, requires the use of certain additional constructions and algorithms. Note that some of the ideas behind this method for solving time-optimization problem appeared earlier in [7].

The methods in [1, 2], also discussed here, propose solving only low-dimensional convex problems in which the objective is quite simple. Moreover, the construction of an optimal process is carried out simultaneously with the calculation of the optimal time.

2. STATEMENT OF THE PROBLEMS

Consider a linear nonstationary discrete-time system

$$x(k+1) = A(k)x(k) + B(k)u(k), \quad k = 0, 1, \dots, \tag{1}$$

where $x(k) \in \mathbb{R}^n$ is the state of the system, $u(k) \in U(k)$ is the control, $U(k)$ are convex compact sets in $\mathbb{R}^m$ containing the origin, $A(k) \in \mathbb{R}^{n\times n}$ and $B(k) \in \mathbb{R}^{n\times m}$ are arbitrary given matrices. The initial condition for the system is fixed:

$$x(0) = x_0 \in \mathbb{R}^n. \tag{2}$$

First, let us discuss the formulation of terminal stabilization problem for system (1) with initial condition (2). Let us fix some value $N > 0$ of the discrete time at which the controllable process terminates. Consider the problem

$$\|x(N)\| \to \min_{u\in\mathcal{U}_N}, \tag{3}$$

where $\mathcal{U}_N := \{k \mapsto u(k) : \{0, \dots, N-1\} \to \mathbb{R}^m \mid u(k) \in U(k)\}$, and $x(N) \in \mathbb{R}^n$ is determined by the system (1)–(2) for each $u \in \mathcal{U}_N$. Here and below, $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^n$.

In explicit form, the resulting problem (1)–(3) can be rewritten as follows:

$$J_N(u) := \left\| \mathcal{A}_N(0)x_0 + \sum_{k=0}^{N-1} \mathcal{A}_N(k+1)B(k)u(k) \right\| \to \min_{u\in\mathcal{U}_N}, \tag{4}$$

where $\mathcal{A}_N(k) := A(N-1)\dots A(k)$ for $0 \leqslant k \leqslant N-1$, $\mathcal{A}_N(N) := I$, I is the identity matrix of dimensions $n \times n$. Since the elements of the set $\mathcal{U}_N$ are naturally identified with the corresponding vectors in the Euclidean space $\mathbb{R}^{mN}$, $\mathcal{U}_N$ itself can be represented as the Cartesian product $U(0)\dots\times U(N-1)$ of the initially given convex and compact sets $U(k) \subset \mathbb{R}^m$.

Problem (4) has a solution, since for any $N > 0$ the function J_N is continuous on the compact set $\mathcal{U}_N$ in $\mathbb{R}^{mN}$.

Now let us consider time-optimization problem. In this problem, we need to calculate the minimum number of steps $N_{\min}$ required to transfer the system (1) from the given initial state x_0 to the origin, and construct an optimal process $\{\overline{x}(k), \overline{u}(k-1)\}_{k=1}^{N_{\min}}$ satisfying the condition $\overline{x}(N_{\min}) = 0$. The number $N_{\min}$ is usually called the optimal time of system (1) with initial condition (2).

In general, time-optimization problem may have no solution. For example, let $n = m = 1$ and the system (1)–(2) have the form

$$x(k+1) = x(k) + u(k), \quad x(0) = x_0 \in \mathbb{R}, \quad u(k) \in U(k) = [-1/2^k; 1/2^k].$$

For this system, time-optimization problem is solvable precisely when $|x_0| < 2$.

In the paper, we will assume that time-optimization problem for the system (1)–(2) is solvable and that the corresponding finite value $N_{\min}$ exists. Excluding the trivial case $x_0 = 0$, we will also assume that $N_{\min} > 0$.

Since, by hypothesis, we have $0 \in U(k)$ for all $k = 0, 1, \ldots$, the optimal value $\overline{J_N}$ in problem (4) is zero if and only if $N \geqslant N_{\min}$. Accordingly, under the assumption $0 < N_{\min} < \infty$, time-optimization problem can be formulated as follows:

$$N \to \min, \quad N \in \left\{ M > 0 \;\middle|\; \overline{J_M} := \min_{u \in \mathcal{U}_M} J_M(u) = 0 \right\}. \tag{5}$$

3. BLOCK COORDINATE DESCENT

The problems (4) and (5) are related to the minimization of the function J_N on the set $\mathcal{U}_N$ in $\mathbb{R}^{mN}$. The function $J_N : \mathbb{R}^{mN} \to \mathbb{R}_+$ and the set $\mathcal{U}_N \subset \mathbb{R}^{mN}$ have a number of useful properties. In particular, $u \mapsto J_N(u)^2$ is a convex and smooth function on $\mathbb{R}^{mN}$, and the set $\mathcal{U}_N$ is compact and convex. Moreover, $\mathcal{U}_N$ can be decomposed into a Cartesian product of sets $U(0) \times \ldots \times U(N-1)$, where all $U(k)$ are also convex and compact.

Note that the dimension of the set $\mathcal{U}_N$ can be significantly higher than the dimension of each of the sets $U(k)$. Moreover, it is the sets $U(k)$ that are part of the original problem statement, while $\mathcal{U}_N$ is an artificial construction. For these reasons, minimization on sets $U(k)$ can be significantly simpler rather than minimization on the entire set $\mathcal{U}_N$. A suitable minimization algorithm that takes these features into account is the block coordinate descent method, for which the theoretical foundations of convergence were laid in [3]. We present its general formulation below.

3.1. Formulation of the Method

Let $K_1, \ldots, K_\nu$ be nonempty compact sets in finite-dimensional Euclidean spaces $V_1, \ldots, V_\nu$ respectively. Let a continuous function $f : V \to \mathbb{R}$ be defined on the Cartesian product $V := V_1 \times \ldots \times V_\nu$. Consider the problem

$$f(v) = f(v_1, \ldots, v_\nu) \to \min_{v \in K}, \quad K := K_1 \times \ldots \times K_\nu. \tag{6}$$

Fix an arbitrary point $v^{(0)} = (v_1^{(0)}, \ldots, v_\nu^{(0)}) \in K$ and write the following iterative procedure. Suppose that for some $l \geqslant 0$ the point $v^{(l)} \in K$ is given. Then we define a point $v^{(l+1)} \in K$ by the rule

$$v_i^{(l+1)} \in \operatorname{Arg}\min_{\mu \in K_i} f(v_1^{(l+1)}, \ldots, v_{i-1}^{(l+1)}, \mu, v_{i+1}^{(l)}, \ldots, v_\nu^{(l)}), \quad i = 1, \ldots, \nu. \tag{7}$$

In the case $V_1 = \ldots = V_\nu = \mathbb{R}$, rule (7) defines a step of the classical numerical Gauss–Seidel method (or coordinate descent method) for solving the problem of conditional finite-dimensional optimization of a function f depending on ν scalar variables. In the general case, the resulting

algorithm is usually called the block coordinate descent method or the generalized (nonlinear) Gauss-Seidel method [4, 5].

By construction, it is clear that for any choice of $v^{(0)} \in K$,

$$f(v^{(l+1)}) \leqslant f(v^{(l)}), \quad l = 0, 1, 2, \ldots.$$

However, under the assumptions made, there can be no convergence of the sequence of numbers $f(v^{(l)})$ to the minimum value in the problem (6). Moreover, as was shown in [3], even the convexity conditions for all sets K_i and the function f itself are insufficient. Especially, it is impossible to speak of any reasonable convergence of the sequence of elements $v^{(l)}$ to a solution in problem (6). At the same time, a number of known results guarantee this convergence in one form or another under additional assumptions.

3.2. Convergence

The main known result on the convergence of the method (7) is the following.

Theorem 1. *Let all compact sets K_i in problem* (6) *be convex, and let the continuous function f be convex on the set $K = K_1 \times \ldots \times K_\nu$. Assume, moreover, that the following conditions hold:*

(i) *function f is continuously differentiable on the set K;*

(ii) *function $\mu \mapsto f(v_1, \ldots, v_{i-1}, \mu, v_{i+1}, \ldots, v_\nu)$ has a unique minimum on the set K_i for each i and arbitrary values $v_j \in K_j$, $j \neq i$.*

Then the following is true. If the point $\overline{v}$ is a limit point for the sequence $\{v^{(l)}\}$ constructed according to the rule (7) *from an arbitrary initial point $v^{(0)} \in K$, then*

$$f(\overline{v}) = \min_{v \in K} f(v).$$

Remark 1. In the statement of Theorem 1 and throughout the text, a limit point for a sequence $\{v^{(l)}\}$ with elements in a finite-dimensional Euclidean space V is understood to be a limit point of the set $\{v^{(l)}\} \subset V$, i.e., any point $\overline{v} \in V$ for which there exists a subsequence of $\{v^{(l)}\}$ converging to it.

The validity of Theorem 1 follows from [4, Proposition 2.7.1] and [4, Proposition 2.1.2]. Note that in [3], the assertion of Theorem 1 is obtained under somewhat different assumptions. Specifically, condition (i) of Theorem 1 is weakened in [3] to local continuous differentiability of f in a neighborhood of every fixed point of the method (7), and condition (ii) is strengthened to the strict convexity of all functions $\mu \mapsto f(v_1, \ldots, v_{i-1}, \mu, v_{i+1}, \ldots, v_\nu)$.

As will be shown below, condition (ii) of Theorem 1 can be weakened, but apparently cannot be abandoned completely. The importance of condition (i) in its local version is demonstrated by the following simple example from [3].

Example 1. On the set $K = [0; 1]^2 \subset \mathbb{R}^2$, consider the function

$$f(v_1, v_2) = \begin{cases} v_1 - 2v_2, & v_2 \leqslant v_1, \\ v_2 - 2v_1, & v_1 < v_2. \end{cases}$$

In this example, the sets $K_i = [0; 1] \subset \mathbb{R}$ are convex and the function $f(v_1, v_2) = |v_1 - v_2| - \min\{v_1, v_2\}$ is convex on K, with $\operatorname{Arg\,min}_{0 \leqslant \mu \leqslant 1} f(v_1, \mu) = \{v_1\}$ and $\operatorname{Arg\,min}_{0 \leqslant \mu \leqslant 1} f(\mu, v_2) = \{v_2\}$, however, condition (i) of Theorem 1 is not satisfied in any neighborhood of any of the fixed points of the method (7) defined by the equality $v_1 = v_2$, and for any initial point $v^{(0)} = (v_1^{(0)}, v_2^{(0)})$, we have $v^{(l)} = (v_2^{(0)}, v_2^{(0)})$ and $f(v^{(l)}) = -v_2^{(0)}$ for $l \geqslant 1$. Consequently, the assertion of Theorem 1 also fails for $v_2^{(0)} \neq 1 = -\min_{v \in K} f(v)$. The example is complete.

In the context of applications of the above results to solving problems in the previous section, we will need the following minor generalization of Theorem 1, which weakens condition (ii).

Corollary 1. *Let all compact sets K_i in problem* (6) *be convex, and let the continuous function f have the form*

$$f(v) = g(\Lambda_1 v_1 + \lambda_1, \ldots, \Lambda_\nu v_\nu + \lambda_\nu),$$

where Λ_i is a linear operator from the finite-dimensional Euclidean space V_i to a finite-dimensional Euclidean space V_i', $\lambda_i \in V_i'$, $i = 1, \ldots, \nu$, $g : V_1' \times \ldots \times V_\nu' \to \mathbb{R}$. Let function g satisfy the conditions of Theorem 1 on the set

$$\Lambda K + \lambda := (\Lambda_1 K_1 + \lambda_1) \times \ldots \times (\Lambda_\nu K_\nu + \lambda_\nu).$$

Then the following is true. If the point $\overline{v}$ is a limit point for the sequence $\{v^{(l)}\}$ constructed according to the rule (7) *from an arbitrary initial point $v^{(0)} \in K$, then*

$$f(\overline{v}) = \min_{v \in K} f(v).$$

Proof. Consider the point $z^{(0)} = \Lambda v^{(0)} + \lambda$, where $\Lambda v + \lambda := (\Lambda_1 v_1 + \lambda_1, \ldots, \Lambda_\nu v_\nu + \lambda_\nu) \in \Lambda K + \lambda$ for $v \in K$. Every sequence $\{v^{(l)}\} \subset K$ constructed according to the rule (7) from the point $v^{(0)} \in K$ generates a sequence $\{z^{(l)}\} = \{\Lambda v^{(l)} + \lambda\} \subset \Lambda K + \lambda$ satisfying the rule

$$z_i^{(l+1)} = \arg \min_{\mu \in \Lambda K_i + \lambda_i} g(z_1^{(l+1)}, \ldots, z_{i-1}^{(l+1)}, \mu, z_{i+1}^{(l)}, \ldots, z_\nu^{(l)}), \quad i = 1, \ldots, \nu, \quad l \geqslant 0.$$

Since the sets $\Lambda_i K_i + \lambda_i$ are nonempty, compact and convex, and the function g on $\Lambda K + \lambda$ satisfies the conditions of Theorem 1, we have

$$g(\overline{z}) = \min_{z \in \Lambda K + \lambda} g(z)$$

for every limit point $\overline{z}$ of the sequence $\{z^{(l)}\}$. Now setting $\overline{z} = \Lambda \overline{v} + \lambda$ and using the continuity of finite-dimensional linear operators Λ_i, we obtain the desired result. Corollary 1 is proved.

We demonstrate the result of Corollary 1 with the following important example.

Example 2. Consider the square of the values of J_N from the problem (4). Let $\nu = N$, $v_i = u(i-1)$, $K_i = U(i-1)$. We obtain the function

$$f(v) = \left\| \mathcal{A}_\nu(0) x_0 + \sum_{i=1}^{\nu} \mathcal{A}_\nu(i) B(i-1) v_i \right\|^2.$$

Here $\mathcal{A}_\nu(i)$ and $B(i)$ are known matrices, x_0 is a known vector. Let $\Lambda_i = \mathcal{A}_\nu(i) B(i-1)$, $\lambda_1 = \mathcal{A}_\nu(0) x_0$, $\lambda_2 = \ldots = \lambda_\nu = 0$. Then the function f and the sets K_i satisfy the conditions of Corollary 1. However, they may not satisfy the conditions of Theorem 1. Indeed, if any of the matrices $\mathcal{A}_\nu(i)$ is singular, then the minimum in the problem $f(v_1, \ldots, v_{i-1}, \mu, v_{i+1}, \ldots, v_\nu) \to \min_{\mu \in K_i}$ may not be unique. At the same time, a function g of the form

$$g(z) = \left\| \sum_{i=1}^{\nu} z_i \right\|^2$$

is continuously differentiable on $\mathbb{R}^{n\nu}$, and each of the functions

$$\mu \mapsto g(z_1, \ldots, z_{i-1}, \mu, z_{i+1}, \ldots, z_\nu) = \left\| \mu + \sum_{j \neq i} z_j \right\|^2$$

is strictly convex on $\Lambda_i K_i + \lambda_i$ for any fixed values $z_j \in \Lambda_j K_j + \lambda_j$, $j \neq i$, where $i = 1, \ldots, \nu$. Therefore, g satisfies the conditions of Theorem 1. Consequently, for $f(v) = g(\Lambda v + \lambda)$, the assertion of Corollary 1 holds. The example is complete.

Note that the smoothness of the function $g(z)$ from Example 2 is not so important. In particular, it can be verified directly that for the function

$$f(v) = \left\| \sum_{i=1}^{\nu} v_i \right\|,$$

where the Euclidean norm can be replaced by any norm in the space $\mathbb{R}^n$, and an arbitrary set of nonempty convex compact sets $K_i \subset \mathbb{R}^n$, the block coordinate descent method for the problem $f(v) \to \min_{v \in K}$, $K = K_1 \times \ldots \times K_\nu$, converges in the same sense as specified in Theorem 1 and Corollary 1, despite the fact that condition (i) of Theorem 1 may not be satisfied even locally in a neighborhood of the solution. We will not dwell on this in detail here.

For convenience, introducing the notation

$$\overline{f} := \min_{v \in K} f(v), \quad \overline{K} := \operatorname{Arg}\min_{v \in K} f(v), \quad \operatorname{dist}(\mu, \overline{K}) := \min_{v \in \overline{K}} \|\mu - v\|,$$

we obtain the following results.

Corollary 2. *Under the conditions of Corollary 1, for any sequence $\{v^{(l)}\}$ constructed according to the rule* (7) *from an arbitrary initial point $v^{(0)} \in K$, the following convergence holds:*

$$f(v^{(l)}) \to \overline{f}, \quad \operatorname{dist}(v^{(l)}, \overline{K}) \to 0, \quad l \to \infty.$$

Proof. Suppose that $f(v^{(l)}) \not\to \overline{f}$. Then there exists a subsequence $\{v^{(l_m)}\}$ with the property $|f(v^{(l_m)}) - \overline{f}| > \delta > 0$ for some $\delta > 0$. Passing to a subsequence again if needed, we have $\{v^{(l_m)}\}$ converges to some $\overline{v} \in K$. Then, by Corollary 1, we obtain $f(\overline{v}) = \overline{f}$. But by the continuity of f on K and the assumption made, we have $|f(\overline{v}) - \overline{f}| \geqslant \delta > 0$ — a contradiction. Similar arguments yield the convergence of $\operatorname{dist}(v^{(l)}, \overline{K}) \to 0$. Corollary 2 is proved.

Corollary 3. *Let the assumptions of Corollary 1 be satisfied. Then a necessary and sufficient condition for the optimality of a point $\overline{v}$ in problem* (6) *is the relation*

$$\overline{v}_i \in \operatorname{Arg}\min_{\mu \in K_i} f(\overline{v}_1, \ldots, \overline{v}_{i-1}, \mu, \overline{v}_{i+1}, \ldots, \overline{v}_\nu), \quad i = 1, \ldots, \nu.$$

Proof. The necessity of the condition is obvious. Assume that it holds. Then, for $v^{(0)} = \overline{v}$, the sequence of points $v^{(l)} \equiv \overline{v}$ satisfies the rule (7). By Corollary 2, this means that $f(\overline{v}) = \overline{f}$, as required. Corollary 3 is proved.

To conclude this subsection, we briefly discuss the convergence of the sequence $\{v^{(l)}\}$, constructed according to the rule (7), to a single point $\overline{v} \in \overline{K}$. The classical result on the presence of such convergence under the conditions of Theorem 1 was obtained in [8] for the case $V_1 = \ldots = V_\nu = \mathbb{R}$ and $K_i = [a_i; b_i]$. It is also noted there that it naturally generalizes to the case where the sets K_i are closed parallelepipeds in $\mathbb{R}^{n_i}$. However, in the general situation, verifying this kind of convergence remains an open question.

The following example shows that in a one-dimensional problem of the form (4), the method (7) converges to a single solution in one iteration.

Example 3. Consider the problem

$$|\lambda_1 v_1 + \ldots + \lambda_\nu v_\nu + \gamma| \to \min, \quad v_i \in [a_i; b_i].$$

Let the numbers $v_i^{(0)} \in [a_i; b_i]$ be given. Then for all i such that $\lambda_i \neq 0$, the numbers $v_i^{(1)}$ are determined according to (7) from the conditions

$$v_i^{(1)} = \arg\min_{\mu \in [a_i; b_i]} |\lambda_i \mu + \sigma_i|, \quad \sigma_i = \lambda_1 v_1^{(1)} + \ldots + \lambda_{i-1} v_{i-1}^{(1)} + \lambda_{i+1} v_{i+1}^{(0)} + \ldots + \lambda_\nu v_\nu^{(0)} + \gamma.$$

For $i = 1, \dots, \nu$ in the case $\lambda_i \neq 0$ we put

$$v_i^{(1)} = \begin{cases} \max\{\lambda_i a_i, \lambda_i b_i\}/\lambda_i, & \sigma_i < -\max\{\lambda_i a_i, \lambda_i b_i\}, \\ \min\{\lambda_i a_i, \lambda_i b_i\}/\lambda_i, & \sigma_i > -\min\{\lambda_i a_i, \lambda_i b_i\}, \\ -\sigma_i/\lambda_i & \text{otherwise.} \end{cases}$$

For the remaining i, we choose $v_i^{(1)} \in [a_i; b_i]$ arbitrarily. It is directly verified that the points $v_i^{(1)}$ constructed step by step in this way satisfy the rule (7) and provide a minimum in the problem. The example is complete.

3.3. Stopping Rules

The usual stopping rule for the classical coordinate descent method is the condition

$$f(v^{(l)}) - f(v^{(l+1)}) < \varepsilon \tag{8}$$

for a pre-selected small value $\varepsilon > 0$. If the convergence of the sequence $\{v^{(l)}\}$ is proven, one can also apply the alternative condition $\|v^{(l+1)} - v^{(l)}\| < \varepsilon$. However, in the general case, such convergence does not hold for the procedure (7), so only condition (8) remains. An undoubted advantage of the condition (8) is the simplicity of its verification. At the same time, inequality (8) in the absence of additional assumptions cannot guarantee that the value $f(v^{(l+1)})$ is close to the unknown optimal value $\overline{f}$.

A more reliable stopping rule can be obtained by exploiting the following property of the class of problems under consideration. Suppose that in problem (6) the sets K_i are convex, and the function f is continuously differentiable and convex on K (i.e., for example, the convergence conditions from Theorem 1 or Corollary 1 are satisfied). Then for any $w \in K$ we have the two-sided estimate

$$f(w) \geqslant \overline{f} \geqslant f(w) + \min_{w+h \in K} \langle f'(w), h \rangle = f(w) - \sum_{i=1}^{\nu} \max_{\mu \in K_i} \left\langle -\frac{\partial f(w)}{\partial v_i}, \mu - w_i \right\rangle.$$

Indeed, the first inequality follows from the definition of the minimum value $\overline{f}$, the second inequality follows from the fact that the epigraph of a proper convex function f is located above any of its supporting and, in particular, tangent hyperplanes, and the last equality follows from the decomposition

$$\langle f'(w), h \rangle = \sum_{i=1}^{\nu} \left\langle \frac{\partial f(w)}{\partial v_i}, h_i \right\rangle.$$

Now we can take as a stopping rule the condition

$$\sum_{i=1}^{\nu} \max_{\mu \in K_i} \left\langle \frac{\partial f(v^{(l)})}{\partial v_i}, v_i^{(l)} - \mu \right\rangle < \varepsilon. \tag{9}$$

The fulfillment of this condition (9) guarantees the validity of the inequality $f(v^{(l)}) < \overline{f} + \varepsilon$ for a predetermined error value $\varepsilon > 0$. Moreover, maximization in (9) is performed over the same sets K_i that appear in (7). Therefore, the stopping rule (9) is consistent with the method (7).

Note also that the equality

$$\max_{v \in K} \langle f'(\overline{v}), \overline{v} - v \rangle = \sum_{i=1}^{\nu} \max_{\mu \in K_i} \left\langle \frac{\partial f(\overline{v})}{\partial v_i}, \overline{v}_i - \mu \right\rangle = 0$$

coincides with the classical necessary and sufficient condition for the optimality of a point $\overline{v} \in K$ in a smooth and convex problem (6) [10, Theorem 7.1.3].

The stopping rules (8) and (9) can be used together. The computationally simpler condition (8) can be checked first, and then, if it holds, we check the guaranteeing condition (9).

Finally, the two-sided estimate presented above allows us to refine the interval of values to which the number $\overline{f}$ belongs at each step when constructing the sequence $v^{(l)}$. Namely, for every $l \in \{0, 1, \dots\}$, holds the inclusion

$$\overline{f} \in \left[f(v^{(l)}) - \sum_{i=1}^{\nu} \max_{\mu \in K_i} \left\langle \frac{\partial f(v^{(l)})}{\partial v_i}, v_i^{(l)} - \mu \right\rangle ; f(v^{(l)}) \right]. \tag{10}$$

If the conditions of Theorem 1 or Corollary 1 are satisfied, the boundaries of the intervals specified here converge, as l increases, to the point $\overline{f}$. Furthermore, this interval estimate can provide an early answer to the question of what values the unknown $\overline{f}$ can and cannot take. The latter is relevant, for example, when considering the previously posed time-optimization problem (5), in which the value $\overline{f} = 0$ plays a key role.

We now turn directly to solving the problems (4) and (5).

4. PROBLEM SOLVING

4.1. Terminal Stabilization Problem

We apply the block coordinate descent method to solve problem (4). As noted in Example 2, problem (4) satisfies the conditions of Corollary 1, which, thanks to Corollary 2, guarantees the convergence of the objective values obtained by method (7) to minimum, as well as the convergence of the control sequence constructed by it to the set of solutions. As the stopping rules we can use (8) and (9). It remains to specify the formulas (7)–(9).

In problem (4) we have $\nu = N$, $K_i = U(i-1)$, $V_i = \mathbb{R}^m$, $f = J_N^2$.

Formula (7) takes the form

$$u^{(l+1)}(k) \in \operatorname{Arg}\min_{\mu \in U(k)} \Bigg\| \mathcal{A}_N(0)x_0 + \sum_{i=0}^{k-1} \mathcal{A}_N(i+1)B(i)u^{(l+1)}(i) + \mathcal{A}_N(k+1)B(k)\mu + \sum_{i=k+1}^{N-1} \mathcal{A}_N(i+1)B(i)u^{(l)}(i) \Bigg\|, \quad k = 0, \dots, N-1.$$

By the definition of the matrix $\mathcal{A}_N(k)$, we have

$$\mathcal{A}_N(0)x_0 + \sum_{i=0}^{k-1} \mathcal{A}_N(i+1)B(i)u^{(l+1)}(i) = \mathcal{A}_N(k) \left(\mathcal{A}_k(0)x_0 + \sum_{i=0}^{k-1} \mathcal{A}_k(i+1)B(i)u^{(l+1)}(i) \right).$$

Hence, using the values $x^{(l)}(k)$ and $\xi^{(l)}(k)$, calculated by the formulas

$$x^{(l)}(k+1) = A(k)x^{(l)}(k) + B(k)u^{(l)}(k), \quad x^{(l)}(0) = x_0,$$

$$\xi^{(l)}(k) = \xi^{(l)}(k+1) - \mathcal{A}_N(k+1)B(k)u^{(l)}(k), \quad \xi^{(l)}(N) = 0,$$

we find that method (7) in problem (4) is written as

$$u^{(l+1)}(k) \in \operatorname{Arg}\min_{\mu \in U(k)} \left\| \mathcal{A}_N(k+1)B(k)\mu + \mathcal{A}_N(k)x^{(l+1)}(k) - \xi^{(l)}(k+1) \right\| \tag{11}$$

for $k = 0, \dots, N-1$. Here $x^{(l+1)}(k)$ are determined by the values $u^{(l+1)}(i)$, $i < k$, and $\xi^{(l)}(k+1)$ are determined by the values $u^{(l)}(i)$, $i > k$, so the formula (11) correctly implements method (7).

Formula (8) takes the form

$$\|x^{(l)}(N)\|^2 - \|x^{(l+1)}(N)\|^2 < \varepsilon. \tag{12}$$

Finally, we write formula (9) using the classical dual variable $\psi(k)$ in problem (1)–(3). Let $\psi^{(l)}(k)$ satisfy the equalities

$$\psi^{(l)}(k) = A(k)^{\mathrm{T}}\psi^{(l)}(k+1), \quad \psi^{(l)}(N) = -2x^{(l)}(N).$$

Then for $\mu \in \mathbb{R}^m$ we have

$$\begin{aligned}\left\langle \frac{\partial f(v^{(l)})}{\partial v_{k+1}}, \mu \right\rangle &= \left\langle \frac{\partial J(u^{(l)})^2}{\partial u(k)}, \mu \right\rangle \\ &= 2\left\langle \mathcal{A}_N(0)x_0 + \sum_{i=0}^{N-1} \mathcal{A}_N(i+1)B(i)u^{(l)}(i), \mathcal{A}_N(k+1)B(k)\mu \right\rangle \\ &= \langle 2x^{(l)}(N), \mathcal{A}_N(k+1)B(k)\mu\rangle = -\langle \mathcal{A}_N(k+1)^{\mathrm{T}}\psi^{(l)}(N), B(k)\mu\rangle = -\langle \psi^{(l)}(k), B(k)\mu\rangle.\end{aligned}$$

Therefore, the condition (9) takes the form

$$\sum_{k=0}^{N-1} \max_{\mu\in U(k)} \left\langle \psi^{(l)}(k), B(k)(\mu - u^{(l)}(k)) \right\rangle < \varepsilon. \tag{13}$$

Let us consider two ways of constructing the algorithm. The first is to use method (7) with the stopping rules (12) and (13) without any additional transformations.

Algorithm 1.

0. Set the permissible computational error $\varepsilon > 0$.
1. Put $l = 0$, define the initial approximation by the equalities $u^{(0)}(k) = 0 \in \mathbb{R}^m$, $k = 0, \dots, N-1$, calculate the matrices $\mathcal{A}_N(k) = A(N-1)\dots A(k)$, $k = 0, \dots, N-1$, put the matrix $\mathcal{A}_N(N)$ to be the identity matrix of size $n \times n$.
2. Calculate the values $\xi^{(l)}(k)$, $k = 1, \dots, N$, by the formulas

$$\xi^{(l)}(k) = \xi^{(l)}(k+1) - \mathcal{A}_N(k+1)B(k)u^{(l)}(k), \quad \xi^{(l)}(N) = 0.$$

3. For each $k = 0, \dots, N-1$, sequentially find and fix some solution $u^{(l+1)}(k)$ to the extremum problem

$$\|\mathcal{A}_N(k+1)B(k)\mu + \mathcal{A}_N(k)x^{(l+1)}(k) - \xi^{(l)}(k+1)\| \to \min_{\mu\in U(k)},$$

where the values $x^{(l+1)}(k)$ are calculated step by step using the formulas

$$x^{(l+1)}(k+1) = A(k)x^{(l+1)}(k) + B(k)u^{(l+1)}(k), \quad x^{(l+1)}(0) = x_0.$$

4. Check the first stopping condition

$$\|x^{(l)}(N)\|^2 - \|x^{(l+1)}(N)\|^2 < \varepsilon,$$

if true, go to step 5; otherwise, increment l by one and go to step 2.
5. Calculate the values $\psi^{(l+1)}(k)$, $k = 1, \dots, N$, by the formulas

$$\psi^{(l+1)}(k) = A(k)^{\mathrm{T}}\psi^{(l+1)}(k+1), \quad \psi^{(l+1)}(N) = -2x^{(l+1)}(N).$$

6. Calculate the value $D_N^{(l+1)}$ by the formula

$$D_N^{(l+1)} = \sum_{k=0}^{N-1} \max_{\mu \in U(k)} \langle \psi^{(l+1)}(k+1), B(k)(\mu - u^{(l+1)}(k)) \rangle.$$

7. Check the second stopping condition

$$D_N^{(l+1)} < \varepsilon,$$

if true, terminate the calculations with the answer $u_\varepsilon = u^{(l+1)}$; otherwise, increase l by one and go to step 2.

The convergence of Algorithm 1 follows from Theorem 1 and its Corollaries 1–3. Steps 0–4 were used in [1] for finding an approximate solution to time-optimization problem (5) for $m = n$, $A(k) \equiv A$, $\det A \neq 0$, $B(k) \equiv I$ and a known two-sided estimate on optimal time.

The second way of constructing the algorithm involves changing the variables in the problem, which simplifies the iterations from (7). However, to find the desired solution based on the results of applying this method, it is necessary to perform the inverse transformation, which is not uniquely defined and leads to a separate subproblem.

The main idea here is to pass from system (1)–(2) to a linear system of the simplest form

$$y(k+1) = y(k) + \theta(k), \quad y(0) = y_0$$

through the degenerate change of variables $\theta(k) = \mathcal{A}_N(k+1)B(k)u(k)$, $y_0 = \mathcal{A}_N(0)x_0$. In this case, $\theta(k) \in \Theta(k) := \mathcal{A}_N(k+1)B(k)U(k) \subset \mathbb{R}^n$ and $y(k) = \mathcal{A}_N(k)x(k)$. In particular, $y(N) = x(N)$. From the latter it follows that if $\overline{\theta}(k) \in \Theta(k)$ minimize the value $\|y(N)\|$, then any element $\overline{u} \in \mathcal{U}_N$ satisfying the system of equations

$$\mathcal{A}_N(k+1)B(k)\overline{u}(k) = \overline{\theta}(k), \quad k = 0, \ldots, N-1, \tag{14}$$

is a solution to problem (4). If $m = n$ and for some k the matrix $\mathcal{A}_N(k+1)B(k)$ is invertible, then $\overline{u}(k)$ is uniquely determined. In the general case, we are faced with a system of equations (14) with constraints $\overline{u}(k) \in U(k)$. By construction, this problem is solvable. The values $\overline{\theta}(k)$ can be found using algorithm 1, since the sets $\Theta(k)$ are convex, compact, and contain the origin in $\mathbb{R}^n$. The difference is that $\overline{\theta}(k)$ are uniquely determined in this case.

Algorithm 2.

0. Set the permissible computational error $\varepsilon > 0$.
1. Define the initial approximation by the equalities $\theta^{(0)}(k) = 0 \in \mathbb{R}^n$, $k = 0, \ldots, N-1$, put $l = 0$ and, given the sets $U(k)$, determine the sets $\Theta(k) = \mathcal{A}_N(k+1)B(k)U(k)$, where $\mathcal{A}_N(k) = A(N-1)\ldots A(k)$, $k = 0, \ldots, N-1$, $\mathcal{A}_N(N)$ is the identity $n \times n$-matrix.
2. Calculate the values $\eta^{(l)}(k)$, $k = 1, \ldots, N$, by the formulas

$$\eta^{(l)}(k) = \eta^{(l)}(k+1) - \theta^{(l)}(k), \quad \eta^{(l)}(N) = 0.$$

3. For each $k = 0, \ldots, N-1$, sequentially find and fix the unique solution $\theta^{(l+1)}(k)$ to the extremum problem

$$\|\mu + y^{(l+1)}(k) - \eta^{(l)}(k+1)\| \to \min_{\mu \in \Theta(k)},$$

where the values $y^{(l+1)}(k)$ are calculated step by step using the formulas

$$y^{(l+1)}(k+1) = y^{(l+1)}(k) + \theta^{(l+1)}(k), \quad k = 0, \ldots, N-1, \quad y^{(l+1)}(0) = \mathcal{A}_N(0)x_0.$$

4. Check the first stopping condition

$$\|y^{(l)}(N)\|^2 - \|y^{(l+1)}(N)\|^2 < \varepsilon,$$

if true, go to step 5; otherwise, increment l by one and go to step 2.

5. Calculate the value $D_N^{(l+1)}$ by the formula

$$D_N^{(l+1)} = 2\sum_{k=0}^{N-1} \max_{\mu\in\Theta(k)} \langle y^{(l+1)}(N), \theta^{(l+1)}(k) - \mu\rangle.$$

6. Check the second stopping condition

$$D_N^{(l+1)} < \varepsilon,$$

if true, terminate the calculations and determine the answer u_ε by solving the system

$$\mathcal{A}_N(k+1)B(k)u_\varepsilon(k) = \theta^{(l+1)}(k), \quad u_\varepsilon(k) \in U(k), \quad k = 0, \dots, N-1;$$

otherwise, increase l by one and go to step 2.

Although the inverse change at step 6 of Algorithm 2 is a separate problem, it is actually implicitly present in Algorithm 1 as well, and must be solved at each iteration. It is natural to move this kind of nontrivial transformation outside the iterative process. At the same time, applying Algorithm 2 requires constructing transformed sets $\Theta(k)$, which in itself can be challenging.

4.2. Time-Optimization Problem

As the main algorithm for solving time-optimization problem (5), we will consider the sequential application of method (7) to construct control improvements in problem (4) for $N = 1, 2, \dots$ and the complementary calculation of the interval estimate, ensuring an effective transition to the next value of N. Let us just clarify that the estimate (10) in accordance with the above specification of the formulas (7)–(9) now takes the form

$$\overline{J}_N^2 \in \left[J_N(u^{(l)})^2 - \sum_{k=0}^{N-1} \max_{u\in U(k)} \langle \psi^{(l)}(k+1), B(k)(u - u^{(l)}(k))\rangle; J_N(u^{(l)})^2 \right].$$

Algorithm 3.

0. Set the permissible computational error $\varepsilon > 0$, put $N = 1$.
1. Put $l = 0$, define/supplement the initial approximation with the equality $u^{(0)}(N-1) = 0 \in \mathbb{R}^n$ and calculate the matrices $\mathcal{A}_N(k) = A(N-1)\dots A(k)$, $k = 0, \dots, N-1$, put the matrix $\mathcal{A}_N(N)$ to be the identity matrix of size $n \times n$.
2. Calculate the values $\xi^{(l)}(k)$, $k = 1, \dots, N$, by the formulas

$$\xi^{(l)}(k) = \xi^{(l)}(k+1) - \mathcal{A}_N(k+1)B(k)u^{(l)}(k), \quad \xi^{(l)}(N) = 0.$$

3. For each $k = 0, \dots, N-1$, sequentially find and fix some solution $u^{(l+1)}(k)$ to the extremum problem

$$\|\mathcal{A}_N(k+1)B(k)\mu + \mathcal{A}_N(k)x^{(l+1)}(k) - \xi^{(l)}(k+1)\| \to \min_{\mu\in U(k)},$$

where the values $x^{(l+1)}(k)$ are calculated step by step using the formulas

$$x^{(l+1)}(k+1) = A(k)x^{(l+1)}(k) + B(k)u^{(l+1)}(k), \quad x^{(l+1)}(0) = x_0.$$

4. Check the external stopping condition

$$\|x^{(l+1)}(N)\|^2 < \varepsilon,$$

if true, terminate the calculations with the answer $N_\varepsilon = N$, $u_\varepsilon = u^{(l+1)}$.

5. Calculate the values $\psi^{(l+1)}(k)$, $k = 1, \dots, N$, by the formulas

$$\psi^{(l+1)}(k) = A(k)^{\mathrm{T}}\psi^{(l+1)}(k+1), \quad \psi^{(l+1)}(N) = -2x^{(l+1)}(N).$$

6. Calculate the estimate $E_N^{(l+1)}$ by the formula

$$E_N^{(l+1)} = \|x^{(l+1)}(N)\|^2 - \sum_{k=0}^{N-1} \max_{\mu \in U(k)} \langle \psi^{(l+1)}(k+1), B(k)(\mu - u^{(l+1)}(k)) \rangle.$$

7. Check the internal stopping condition

$$E_N^{(l+1)} > 0,$$

if true, put the found $u^{(l+1)}$ as the new initial approximation, i.e. put $u^{(0)}(k) = u^{(l+1)}(k)$, $k = 0, \dots, N-1$, increment N by one and go to step 1; otherwise increment l by one and go to step 2.

Algorithm 3 is identical to the algorithm for solving time-optimization problem proposed in [2] (to simplify the calculations, in [2] the case $m = n$ and $B(k) \equiv I$ was considered). From the construction, it is clear that the convergence of this algorithm established in [2, Theorem 5], as well as the related results on optimality and improvement (see [2, Theorems 1–4]), follow in the general case from the well-known Theorem 1 and its Corollaries 1–3.

Note that the change of variables discussed in the previous subsection is less relevant here. This is because, for every N, except for large values close to $N_{\min}$, due to the estimate-based stopping conditions, only a small number of transformed problems are solved, and switching to new variables does not provide any significant speedup.

As an alternative approach to solving time-optimization problem, we consider the following algorithm based on the results of [6, 7]. We will apply the numerical method from [6] to calculate the optimal time $N_{\min}$ and then solve the remaining terminal stabilization problem as in the previous subsection. For completeness, we present here the formulation and brief justification of the method for calculating $N_{\min}$ from [6, 7].

For an arbitrary compact set K in $\mathbb{R}^n$, consider its support function

$$s_K(p) := \max_{\mu \in K} \langle p, \mu \rangle.$$

First of all, we consider the support functions of the sets $U(k)$. Since $U(k)$ are convex and contain the origin, $s_{U(k)}$ take finite non-negative values, are convex, continuous, and positively homogeneous [11, Corollary 13.2.2]. Since all sets $U(k)$ are predetermined, the functions $s_{U(k)}$ can, with certain reservations, be considered as known.

Consider the reachability set X_N of the system (1)–(2) at time $N \geqslant 0$. Using the matrices $\mathcal{A}_N(k)$ and the Minkowski set addition operation, it can be represented as

$$X_N = \mathcal{A}_N(0)x_0 + \sum_{k=0}^{N-1} \mathcal{A}_N(k+1)B(k)U(k).$$

The set X_N is convex and compact, $\mathcal{A}_N(0)x_0 \in X_N$. Since $0 \in U(k)$, then

$$0 \in X_N \quad \Leftrightarrow \quad N \geqslant N_{\min}.$$

The support function of the set X_N has the form

$$s_{X_N}(z) = \max_{x\in X_N}\langle z,x\rangle = \max_{u\in\mathcal{U}_N}\left\langle z, \mathcal{A}_N(0)x_0 + \sum_{k=0}^{N-1}\mathcal{A}_N(k+1)B(k)u(k)\right\rangle$$
$$= \langle z,\mathcal{A}_N(0)x_0\rangle + \sum_{k=0}^{N-1}\max_{\mu\in U(k)}\langle z,\mathcal{A}_N(k+1)B(k)\mu\rangle$$
$$= \langle z,\mathcal{A}_N(0)x_0\rangle + \sum_{k=0}^{N-1} s_{U(k)}\left(B(k)^{\mathrm{T}}\mathcal{A}_N(k+1)^{\mathrm{T}}z\right).$$

The function $s_{X_N}:\mathbb{R}^n\to\mathbb{R}$ is convex, continuous and positively homogeneous, and

$$s_{X_N}(z)\geqslant 0\ \ \forall z\in\mathbb{B}^n \quad\Leftrightarrow\quad 0\in X_N \quad\Leftrightarrow\quad N\geqslant N_{\min},$$

where $\mathbb{B}^n$ denotes the closed unit ball in $\mathbb{R}^n$. Therefore, the problem of finding the optimal time for the system (1)–(2) can be reformulated in terms of known functions $s_{U(k)}$ and given point x_0:

$$N\to\min,\quad \min_{z\in\mathbb{B}^n}\left(\langle z,\mathcal{A}_N(0)x_0\rangle + \sum_{k=0}^{N-1} s_{U(k)}\left(B(k)^{\mathrm{T}}\mathcal{A}_N(k+1)^{\mathrm{T}}z\right)\right) = 0. \tag{15}$$

Thus, the question of calculating the optimal time for system (1) with initial condition (2) is reduced to solving a sequence of convex finite-dimensional optimization problems of the same dimension n. Since all functions $s_{U(k)}$ take non-negative values, are continuous, and are equal to zero at the origin, the minimization can be performed on a smaller convex and compact set $\mathbb{B}^n_- := \{z\in\mathbb{B}^n : \langle z,\mathcal{A}_N(0)x_0\rangle\leqslant 0\}$. Moreover, due to the positive homogeneity of the functions $s_{U(k)}$, the solution to problem (15) will not change if $\mathbb{B}^n$ is replaced by the unit sphere $\mathbb{S}^n$ or the compact set $\mathbb{S}^n_- := \{z\in\mathbb{S}^n : \langle z,\mathcal{A}_N(0)x_0\rangle\leqslant 0\}$. The latter circumstance is used in [6] to construct an optimal process. Here, for these purposes, we will apply the block coordinate descent method.

So, another way to solve time-optimization problem is to directly determine the value $N_{\min}$ by solving problem (15), and then find an optimal control using method (7) for $N = N_{\min}$. Note that with this approach, the above-mentioned change of variables can also be effective, i.e., we are talking about the possibility of using both Algorithms 1 and 2. Moreover, there is no need to use the stopping rules (12) and (13). Instead, the calculations are terminated when the required proximity of $\|x(N)\|$ to zero is reached. At the same time, it should be noted that this approach can lead to significant difficulties in those frequent cases when the support functions $s_{U(k)}$ cannot be constructed explicitly and the optimization problems in (15) become minimax.

As a result, we obtain the following two variations of the algorithm.

Algorithm 4.

0. Set the permissible computational error $\varepsilon > 0$, put $N = 1$.
1. Calculate matrices $\mathcal{A}_N(k) = A(N-1)\dots A(k)$, $k = 0,\dots,N-1$, put the matrix $\mathcal{A}_N(N)$ to be the identity matrix of size $n\times n$.
2. Calculate the value

$$\overline{S}_N = \min_{z\in\mathbb{B}^n}\left(\langle z,\mathcal{A}_N(0)x_0\rangle + \sum_{k=0}^{N-1} s_{U(k)}\left(B(k)^{\mathrm{T}}\mathcal{A}_N(k+1)^{\mathrm{T}}z\right)\right).$$

3. Check the condition $\overline{S}_N < 0$. If true, increase N by one and go to step 1. Otherwise, go to step 4.

4. Set the initial approximation to $u^{(0)}(k) = 0 \in \mathbb{R}^m$, $k = 0, \dots, N-1$, set $l = 0$.
5. Calculate the values $\xi^{(l)}(k)$, $k = 1, \dots, N$, by the formulas

$$\xi^{(l)}(k) = \xi^{(l)}(k+1) - \mathcal{A}_N(k+1)B(k)u^{(l)}(k), \quad \xi^{(l)}(N) = 0.$$

6. For each $k = 0, \dots, N-1$, sequentially find and fix some solution $u^{(l+1)}(k)$ to the extremum problem

$$\|\mathcal{A}_N(k+1)B(k)\mu + \mathcal{A}_N(k)x^{(l+1)}(k) - \xi^{(l)}(k+1)\| \to \min_{\mu \in U(k)},$$

where the values $x^{(l+1)}(k)$ are calculated step by step using the formulas

$$x^{(l+1)}(k+1) = A(k)x^{(l+1)}(k) + B(k)u^{(l+1)}(k), \quad x^{(l+1)}(0) = x_0.$$

7. Check stopping condition

$$\|x^{(l+1)}(N)\|^2 < \varepsilon,$$

if true, terminate the calculations with the answer $N_\varepsilon = N$, $u_\varepsilon = u^{(l+1)}$; otherwise, increase l by one and go to step 2.

Algorithm 5.

0. Set the permissible computational error $\varepsilon > 0$, put $N = 1$.
1. Calculate matrices $\mathcal{A}_N(k) = A(N-1)\dots A(k)$, $k = 0, \dots, N-1$, put the matrix $\mathcal{A}_N(N)$ to be the identity matrix of size $n \times n$.
2. Calculate the value

$$\overline{S}_N = \min_{z \in \mathbb{B}^n} \left(\langle z, \mathcal{A}_N(0)x_0 \rangle + \sum_{k=0}^{N-1} s_{U(k)} \left(B(k)^{\mathrm{T}} \mathcal{A}_N(k+1)^{\mathrm{T}} z \right) \right).$$

3. Check the condition $\overline{S}_N < 0$. If true, increase N by one and go to step 1. Otherwise, go to step 4.
4. Define the initial approximation with the equalities $\theta^{(0)}(k) = 0 \in \mathbb{R}^n$, $k = 0, \dots, N-1$, put $l = 0$ and, given the sets $U(k)$, determine the sets $\Theta(k) = \mathcal{A}_N(k+1)B(k)U(k)$.
5. Calculate the values $\eta^{(l)}(k)$, $k = 1, \dots, N$, by the formulas

$$\eta^{(l)}(k) = \eta^{(l)}(k+1) - \theta^{(l)}(k), \quad \eta^{(l)}(N) = 0.$$

6. For each $k = 0, \dots, N-1$, sequentially find and fix the unique solution $\theta^{(l+1)}(k)$ to the extremum problem

$$\|\mu + y^{(l+1)}(k) - \eta^{(l)}(k+1)\| \to \min_{\mu \in \Theta(k)},$$

where the values $y^{(l+1)}(k)$ are calculated step by step using the formulas

$$y^{(l+1)}(k+1) = y^{(l+1)}(k) + \theta^{(l+1)}(k), \quad k = 0, \dots, N-1, \quad y^{(l+1)}(0) = \mathcal{A}_N(0)x_0.$$

7. Check stopping condition

$$\|y^{(l+1)}(N)\|^2 < \varepsilon,$$

if true, terminate the calculations with the answer $N_\varepsilon = N$ and u_ε found from solving the system

$$\mathcal{A}_N(k+1)B(k)u_\varepsilon(k) = \theta^{(l+1)}(k), \quad u_\varepsilon(k) \in U(k), \quad k = 0, \dots, N-1;$$

otherwise, increase l by one and go to step 2.

Note that, by construction, applying Algorithms 4 and 5 yields the exact equality $N_\varepsilon = N_{\min}$ for any value of $\varepsilon > 0$.

5. CONCLUSION

The use of the algorithms constructed in this paper appears to be relevant in problems with non-strictly convex sets of geometric constraints, with constraints described by mixed systems of inequalities, and in problems with singular matrices. In other situations, the most efficient way should be using classical finite-dimensional numerical optimization algorithms, as well as the above-mentioned method based on the support function framework from [6].

REFERENCES

1. Ibragimov, D.N. and Tsarkov, K.A., On an Approach to Solving the Time-Optimization Problem for Linear Discrete-Time Systems Based on Krotov Method, *Autom. Remote Control*, 2024, vol. 85, no. 11, pp. 1053–1078.
2. Ibragimov, D.N. and Tsarkov, K.A., Numerical Method for Solving the Time-Optimization Problem for Linear Non-Stationary Discrete-Time Systems of General Form, *Autom. Remote Control*, 2025, vol. 86, no. 11, pp. 989–1014.
3. Warga, J., Minimizing Certain Convex Functions, *J. Soc. Indust. Appl. Math.*, 1963, vol. 11, no. 3, pp. 588–593.
4. Bertsekas, D.P., *Nonlinear Programming*, Athena Scientific, 1999.
5. Peng, L. and Vidal, R., Block Coordinate Descent on Smooth Manifolds: Convergence Theory and Twenty-One Examples, *arxiv.org*, 2023 (arXiv:2305.14744).
6. Bortakovskii, A.S., Applying Support Functions to Solve a Problem of Fast-Acting Discrete Linear Systems with a Convex Set of Controls, *J. Comp. Sys. Sci. Int.*, 2025, vol. 64, no. 2, pp. 203–220.
7. Ibragimov, D.N., On the Optimal Speed Problem for the Class of Linear Autonomous Infinite-Dimensional Discrete-Time Systems with Bounded Control and Degenerate Operator, *Autom. Remote Control*, 2019, vol. 80, no. 3, pp. 393–412.
8. Luo, Z. and Tseng, P., On the Convergence of the Coordinate Descent Method for Convex Differentiable Minimization, *J. Optim. Theory Appl.*, 1992, vol. 72, no. 1, pp. 7–35.
9. Panteleev, A.V. and Letova T.A., *Metody optimizatsii v primerakh i zadachakh* (Optimization Methods in Examples and Problems), St. Petersburg: Lan, 2022.
10. Polyak, B.T., *Vvedenie v optimizatsiyu* (Introduction to Optimization), Moscow: Nauka, 1983.
11. Rockafellar, R.T., *Convex Analysis*, Princeton University Press, 1970.

This paper was recommended for publication by A.I. Kibzun, *a member of the Editorial Board*

ISSN 0005-1179 (print), ISSN 1608-3032 (online), Automation and Remote Control, 2026, Vol. 87, No. 9, pp. 846–858.
© The Author(s), 2026 published by Trapeznikov Institute of Control Sciences, Russian Academy of Sciences, 2026.
Russian Text © The Author(s), 2026, published in Avtomatika i Telemekhanika, 2026, No. 9, pp. 110–126.

NONLINEAR SYSTEMS

Constructing Lyapunov Functions in Partial Stability Problems

V. I. Vorotnikov[*,a] **and Yu. G. Martyshenko**[**,b]

[*] *Peoples' Friendship University (RUDN), Sochi Institute, Sochi, Russia*
[**] *Gubkin Russian State University of Oil and Gas, Moscow, Russia*
e-mail: [a] *vorotnikov-vi@rambler.ru,* [b] *j-mart@mail.ru*

Received November 20, 2025
Revised April 30, 2026
Accepted May 4, 2026

Abstract—This paper considers systems of nonlinear ordinary differential equations with a continuous right-hand side. For such systems, stability analysis of the trivial equilibrium with respect to part of the variables is performed in the context of the direct Lyapunov method. Lyapunov functions are constructed by choosing an appropriate pair of a main Lyapunov function and an auxiliary function: the latter function is used to correct the domain of the state space of the system where the former function is introduced and analyzed. Illustrative examples are provided in which such pairs are constructed for some nonlinear systems of two and three differential equations.

Keywords: system of nonlinear ordinary differential equations, partial stability, Lyapunov function method, construction of Lyapunov functions

DOI: 10.7868/S1608303226090049

1. INTRODUCTION

Along with stability problems with respect to all variables, modern control theory also addresses more general ones with respect to part of the variables (partial stability), going back to the classical works of A.M. Lyapunov [1] and V.V. Rumyantsev [2, 3]. These problems naturally arise in applications and are considered for models with various mathematical descriptions, e.g., continuous- and discrete-time dynamics, the hereditarity effect (delays) and random disturbances, etc., see the reviews in [3–5]. New stability problems have also been investigated: input-output stability [6, 7] and finite-time output stability [8–10].

As in the case of stability with respect to all variables, the main rigorous method for analyzing partial stability is the Lyapunov function method with an appropriate modification. It has been successfully used, e.g., in the stability analysis of complex mechanical systems [3, 11]. However, despite the proven reversibility of several theorems underlying this method and its further development, the most urgent challenge still is to construct Lyapunov functions both for general classes of systems and for particular systems.

In many cases, stability with respect to all variables "in the small" is analyzed based on the linear approximation. In contrast, partial stability analysis even in this situation is *nonlocal* due to "uncontrolled" variables and depends, to a much greater extent, on the structure and parameters of the systems. Such an analysis can be successfully performed by using a suitable Lyapunov function.

This paper considers a possible approach to finding Lyapunov functions in the partial stability problem of the trivial equilibrium for a system of nonlinear differential equations with a continuous right-hand side. It is based on a transition to the analysis of "*extended*" partial stability, i.e.,

not only with respect to an initially specified part of the variables but also with respect to some additional variables (functions of the state variables of the system of differential equations under study). This transition neither changes nor refines the essential meaning of the original partial stability problem but has another goal: to *narrow* the domain of the system's state space where a suitable Lyapunov function is searched, thereby facilitating its choice to ensure required properties. Naturally, according to the meaning of the problem being solved, extended partial stability is an intermediate concept between stability with respect to an initially specified part of the variables and stability with respect to all variables (full stability).

This approach is formalized by finding, along with a main Lyapunov function, one more (generally speaking, vector-valued) auxiliary function that defines a set of additional variables and considering both as a *pair* (bundle) of functions for solving the original partial stability problem. In this case, the auxiliary function is used to correct the domain of the system's state space where the main Lyapunov function is introduced and its properties are analyzed. The requirements imposed on the main Lyapunov function are *weaker* than in the stability analysis with respect to an initially specified part of the variables.

Structural forms of nonlinear systems of two and three differential equations are identified for which the indicated pairs can be constructed and the partial stability domains can be found in the parameter space. The extended partial stability analysis of these structural forms reduces to the stability analysis with respect to all variables based on the linear approximation of equilibria of special auxiliary systems of differential equations obtained from the original system by corresponding transformations of the state variables. These structural forms are used below as a nonlinear approximation for partial stability analysis of systems with polynomial nonlinearities.

2. PROBLEM STATEMENT

Consider a nonlinear system of ordinary differential equations of the form

$$\begin{gathered}\mathbf{x}' = \mathbf{X}(t,\mathbf{x}), \quad \mathbf{X}(t,\mathbf{0}) \equiv \mathbf{0},\\ \mathbf{x} = (x_1,\dots,x_n) = (y_1,\dots,y_m,z_1,\dots,z_p) = (\mathbf{y},\mathbf{z}), \quad m+p=n,\end{gathered} \tag{1}$$

with two groups of variables in the state vector $\mathbf{x}$, namely, 1) the variables $\mathbf{y}$, with respect to which the stability of the equilibrium $\mathbf{x}=\mathbf{0}$ is studied, and 2) the other (uncontrolled) variables $\mathbf{z}$ [3, 4, 12]. From this point onwards, $\|\mathbf{x}\|$ indicates the Euclidean norm of a vector $\mathbf{x}$.

We introduce conventional assumptions from the theory of partial stability ($\mathbf{y}$-stability) [3, 4], i.e., the continuity of the vector function $\mathbf{X}$ in the domain

$$t \geqslant 0, \quad \|\mathbf{y}\| \leqslant h, \quad \|\mathbf{z}\| < \infty, \tag{2}$$

as well as the uniqueness and $\mathbf{z}$-continuability of solutions of system (1). (In other words, each solution $\mathbf{x}(t)$ is defined for all $t \geqslant 0$ such that $\|\mathbf{y}(t)\| \leqslant h$.) Let $\mathbf{x}(t) = \mathbf{x}(t;t_0,\mathbf{x}_0)$ denote the solution of system (1) with an initial condition $\mathbf{x}_0 = \mathbf{x}(t_0;t_0,\mathbf{x}_0)$.

Definitions. The equilibrium $\mathbf{x}=\mathbf{0}$ of system (1) is said to be:

1) $\mathbf{y}$-*stable* if, for any $\varepsilon > 0$ and $t_0 \geqslant 0$, there exists a number $\delta(\varepsilon,t_0) > 0$ such that $\|\mathbf{x}_0\| < \delta$ implies $\|\mathbf{y}(t;t_0,\mathbf{x}_0)\| < \varepsilon$ for all $t \geqslant t_0$;

2) *uniformly* $\mathbf{y}$-*stable* if $\delta = \delta(\varepsilon)$ in definition 1);

3) *asymptotically* $\mathbf{y}$-*stable* if it is $\mathbf{y}$-stable in the sense of definition 1) and there exists $\Delta(t_0) > 0$ such that each solution $\mathbf{x}(t;t_0,\mathbf{x}_0)$ of system (1) with $\|\mathbf{x}_0\| < \Delta$ satisfies the limit relation

$$\|\mathbf{y}(t;t_0,\mathbf{x}_0)\| \to 0, \quad t \to \infty; \tag{3}$$

here, the inequality $\|\mathbf{x}_0\| < \Delta$ determines the domain of $\mathbf{y}$-*attraction* of the equilibrium $\mathbf{x}=\mathbf{0}$ of system (1) for the initial time instant t_0;

4) *uniformly asymptotically* $\mathbf{y}$*-stable* if it is uniformly $\mathbf{y}$-stable in the sense of definition 2), Δ is independent of t_0, and the limit relation (3) holds uniformly in $\{t_0, \mathbf{x}_0\}$ from the domain $t_0 \geqslant 0$, $\|\mathbf{x}_0\| < \Delta$.

For analyzing the $\mathbf{y}$-stability of the equilibrium $\mathbf{x} = \mathbf{0}$ of system (1) in the context of the direct Lyapunov method, we consider the following functions:

1) a scalar function $V(t, \mathbf{x})$, $V(t, \mathbf{0}) \equiv 0$, continuously differentiable in the domain (2), as well as its derivative V' along the trajectories of system (1), as a Lyapunov function;

2) a vector function $\mathbf{w}(\mathbf{x})$, $\mathbf{w}(\mathbf{0}) \equiv \mathbf{0}$, continuous in the domain (2), used to estimate from above the Lyapunov function and its derivative;

3) $a(r)$, $b(r)$, and $c(r)$, standard continuous functions used in stability theory, monotonically increasing for $r \in [0, h]$, such that $a(0) = b(0) = c(0) = 0$ (Hahn functions [3]).

According to Rumyantsev [2], if system (1) in the domain

$$t \geqslant 0, \quad \|\mathbf{y}\| < h_1 \leqslant h, \quad \|\mathbf{z}\| < \infty \tag{4}$$

with a sufficiently small number $h_1 > 0$ has a function V such that

$$V(t, \mathbf{x}) \geqslant a(\|\mathbf{y}\|), \quad V'(t, \mathbf{x}) \leqslant 0, \tag{5}$$

then the equilibrium $\mathbf{x} = \mathbf{0}$ of system (1) is $\mathbf{y}$-stable.

With conditions (5) replaced by

$$a(\|\mathbf{y}\|) \leqslant V(t, \mathbf{x}) \leqslant b(\|\mathbf{u}(\mathbf{x})\|), \quad V'(t, \mathbf{x}) \leqslant -c(\|\mathbf{u}(\mathbf{x})\|),$$
$$\mathbf{u}(\mathbf{x}) = [\mathbf{y}, \mathbf{w}(\mathbf{x})],$$

the equilibrium $\mathbf{x} = \mathbf{0}$ becomes uniformly asymptotically $\mathbf{y}$-stable [3, 4, 12].

In stability analysis with respect to all variables, the construction of a Lyapunov function not only settles the stability issue itself but also gives some estimates for the domain of attraction. They are obtained by approximating the domain of attraction by sets lying inside level surfaces of Lyapunov functions.

Several analytical and numerical methods for constructing Lyapunov functions and estimating the domains of attraction were described, e.g., in the monographs [13–16], the reviews [17–20], and the works [21–27]; systems with a discontinuous right-hand side were considered in [28]. We also mention the *divergent* method for stability analysis [29], an alternative to the Lyapunov function method.

The stability problem with respect to part of the variables ($\mathbf{y}$-stability) has certain peculiarities. Even the $\mathbf{y}$-stability analysis of the equilibrium $\mathbf{x} = \mathbf{0}$ of system (1) "in the small" leads to a nonlocal problem of investigating the properties of the Lyapunov function in the cylindrical domain (4) of the state space. Moreover, $\mathbf{y}$-stability analysis "in the small" (with respect to $\mathbf{y}_0$) may allow for large initial perturbations of the uncontrolled variables $\mathbf{z}$ (with respect to $\mathbf{z}_0$). Finding Lyapunov functions for such situations is a topical and little-studied problem in the theory of partial stability.

3. THE PROPOSED APPROACH TO FINDING LYAPUNOV FUNCTIONS

In the general case, when investigating the $\mathbf{y}$-stability problem of the equilibrium $\mathbf{x} = \mathbf{0}$ of system (1) by the direct Lyapunov method, Lyapunov functions depend on t, $\mathbf{y}$, and, moreover, on the uncontrolled variables $\mathbf{z}$.

Therefore, the assumption $\|\mathbf{z}\| < \infty$, which defines the domain (4) to select and analyze the Lyapunov function, may turn out to be overly restrictive, significantly complicating the verification of the sign definiteness properties of the functions and their derivatives along the trajectories of system (1).

Additional possibilities appear in the transition from the original $\mathbf{y}$-stability problem to the analysis of extended partial stability, i.e., $(\mathbf{y}, \boldsymbol{\mu})$-stability of the equilibrium $\mathbf{x} = \mathbf{0}$ of system (1). In this case, the domain (4) is replaced by a narrower one in the state space:

$$t \geqslant 0, \quad \|\mathbf{y}\| + \|\boldsymbol{\mu}(\mathbf{x})\| < h_1 \leqslant h, \quad \|\mathbf{z}\| < \infty, \tag{6}$$

where $\boldsymbol{\mu}(\mathbf{x})$ is a function continuous in the domain (2) such that $\boldsymbol{\mu}(\mathbf{0}) \equiv \mathbf{0}$.

This approach is motivated by the following fact: in the $\mathbf{y}$-stability analysis of the equilibrium $\mathbf{x} = \mathbf{0}$ of system (1), extended $(\mathbf{y}, \boldsymbol{\mu})$-stability always holds for some combinations of the state variables defining the function $\boldsymbol{\mu}(\mathbf{x})$. Specifying a particular function $\boldsymbol{\mu}$, however, requires a separate analysis; therefore, this approach is formalized by finding, along with the *main Lyapunov function* $V(t, \mathbf{x})$, one more (generally speaking, vector-valued) *auxiliary* function $\boldsymbol{\mu}(\mathbf{x})$ for the corresponding correction of the domain (6), where the main Lyapunov function is introduced. These functions are treated as a *pair* $\{V(t, \mathbf{x}), \boldsymbol{\mu}(\mathbf{x})\}$ for solving the $\mathbf{y}$-stability problem of the equilibrium $\mathbf{x} = \mathbf{0}$ of system (1).

With such a pair, during the solution procedure one can use Lyapunov functions $V(t, \mathbf{x})$ whose derivatives along the trajectories of system (1) may be *indefinite* (of alternating sign) in the domain (4). (And even the functions themselves may have this property.)

Theorem 1. *Suppose that for system* (1)*, it is possible to specify a pair* $\{V(t, \mathbf{x}), \boldsymbol{\mu}(\mathbf{x})\}$ *such that*

$$V(t, \mathbf{x}) \geqslant a(\|\mathbf{y}\| + \|\boldsymbol{\mu}(\mathbf{x})\|), \quad V'(t, \mathbf{x}) \leqslant 0 \tag{7}$$

for sufficiently small $h_1 > 0$ *in the domain* (6)*.*

Then the equilibrium $\mathbf{x} = \mathbf{0}$ *of system* (1) *is* $\mathbf{y}$*-stable.*

If conditions (7) *are replaced by*

$$a(\|\mathbf{y}\| + \|\boldsymbol{\mu}(\mathbf{x})\|) \leqslant V(t, \mathbf{x}) \leqslant b(\|\mathbf{u}(\mathbf{x})\|), \quad V'(t, \mathbf{x}) \leqslant -c(\|\mathbf{u}(\mathbf{x})\|), \tag{8}$$

then the equilibrium $\mathbf{x} = \mathbf{0}$ *is uniformly asymptotically* $\mathbf{y}$*-stable.*

This result is proved via the scheme indicated in [2–4, 12].

Within the proposed approach to the $\mathbf{y}$-stability analysis of the equilibrium $\mathbf{x} = \mathbf{0}$ of system (1), as a candidate for the main function $V(t, \mathbf{x})$ in the pair, it is natural to take the quadratic form

$$V(t, \mathbf{x}) = \mathbf{y} A_1(t) \mathbf{y}^{\mathrm{T}} + \boldsymbol{\mu}(\mathbf{x}) A_2(t) \boldsymbol{\mu}^{\mathrm{T}}(\mathbf{x}) \tag{9}$$

of the variables $\mathbf{y}$ and $\boldsymbol{\mu}$ with appropriately chosen elements of the matrix functions A_1 and A_2 of the corresponding dimensions, where the superscript T means transpose. (In particular, these elements may be independent of t.) Alternatively, a higher-order matrix form of the variables $\mathbf{y}$ and $\boldsymbol{\mu}$ can be adopted: $V(t, \mathbf{x}) \equiv V^*(t, \mathbf{y}, \boldsymbol{\mu}(\mathbf{x}))$.

This pair design scheme is constructively implemented if system (1) can be represented as

$$\mathbf{y}' = \mathbf{Y}^*(t, \mathbf{y}, \boldsymbol{\mu}) + \mathbf{Y}^{**}(t, \mathbf{x}), \quad \boldsymbol{\mu}' = \mathbf{M}(t, \mathbf{y}, \boldsymbol{\mu}) + \mathbf{M}^*(t, \mathbf{x}), \quad \mathbf{z}' = \mathbf{Z}(t, \mathbf{x});$$

and the issue of the $(\mathbf{y}, \boldsymbol{\mu})$-stability of the trivial equilibrium of the resulting system is settled by investigating stability with respect to *all* variables of the trivial equilibrium of the first (linear or nonlinear) approximation system

$$\mathbf{y}' = \mathbf{Y}^*(t, \mathbf{y}, \boldsymbol{\mu}), \quad \boldsymbol{\mu}' = \mathbf{M}(t, \mathbf{y}, \boldsymbol{\mu}). \tag{10}$$

Then Lyapunov functions of the form (9) can be constructed as Lyapunov functions for system (10) by using well-known (analytical or numerical) stability analysis methods with respect to *all* variables.

Note that several partial stability conditions based on the linear and nonlinear approximation were given in [2–5, 30, 31].

Another way of choosing a candidate for the main function $V(t,\mathbf{x})$ involves the first integrals of system (1). Starting from the works of N.G. Chetaev [32], the construction of Lyapunov functions in the form of a bundle of first integrals has been used in the stability analysis of mechanical systems with respect to all or part of the variables [3, 5]. In the $\mathbf{y}$-stability problem of the equilibrium $\mathbf{x}=\mathbf{0}$ of system (1), the introduction of an additional function $\boldsymbol{\mu}(\mathbf{x})$ allows simplifying, in a number of cases, the Lyapunov function structure via the representation

$$V(t,\mathbf{x})=V^*(t,\mathbf{y},\boldsymbol{\mu}(\mathbf{x}))+W(t,\mathbf{x}), \tag{11}$$

where $V^*(t,\mathbf{y},\boldsymbol{\mu}(\mathbf{x}))$ is a sign definite function (in $\mathbf{y}$ and $\boldsymbol{\mu}$) in the domain (6), and the function $W(t,\mathbf{x})$ satisfies the condition $W(t,\mathbf{x})\geqslant 0$ in the domain (6).

For linear systems with constant coefficients, an algorithm for choosing the function $\boldsymbol{\mu}(\mathbf{x})$ and constructing Lyapunov functions in the form (9) was considered in [12].

Remark 1. In pairs, a scalar main Lyapunov function $V(t,\mathbf{x})$ may be replaced by a vector-valued main function $\mathbf{V}=[V_1(t,\mathbf{x}),\dots,V_k(t,\mathbf{x})]$, and such a function is analyzed in the domain (6) by the comparison method [33].

Remark 2. The Lyapunov function–feedback pair (stabilizing pair) was introduced for stabilization of control systems [34]; the Lyapunov function–vector field pair [35] and a correcting auxiliary function to construct a Lyapunov function directly from the right-hand side of the system were applied for stability analysis with respect to all variables [36].

The idea of constructing the pair $\{V(t,\mathbf{x}),\boldsymbol{\mu}(\mathbf{x})\}$, which can be regarded as a solution of the Lyapunov function problem in partial stability analysis, was not stated in general form. However, the $\mathbf{y}$-stability conditions (7), to be verified in the domain (6), were indicated and discussed in [4, 5, 12]; and the uniform asymptotic $\mathbf{y}$-stability conditions (8) were announced in [5, 12]. In this connection, we note that within the proposed approach, the pair can also be chosen under other conditions on the Lyapunov function in the domain (6), different from (8).

Remark 3. For a chosen auxiliary function $\boldsymbol{\mu}(\mathbf{x})$, verification of conditions (7) and (8) in the domain (6) gets simplified if the Lyapunov function and the nonlinearities in system (1) are *polynomial* and the solutions of system (1) starting in a small neighborhood of the equilibrium $\mathbf{x}=\mathbf{0}$ are known to be bounded in the variable $\mathbf{z}$. If the auxiliary system (10) can be *identified* from the original one (1), then it suffices to verify the sign definiteness of the constructed Lyapunov function $V(t,\mathbf{y},\boldsymbol{\mu}(\mathbf{x}))$ and its derivative along the auxiliary system with respect to *all* variables.

Example 1. The motion of a holonomic mechanical system in a potential force field is governed by the Lagrange equations of the second kind:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial x_i'}\right)-\frac{\partial T}{\partial x_i}=-\frac{\partial \Pi}{\partial x_i},$$

where T is the kinetic energy, Π is the potential energy, and x_i are generalized coordinates.

Let the imposed links have no explicit dependence on time t, and let the system admit the trivial equilibrium $\mathbf{x}=\mathbf{x}'=\mathbf{0}$. We divide the components of the generalized coordinate vector $\mathbf{x}$ into two groups, $\mathbf{x}=(\mathbf{y},\mathbf{z})$, and assume the existence of an auxiliary function $\boldsymbol{\mu}(\mathbf{x})$ such that, in the domain

$$\|\mathbf{y}\|+\|\boldsymbol{\mu}(\mathbf{x})\|+\|\mathbf{x}'\|<h_1\leqslant h,\quad \|\mathbf{z}\|<\infty, \tag{12}$$

the following conditions are satisfied:

1) T is positive definite with respect to the generalized velocities.

2) Π does not depend explicitly on time t and is positive definite with respect to $\mathbf{y}$ and, moreover, with respect to the additional variables $\boldsymbol{\mu}(\mathbf{x})$, i.e., $\Pi(\mathbf{x})\geqslant a(\|\mathbf{y}\|+\|\boldsymbol{\mu}(\mathbf{x})\|)$.

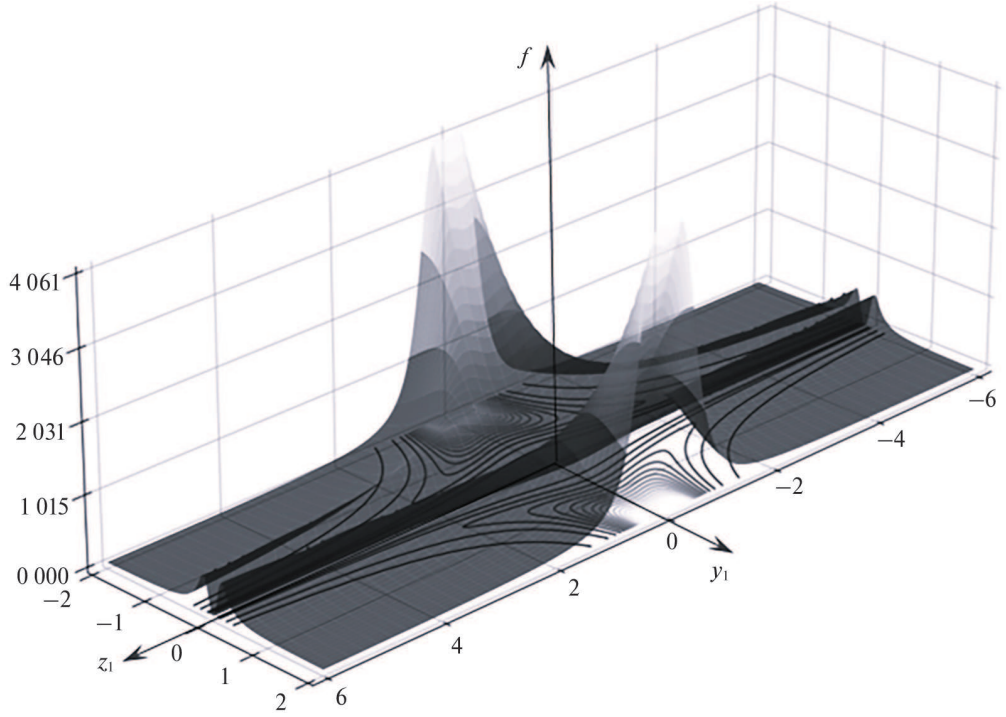


Fig. 1. Surface (13).

Under these assumptions, the energy integral $T + \Pi = \text{const}$ holds and, consequently, the pair $\{V(\mathbf{x}, \mathbf{x}') = T(\mathbf{x}, \mathbf{x}') + \Pi(\mathbf{x}),\ \boldsymbol{\mu} = \boldsymbol{\mu}(\mathbf{x})\}$ solves the partial stability problem of the equilibrium $\mathbf{x} = \mathbf{x}' = \mathbf{0}$: uniform stability with respect to $\mathbf{y}$ and $\mathbf{x}'$ (part of the generalized coordinates and all generalized velocities).

In particular, consider the motion of a unit-mass point on the surface

$$f = \frac{y_1^2(1 + z_1^2)}{1 + y_1^4 z_1^4} \tag{13}$$

above the horizontal plane Oy_1z_1 [4, 12].

In this case, $\mathbf{x} = (y_1, z_1)$,

$$T(\mathbf{x}, \mathbf{x}') = \frac{1}{2}\left\{y_1'^2 + z_1'^2 + \left[\frac{\partial f}{\partial y_1}y_1' + \frac{\partial f}{\partial z_1}z_1'\right]^2\right\}, \quad \text{and} \quad \Pi(\mathbf{x}) = gf(y_1, z_1),$$

where g denotes the acceleration of gravity.

We specify the pair as follows:

$$\{V(\mathbf{x}, \mathbf{x}') = T(\mathbf{x}, \mathbf{x}') + \Pi(\mathbf{x}),\ \mu_1(\mathbf{x}) = y_1z_1\}, \tag{14}$$

so the main Lyapunov function $V(\mathbf{x}, \mathbf{x}')$ has the structure (11) if $\mathbf{y} = (y_1, \mathbf{x}')$, and $W(\mathbf{x}, \mathbf{x}')$ is defined by the bracketed term in the expression for $T(\mathbf{x}, \mathbf{x}')$.

For sufficiently small h_1, we have

$$V(\mathbf{x}, \mathbf{x}') \geqslant a(|y_1| + |\mu_1| + \|\mathbf{x}'\|), \quad V'(\mathbf{x}, \mathbf{x}') \equiv 0$$

in the domain (12). Therefore, the equilibrium $\mathbf{x} = \mathbf{x}' = \mathbf{0}$ of the point is uniformly stable with respect to y_1 and $\mathbf{x}'$.

At the same time, for $\mathbf{y} = (y_1, \mathbf{x}')$ the condition $V(\mathbf{x}, \mathbf{x}') \geqslant a(|y_1| + \|\mathbf{x}'\|)$ fails in the domain (4): $V(\mathbf{x}, \mathbf{x}') \to 0$ as $|z_1| \to \infty$ for $\mathbf{x}' = \mathbf{0}$ and any fixed value of y_1. Hence, the Lyapunov function $V(\mathbf{x}, \mathbf{x}')$ in the pair (14) is indefinite with respect to y_1 and $\mathbf{x}'$ in the sense of [2] and does not satisfy the first condition of (5) in the domain (4).

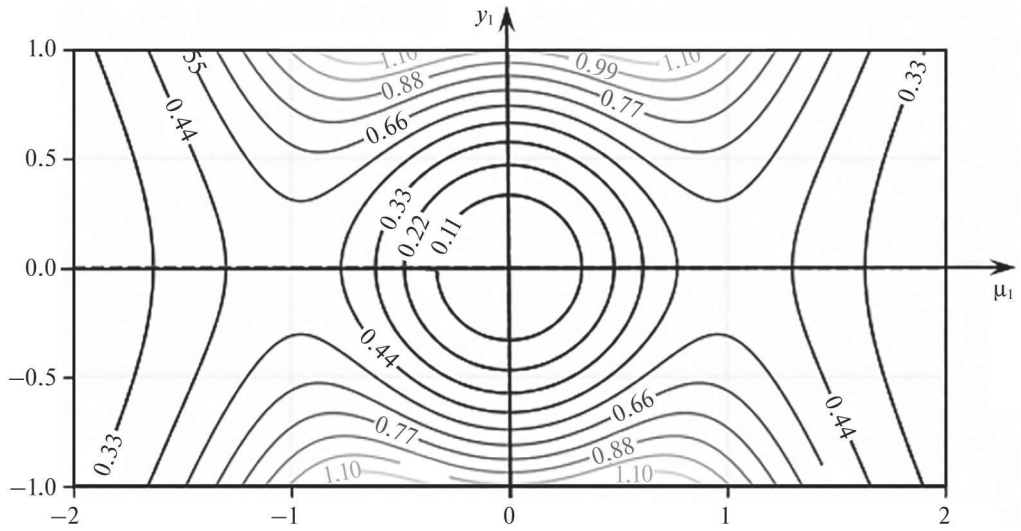


Fig. 2. Level lines of the surface (13) on the plane (y_1, μ_1).

The pair (14) serves to reveal a nonlocal nature of the uniform stability with respect to y_1 and $\mathbf{x}'$ and to estimate the domain of admissible initial perturbations $\mathbf{x}_0$ and $\mathbf{x}'_0$ of the variables $\mathbf{x}$ and $\mathbf{x}'$, respectively. Indeed, as the main function $V(y_1, z_1, \mathbf{x}')$ satisfies the condition $V(0, z_1, \mathbf{0}) \equiv 0$, the uniform stability of the equilibrium $\mathbf{x} = \mathbf{x}' = \mathbf{0}$ of the point with respect to y_1 and $\mathbf{x}'$ will hold not only for sufficiently small values of y_{10} and $\|\mathbf{x}'_0\|$ but also for large (finite) values of z_{10}.

As a supplement to [4, 12], it can be shown that for a sufficiently small value of $\|\mathbf{x}'_0\|$, the domain of admissible initial values y_{10} and z_{10} is given by the inequality

$$\frac{y_{10}^2(1+z_{10}^2)}{1+y_{10}^4 z_{10}^4} < \frac{1}{2}. \tag{15}$$

To prove this fact, we consider the equation $f(y_1, z_1) = c$ $(c = \text{const} > 0)$, defining the level lines of the surface (13). In terms of the variables (y_1, μ_1), this equation can be represented as a biquadratic one in μ_1 :

$$cr^2 - r + (c - y_1^2) = 0 \quad (r = \mu_1^2). \tag{16}$$

For $4cy_1^2 \geqslant 4c^2 - 1$, equation (16) has the two real solutions

$$r_1(c, y_1) = \frac{1 - \sqrt{1 - 4c(c - y_1^2)}}{2c}, \quad r_2(c, y_1) = \frac{1 + \sqrt{1 - 4c(c - y_1^2)}}{2c}.$$

In view of Vieta's formulas, we conclude that 1) $r_1 > 0$ and $r_2 > 0$ for $0 < c < 1/2$, $y_1^2 < c$, and also for $c \geqslant 1/2$ and $c - 1/(4c) \leqslant y_1^2 < c$; 2) $r_2 > 0$ for $y_1^2 > c$ and any $c > 0$. As a consequence, $r_2 > 0$ for $0 < c < 1/2$ and all y_1.

Let us analyze the solution $r_1 > 0$ in the case $0 < c < 1/2$. Since $r_1 \to 0$ as $c \to 0$, and $r_1 \to 1$ as $c \to 1/2$ and $y_1 \to 0$, we can identify the domain $y_1^2 < 1/2$, $\mu_1^2 < 1$ on the plane (y_1, μ_1) where the level lines of the surface (13) are closed elliptic curves encircling the point $y_1 = \mu_1 = 0$ and intersecting the line $y_1 = 0$ at the points $\mu_1 = \pm\sqrt{r_1}$.

In the domain $(y_1^2 < 1/2) \bigcap (y_1^2 < 1/z_1^2)$ on the plane (y_1, z_1), these level lines are associated with the open curves encircling the line $y_1 = 0$ and asymptotically approaching it as $c \to 0$ and $|z_1| \to \infty$.

The second branches of the level lines in the case $0 < c < 1/2$ (corresponding to the solution $r_2 > 0$), as well as the level lines for $c \geqslant 1/2$, are defined by the curves lying outside the indicated

domains. Indeed, $r_2 \to +\infty$ as $c \to 0$, and $r_2 \to 1$ as $c \to 1/2$ and $y_1 \to 0$, and the second branches of the level lines are open curves in the domain $\mu_1^2 \geqslant 1$ on the plane (y_1, μ_1) (intersecting the line $y_1 = 0$ at the points $\mu_1 = \pm\sqrt{r_2}$) and open curves in the domain $y_1^2 \geqslant 1/z_1^2$ on the plane (y_1, z_1). For $c \geqslant 1/2$, the level lines of the surface (13) lie in the domain $y_1^2 \geqslant 1/2$.

As a result, for a sufficiently small value of $\|\mathbf{x}_0'\|$ and values of y_{10} and z_{10} from the domain (15), it is possible to make the value $V(\mathbf{x}_0, \mathbf{x}_0')$ of the function $V(\mathbf{x}, \mathbf{x}')$ less than any arbitrarily small number and, furthermore, to place the point between the level lines of the surface (13) encircling the line $y_1 = 0$. Based on the identity $V'(\mathbf{x}, \mathbf{x}') \equiv 0$, we conclude that for all $t \geqslant t_0$, the point will not leave the indicated level lines of the surface (13).

Due to inequality (15), the uniform stability of the equilibrium $\mathbf{x} = \mathbf{x}' = \mathbf{0}$ of the point with respect to y_1 and $\mathbf{x}'$ is possible for sufficiently small values of z_{10} and $\|\mathbf{x}_0'\|$ and for $y_{10}^2 < 1/2$.

4. IDENTIFICATION OF SOME REFERENCE SYSTEMS

Using the proposed approach, we identify some structural forms of nonlinear systems of two and three equations for which the pairs under consideration can be constructed. Hereinafter, the functions $f_i(y_1)$ $(i = 1, 2)$ either have the form $d_i y_1$ with constants d_i or satisfy the conditions $|f_i(y_1)| \leqslant A|y_1|^{1+\alpha}$ with positive constants A and α in the domain $|y_1| < h_1$.

1. Let system (1) be composed of the equations

$$y_1' = ay_1 + ly_1^2 z_1, \quad z_1' = bz_1 + f_1(y_1)z_1 \tag{17}$$

with constant parameters a, b, and l.

We will establish conditions on the parameters under which the equilibrium $y_1 = z_1 = 0$ of system (17) is uniformly asymptotically y_1-stable.

By introducing the additional variable $\mu_1 = y_1 z_1$, one identifies the system

$$y_1' = ay_1 + ly_1\mu_1, \quad \mu_1' = (a+b)\mu_1 + l\mu_1^2 + \mu_1 f_1(y_1) \tag{18}$$

from the equations

$$y_1' = ay_1 + ly_1\mu_1, \quad \mu_1' = (a+b)\mu_1 + l\mu_1^2 + \mu_1 f_1(y_1), \quad z_1' = bz_1 + f_1(y_1)z_1.$$

If

$$a < 0, \quad a + b < 0, \tag{19}$$

then system (18) can be regarded as one of type (10) in the pair design scheme (Section 3) as applied to the original system (17): the equilibrium $y_1 = \mu_1 = 0$ of system (18) is asymptotically stable with respect to y_1 and μ_1 by Lyapunov's stability theorem based on the linear approximation.

The possibility of identifying system (18) and the admissible choice of $V = y_1^2 + \mu_1^2$ as a Lyapunov function for this system lead to the pair

$$\{V(\mathbf{x}) = y_1^2 + (y_1 z_1)^2,\ \mu_1(\mathbf{x}) = y_1 z_1\}. \tag{20}$$

The main Lyapunov function $V(\mathbf{x})$ in the pair (20) has the structural form (9), and under conditions (19), its derivative V' along the trajectories of system (17),

$$V' = 2\{ay_1^2 + (a+b)\mu_1^2(\mathbf{x}) + l\mu_1(\mathbf{x})[y_1^2 + \mu_1^2(\mathbf{x})] + \mu_1^2(\mathbf{x})f_1(y_1)\},$$

satisfies conditions (8) for $u_1 = [y_1, \mu_1(\mathbf{x})]$ in the domain (6), although it is *indefinite* in the domain (4).

As a result, under conditions (19) the equilibrium $y_1 = z_1 = 0$ of system (17) is uniformly asymptotically y_1-stable by Theorem 1. Moreover, the main function $V(y_1, z_1)$ in the pair (20) also satisfies the condition $V(0, z_1) \equiv 0$ and, consequently, uniform y_1-stability holds not only for sufficiently small values of y_{10} but also for large (finite) values of z_{10}.

The domain of attraction in system (18) can be estimated by known methods [22] using a Lyapunov function from a parametric class of quadratic forms in the variables y_1 and μ_1. In this case, on the plane (y_1, z_1) we obtain the corresponding domain of y_1-attraction of the equilibrium $y_1 = z_1 = 0$ of system (17).

2. Let system (1) be composed of the equations

$$\begin{gathered} y_1' = ay_1 + lz_1z_2, \\ z_1' = bz_1 + f_1(y_1)z_1, \quad z_2' = cz_2 + f_2(y_1)z_2 \end{gathered} \tag{21}$$

with constant parameters a, b, c, and l.

By introducing the variable $\mu_1 = z_1z_2$, one identifies the system

$$y_1' = ay_1 + l\mu_1, \quad \mu_1' = (b + c)\mu_1 + [f_1(y_1) + f_2(y_1)]\mu_1,$$

thereby predetermining the choice of the pair

$$\{V(\mathbf{x}) = y_1^2 + (z_1z_2)^2,\ \mu_1(\mathbf{x}) = z_1z_2\}.$$

Similar to case 1, we can demonstrate that the equilibrium $y_1 = z_1 = z_2 = 0$ of system (21) is uniformly asymptotically y_1-stable under the conditions $a < 0$ and $b + c < 0$. Moreover, uniform y_1-stability holds not only for sufficiently small values of y_{10} and z_{10} (z_{20}) but also for large (finite) values of z_{20} (z_{10}).

3. Let system (1) be composed of the equations

$$\begin{gathered} y_1' = ay_1 + ly_1^2z_1z_2, \\ z_1' = bz_1 + f_1(y_1)z_1, \quad z_2' = cz_2 + f_2(y_1)z_2. \end{gathered} \tag{22}$$

By introducing the variable $\mu_1 = y_1z_1z_2$, one identifies the system

$$y_1' = ay_1 + ly_1\mu_1, \quad \mu_1' = (a + b + c)\mu_1 + l\mu_1^2 + [f_1(y_1) + f_2(y_1)]\mu_1,$$

thereby predetermining the choice of the pair

$$\{V(\mathbf{x}) = y_1^2 + (y_1z_1z_2)^2,\ \mu_1(\mathbf{x}) = y_1z_1z_2\}.$$

Under the conditions $a < 0$ and $a + b + c < 0$, the equilibrium $y_1 = z_1 = z_2 = 0$ of system (22) is uniformly asymptotically y_1-stable. Moreover, uniform y_1-stability holds not only for sufficiently small values of y_{10} but also for large (finite) values of z_{10} and z_{20}.

4. Let system (1) be composed of the equations

$$\begin{gathered} y_1' = ay_1 + l_1y_1^2z_1 + l_2y_1z_1z_2, \\ z_1' = bz_1 + f_1(y_1)z_1, \quad z_2' = cz_2 + f_2(y_1)z_2. \end{gathered} \tag{23}$$

By introducing the variables $\mu_1 = y_1z_1$ and $\mu_2 = z_1z_2$, one identifies the system

$$\begin{gathered} y_1' = ay_1 + l_1y_1\mu_1 + l_2y_1\mu_2, \\ \mu_1' = (a + b)\mu_1 + l_1\mu_1^2 + \mu_1f_1(y_1) + l_2\mu_1\mu_2, \\ \mu_2' = (b + c)\mu_2 + [f_1(y_1) + f_2(y_1)]\mu_2, \end{gathered}$$

thereby predetermining the choice of the pair

$$\{V(\mathbf{x}) = y_1^2 + (y_1z_1)^2 + (z_1z_2)^2,\ \boldsymbol{\mu}(\mathbf{x}) = (y_1z_1, z_1z_2)\} \tag{24}$$

with a vector-valued auxiliary function $\boldsymbol{\mu}(\mathbf{x})$.

Under the conditions $a < 0$, $a + b < 0$, and $b + c < 0$, the equilibrium $y_1 = z_1 = z_2 = 0$ of system (23) is uniformly asymptotically y_1-stable. Moreover, uniform y_1-stability holds not only for sufficiently small values of y_{10} and z_{20} but also for large (finite) values of z_{10}.

5. Let system (1) be composed of the equations

$$\begin{gathered} y_1' = ay_1 + l_1y_1^2z_2 + l_2y_1z_1z_2, \\ z_1' = bz_1 + f_1(y_1)z_1, \quad z_2' = cz_2 + f_2(y_1)z_2. \end{gathered} \tag{25}$$

By introducing the variables $\mu_1 = y_1z_2$ and $\mu_2 = z_1z_2$, one identifies the system

$$\begin{gathered} y_1' = ay_1 + l_1y_1\mu_1 + l_2y_1\mu_2, \\ \mu_1' = (a + c)\mu_1 + l_1\mu_1^2 + \mu_1f_2(y_1) + l_2\mu_1\mu_2, \\ \mu_2' = (b + c)\mu_2 + [f_1(y_1) + f_2(y_1)]\mu_2. \end{gathered}$$

Under the conditions $a < 0$, $a + c < 0$, and $b + c < 0$, the equilibrium $y_1 = z_1 = z_2 = 0$ of system (25) is uniformly asymptotically y_1-stable. Moreover, uniform y_1-stability holds not only for sufficiently small values of y_{10} and z_{10} but also for large (finite) values of z_{20}.

The pair solving the problem is

$$\{V(\mathbf{x}) = y_1^2 + (y_1z_2)^2 + (z_1z_2)^2,\ \boldsymbol{\mu}(\mathbf{x}) = (y_1z_2, z_1z_2)\}.$$

5. USING REFERENCE SYSTEMS AS FIRST-APPROXIMATION SYSTEMS

The identified structural forms can be used as a nonlinear approximation for systems with polynomial nonlinearities.

1. Consider the perturbed system (17) composed of the equations

$$y_1' = ay_1 + ly_1^2z_1 + Y_1(t, \mathbf{x}), \quad z_1' = bz_1 + f_1(y_1)z_1 + Z_1(t, \mathbf{x}). \tag{26}$$

By introducing the variable $\mu_1 = y_1z_1$, the system can be transformed to

$$\begin{gathered} y_1' = ay_1 + ly_1\mu_1 + Y_1(t, \mathbf{x}), \quad \mu_1' = (a + b)\mu_1 + l\mu_1^2 + \mu_1f_1(y_1) + Y_1^*(t, \mathbf{x}), \\ z_1' = bz_1 + f_1(y_1)z_1 + Z_1(t, \mathbf{x}), \\ Y_1^*(t, \mathbf{x}) = y_1Z_1(t, \mathbf{x}) + z_1Y_1(t, \mathbf{x}). \end{gathered} \tag{27}$$

Using the analysis of system (17) and the pair (20), we can show that the equilibrium $y_1 = \mu_1 = z_1 = 0$ of system (27) is uniformly asymptotically (y_1, μ_1)-stable under conditions (19) and

$$|Y_1(t, \mathbf{x})| + |Y_1^*(t, \mathbf{x})| \leqslant \alpha_1|y_1| + \alpha_2|y_1z_1| \tag{28}$$

with sufficiently small positive constants α_1 and α_2 in the domain (6).

Therefore, under the above conditions, the equilibrium $y_1 = z_1 = 0$ of the original system (26) is uniformly asymptotically y_1-stable.

Condition (28) is valid, e.g., for $Y_1 = y_1^2z_1^2$, $Z_1 = y_1z_1^2$, and $Z_2 = y_1^2z_1$.

2. Consider the perturbed system (23) composed of the equations

$$\begin{gathered} y_1' = ay_1 + l_1 y_1^2 z_1 + l_2 y_1 z_1 z_2 + Y_1(t,\mathbf{x}), \\ z_1' = bz_1 + f_1(y_1)z_1 + Z_1(t,\mathbf{x}), \quad z_2' = cz_2 + f_2(y_1)z_2 + Z_2(t,\mathbf{x}). \end{gathered} \tag{29}$$

Using the analysis of system (23) and the pair (24), we can show that the equilibrium $y_1 = z_1 = 0$ of system (29) is uniformly asymptotically y_1-stable under the conditions $a < 0$, $a + b < 0$, $b + c < 0$, and

$$\begin{gathered} |Y_1(t,\mathbf{x})| + |Y_1^*(t,\mathbf{x})| + |Z_1^*(t,\mathbf{x})| \leqslant \alpha_1|y_1| + \alpha_2|y_1 z_1| + \alpha_3|z_1 z_2|, \\ Z_1^*(t,\mathbf{x}) = z_1 Z_2(t,\mathbf{x}) + z_2 Z_1(t,\mathbf{x}), \end{gathered} \tag{30}$$

with sufficiently small positive constants α_i in the domain (6).

Condition (30) is valid, e.g., in the case $Y_1 = y_1^2 z_1^2$, $Z_1 = y_1^2 z_1$, $Z_2 = y_1 z_1 z_2^2$.

6. CONCLUSIONS

This paper has considered systems of nonlinear ordinary differential equations with a continuous right-hand side. A possible approach has been proposed to find Lyapunov functions for the partial stability analysis of the trivial equilibrium of such systems (with respect to part of the variables). Within the approach, a main Lyapunov function–auxiliary function pair is chosen, where the latter function is used to correct the domain of the state space of the system where the former function is introduced.

A method has been presented to construct such pairs by transforming the original system and using partial stability conditions based on the first (linear or nonlinear) approximation.

In several illustrative examples, such pairs have been constructed for some nonlinear systems of two and three differential equations, with the main Lyapunov function defined as a quadratic form of the variables selected for stability analysis and additional variables. For these systems, partial stability domains have been found in the parameter space.

An analogous approach to partial stability analysis was considered for discrete-time systems [37] as well as for stochastic systems with continuous- and discrete-time dynamics [38–40].

REFERENCES

1. Lyapunov, A.M., Investigation of a Special Case of the Motion Stability Problem, *Mat. Sbornik*, 1893, vol. 17, no. 2, pp. 253–333.

2. Rumyantsev, V.V., On the Stability of Motion with Respect to Part of the Variables, *Vestn. Mosk. Gos. Univ. Mat., Mekh., Fiz., Astron., Khim.*, 1957, no. 4, pp. 9–16.

3. Rumyantsev, V.V. and Oziraner, A.S., *Ustoichivost' i stabilizatsiya dvizheniya po otnosheniyu k chasti peremennykh* (Stability and Stabilization of Motion with Respect to Part of the Variables), Moscow: Nauka, 1987.

4. Vorotnikov, V.I. and Rumyantsev, V.V., *Stability and Control with Respect to a Part of the Phase Vector Components of Dynamical Systems: Theory, Methods and Applications*, London: Science Publishing, 2004.

5. Vorotnikov, V.I., Partial Stability and Control: The State of the Art and Developing Prospects, *Autom. Remote Control*, 2005, vol. 66, no. 4, pp. 511–561.

6. Sontag, E.D. and Wang, Y., Notions of Input to Output Stability, *Syst. & Control Lett.*, 1999, vol. 38, no. 4–5, pp. 235–248.

7. Dashkovskiy, S.N., Efimov, D.V., and Sontag, E.D., Input to State Stability and Allied System Properties, *Autom. Remote Control*, 2011, vol. 72, no. 8, pp. 1579–1614.

8. Jammazi, C., On a Sufficient Condition for Finite-Time Partial Stability and Stabilization: Applications, *IMA J. Mathematical Control and Information*, 2010, vol. 27, no. 1, pp. 29–56.

9. Haddad, W.M. and L'Afflitto, A., Finite-Time Partial Stability and Stabilization, and Optimal Feedback Control, *J. Franklin Institute*, 2015, vol. 352, no. 6, pp. 2329–2357.

10. Zimenko, K., Efimov, D., Polyakov, A., and Kremlev, A., On Necessary and Sufficient Conditions for Output Finite-Time Stability, *Automatica*, 2021, vol. 125, art. no. 109427.

11. Zuyev, A.L., *Partial Stabilization and Control of Distributed Parameter Systems with Elastic Elements*, Cham: Springer, 2015.

12. Vorotnikov, V.I., *Partial Stability and Control*, Boston: Birkhauser, 1998.

13. Barbashin, E.A., *Funktsii Lyapunova* (Lyapunov Functions), Moscow: Nauka, 1970.

14. Giesl, P., *Construction of Global Lyapunov Functions Using Radial Basis Functions*, Berlin: Springer Verlag, 2007.

15. Hafstein, S., An Algorithm for Constructing Lyapunov Functions, *Electronic J. Diff. Equat.*, monograph 08, 2007.

16. Malisoff, M. and Mazenc, F., *Constructions of Strict Lyapunov Functions*, Springer: London, 2009.

17. Genesio, R., Tartaglia, M., and Vicino, A., On the Estimation of Asymptotic Stability Regions: State of the Art and New Proposals, *IEEE Trans. Autom. Control*, 1985, vol. AC-30, no. 8, pp. 747–756.

18. Giesl, P. and Hafstein, S., Review on Computational Methods for Lyapunov Functions, *Discrete Contin. Dyn. Syst. Ser. B*, 2015, vol. 20, no. 8, pp. 2291–2331.

19. Anderson, J. and Papachristodoulou, A., Advances in Computational Lyapunov Analysis Using Sum-of-Squares Programming, *Discrete Contin. Dyn. Syst. Ser. B*, 2015, vol. 20, no. 8, pp. 2361–2381.

20. Krasil'nikov, P.S., A Generalized Scheme for Constructing Lyapunov Functions from First Integrals, *J. Appl. Math. Mech.*, 2001, vol. 65, no. 2, pp. 195–204.

21. D'yachenko, I.V., Molchanov, A.P., and Pyatnitski, E.S., A Numerical Method for Constructing Lyapunov Functions and the Analysis of the Stability of Nonlinear Dynamical Systems on a Computer, *Autom. Remote Control*, 1994, vol. 55, no. 4, pp. 475–487.

22. Kamenetskii, V.A., Construction of Domains of Attraction by the Method of Lyapunov Functions, *Autom. Remote Control*, 1994, vol. 55, no. 6, pp. 778–791.

23. Rumyantsev, V.V., A Comparison of Three Methods of Constructing Lyapunov Functions, *J. Appl. Math. Mech.*, 1995, vol. 59, no. 6, pp. 873–877.

24. Johansen, T., Computation of Lyapunov Functions for Smooth, Nonlinear Systems Using Convex Optimization, *Automatica*, 2000, vol. 36, no. 11, pp. 1617–1626.

25. Marinosson, S., Lyapunov Function Construction for Ordinary Differential Equations with Linear Programming, *Dynamical Systems — an Intern. J.*, 2002, vol. 17, no. 2, pp. 137–150.

26. Grune, L., Computing Lyapunov Functions Using Deep Neural Networks, *J. Comput. Dyn.*, 2021, vol. 8, no. 2, pp. 131–152.

27. Zenkin, A.M., Peregudin, A.A., and Bobtsov, A.A., Lyapunov Function Search Method for Analysis of Nonlinear Systems Stability Using Genetic Algorithm, *Scientific and Technical Journal of Information Technologies, Mechanics and Optics*, 2023, vol. 23, no. 5, pp. 886–893.

28. Mendoza-Avila, J., Efimov, D., Ushirobira, R., and Moreno, J.A., Numerical Design of Lyapunov Functions for a Class of Homogeneous Discontinuous Systems, *Int. J. Robust Nonlinear Control*, 2021, vol. 31, no. 9, pp. 3708–3729.

29. Furtat, I.B., Divergent Stability Conditions of Dynamic Systems, *Autom. Remote Control*, 2020, vol. 81, no. 2, pp. 247–257.

30. Shchennikov, V.N., Investigation of Stability with Respect to a Part of Variables of Differential Systems with Homogeneous Right-Hand Sides, *Differential Equations*, 1984, vol. 20, no. 9, pp. 1184–1188.

31. Aleksandrov, A.Yu., Stability Analysis of Solutions for a Class of Complex Systems, *Vestnik St. Petersburg University: Mathematics*, 2011, vol. 44, no. 4, pp. 219–227.

32. Chetaev, N.G., *The Stability of Motion*, New York: Pergamon Press, 1961.

33. Matrosov, V.M., Anapol'skii, L.Yu., and Vasil'ev, S.N., *Metod sravneniya v matematicheskoi teorii sistem* (The Comparison Method in Mathematical System Theory), Novosibirsk: Nauka, 1980.

34. Pyatnitski, E.S., Design of Systems for the Stabilization of Programmed Motions of Nonlinear Control Plants, *Autom. Remote Control*, 1993, vol. 54, no. 7, pp. 1046–1062.

35. Propoi, A.I., On the Construction of Lyapunov Functions. I, *Autom. Remote Control*, 2000, vol. 61, no. 5, pp. 742–748.

36. Zhukov, V.P., On the Conditions Defining Stability Nature for Some Classes of Nonlinear Dynamic System, *Problemy Upravleniya*, 2007, no. 2, pp. 8–10.

37. Vorotnikov, V.I. and Martyshenko, Yu.G., Approach to the Stability Analysis of Partial Equilibrium States of Nonlinear Discrete Systems, *Journal of Computer and Systems Sciences International*, 2022, vol. 61, no. 3, pp. 348–359.

38. Vorotnikov, V.I. and Martyshenko, Yu.G., On the Partial Stability in Probability of Nonlinear Stochastic Systems, *Autom. Remote Control*, 2019, vol. 80, no. 5, pp. 856–866.

39. Vorotnikov, V.I. and Martyshenko, Yu.G., On the Problem of Partial Stability for Discrete-Time Stochastic Systems, *Autom. Remote Control*, 2021, vol. 82, no. 9, pp. 1554–1567.

40. Vorotnikov, V.I. and Martyshenko, Yu.G., Partial Stability in Probability of Nonlinear Stochastic Discrete-Time Systems with Delay, *Autom. Remote Control*, 2024, vol. 85, no. 8, pp. 761–772.

This paper was recommended for publication by A.A. Bobtsov, a member of the Editorial Board

ISSN 0005-1179 (print), ISSN 1608-3032 (online), Automation and Remote Control, 2026, Vol. 87, No. 9, pp. 859–862.
© The Author(s), 2026 published by Trapeznikov Institute of Control Sciences, Russian Academy of Sciences, 2026.
Russian Text © The Author(s), 2026, published in Avtomatika i Telemekhanika, 2026, No. 9, pp. 127–131.

NOTES, CHRONICLES, INFORMATION

The Problem of Optimal Iterative Learning

D. A. Novikov

Trapeznikov Institute of Control Sciences, Russian Academy of Sciences, Moscow, Russia
e-mail: novikov@ipu.ru

Received April 23, 2026
Revised May 26, 2026
Accepted June 1, 2026

Abstract—It is shown that three variants of the optimal iterative learning problem — maximizing the expected result for a fixed total number of trials and time; minimizing the time to achieve a given expected result; minimizing the number of trials required to achieve a given expected result — have either a degenerate (independent of the learning intensity dynamics) or a trivial (maximum permissible learning intensity) solution.

Keywords: iterative learning, learning curve, optimal learning intensity

DOI: 10.7868/S1608303226090059

1. INTRODUCTION

Learning in general is "the process and result of acquiring individual experience" [1–5]. We consider one of the types of learning, namely iterative learning (from the Latin *iterativus* — repetitive) — the repeated repetition by a learning system (living or non-living) of actions, trials, attempts, etc. to achieve a fixed goal under constant external conditions.

Iterative learning as learning in strictly repetitive conditions is one of the simplest types of learning. It occurs in a wide class of phenomena: the formation of various skills, the assimilation of information by humans, animal learning (conditioned reflex formation), and the training of technical and cybernetic systems. Various aspects of iterative learning are studied in pedagogy, psychology and physiology of humans and animals, control theory, and other sciences (see reviews in [4, 6]). Modern mathematical models of iterative learning and a corresponding review can be found in [2].

The constancy of external conditions allows for a quantitative description of iterative learning in the form of graphs — learning curves, which represent the dependence of the learning level criterion on time or on the number of repetitions (iterations) [4, 7]. As a rule, the learning level is interpreted as the probability of correctly performing an action or operation and accordingly takes values from zero to one.

2. PROBLEM STATEMENT

We consider a learning process in continuous time. The following assumptions are introduced.

A.1. The learning intensity $u(t) \geq 0$ is a continuous, non-negative, and bounded above function of time:

$$0 \leq u(t) \leq u_{\max}(t).$$

A.2. Learning is described by a continuous learning curve $L(U(t))$ — a mapping of the total number of attempts, trials, or repetitions made by time t into the interval $[0; 1]$.

A.3. The total number of attempts $U(T) > 0$ that the learning system can make over time $T > 0$ is fixed. Moreover,

$$U(t) = \int_0^t u(\tau)\, d\tau, \quad U(0) = 0.$$

Obviously, $u(t) = U'(t)$.

A.4. The expected number of successful attempts at time t is equal to the product of the learning intensity and the current learning level: $u(t)L(U(t))$. Then the expected number of successful attempts $Z(T)$ over time T is equal to

$$Z(T) = \int_0^T U'(t)\, L(U(t))\, dt. \tag{1}$$

The optimal iterative learning problem consists of maximizing the expected number of successful attempts by choosing the learning intensity dynamics (in [5] this problem was formulated in discrete time):

$$\int_0^T U'(t)\, L(U(t))\, dt \to \max_{U(\cdot)}. \tag{2}$$

3. MAIN RESULT

Let us make a change of variables in the integral (1):

$$Z(T) = \int_0^T U'(t)\, L(U(t))\, dt = \int_{U(0)}^{U(T)} L(U)\, dU.$$

Denote by $W(\cdot)$ the anti-derivative of the learning curve $L(\cdot)$. Then

$$Z(T) = W(U(T)) - W(0). \tag{3}$$

From (3) it is clear that the expected number of successful attempts *does not depend* on the learning intensity dynamics, but is determined only by the shape of the learning curve and the total number of attempts $U(T)$.

Interpretation: in iterative learning (fixed goal, constant external conditions), the learning level depends only on the total number of attempts made and does not depend on their distribution over time. The effects of fatigue, forgetting, warm-up, etc. [3] are not taken into account in this model.

Statement. *If assumptions A.1–A.4 hold, then the value of the efficiency criterion in the optimal iterative learning problem* (2) *does not depend on the learning intensity dynamics.*

4. EXAMPLE: EXPONENTIAL LEARNING CURVE

The most common example of a learning curve is the exponential curve:

$$L(U) = 1 - \exp[-\gamma U(t)], \quad \gamma > 0,$$

where γ is the learning rate. If the intensity u is constant over time, the classical curve $1 - e^{-gt}$ is obtained, with the reduced learning rate $g = \gamma u$.

For the exponential learning curve, expression (1) takes the form

$$Z(T) = \int_0^T U'(t)\big[1 - e^{-\gamma U(t)}\big]dt = U(T) - \int_0^T U'(t)e^{-\gamma U(t)}dt. \tag{4}$$

Problem (2) taking into account (4) reduces to minimizing the number of unsuccessful attempts:

$$\int_0^T U'(t)e^{-\gamma U(t)}dt \to \min_{U(t)}. \tag{5}$$

For the exponential learning curve, we can make the substitution $v(t) = e^{-\gamma U(t)}$, $U(t) = -\frac{1}{\gamma}\ln v(t)$. Then (5) transforms to

$$-\frac{1}{\gamma}\int_0^T v'(t)\,dt \to \min,$$

which is equivalent to $\frac{1}{\gamma}\big[1 - e^{-\gamma U(T)}\big] \to \min$, which also does not depend on the form of $u(\cdot)$.

5. TWO ADDITIONAL PROBLEM FORMULATIONS

By analogy, the following two variants of optimization problems are possible.

(1) **Minimizing time** T to achieve a given expected result Z. By virtue of (3) and the monotonicity of the learning curve, the solution is trivial: the learning time is minimal provided that at each instant the learning intensity is maximal:

$$u(t) \equiv u_{\max}(t).$$

(2) **Minimizing the number of attempts** U required to achieve a given expected result Z. By virtue of (3), this problem has a degenerate solution (independent of the learning intensity dynamics). In substantive terms: the learning system must inevitably make a certain number of errors; otherwise, learning is impossible.

6. OPTIMAL CONTROL PROBLEM (OCP)

6.1. Reduction to an OCP

We introduce state variables $x_1 = U$ and $x_2 = u$. Then the two-point boundary value problem has the form

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = w, \quad w \in [0, 1],$$

$c = (0, ?)^T$, $d = (?, ?)^T$, $-\int_0^T x_2(t)L(x_1(t))\,dt \to \min$.

6.2. Application of Pontryagin's Maximum Principle (PMP)

The problem is autonomous, T is fixed, and the system is linear. Let us apply the local maximum principle. The Pontryagin function with $\lambda_0 = 1$ has the form

$$H = \psi_1 x_2 + \psi_2 w + x_2(t)L(x_1(t)).$$

The adjoint system:

$$\dot{\psi}_1(t) = -x_2(t)L_{x_1}(x_1(t)), \qquad \dot{\psi}_2(t) = -\psi_1(t) - L(x_1(t)).$$

It follows that $\psi_1(t) = L(x_1(0)) - L(x_1(t))$, and $\dot{\psi}_2(t) = -L(x_1(0))$, which is a constant. Then $\psi_2(t) = -L(x_1(0))t + a$ is a linear function.

The optimal control is $u^* = \operatorname{sgn}(\psi_2)$. Since ψ_2 is linear, there is at most one switching of the control. However, since the right endpoint with respect to x_2 is not fixed, the transversality condition implies that $\psi_2(T) = 0$. Hence, there are no control switchings, and only trivial solutions of the OCP exist.

7. CONCLUSION

All three variants of the optimal iterative learning problem have either a degenerate (independent of the learning intensity dynamics) or a trivial (maximum permissible learning intensity) solution. To obtain non-trivial solutions, it is necessary to incorporate into the model the effects of fatigue, forgetting, as well as possibly the limited cognitive and other capabilities of the learning system.

REFERENCES

1. *Kratkii psikhologicheskii slovar'* (A Concise Psychological Dictionary), Moscow: Politizdat, 1985.
2. Belov, M.V. and Novikov, D.A., *Models of Technologies*, Heidelberg: Springer, 2020.
3. Grebenkov, D.I., Kozlova, A.A., Lemtyuzhnikova, D.V., et al., Models of fatigue and rest in learning. Part 1, *Control Sciences*, 2025, no. 3, pp. 24–31.
4. Novikov, D.A., *Zakonomernosti iterativnogo naucheniya* (Patterns of Iterative Learning), Moscow: Institute of Control Sciences of the Russian Academy of Sciences, 1998.
5. Novikov, D.A., Models of learning on the job, *Large-Scale Syst. Control*, 2007, no. 19, pp. 5–22.
6. *Learning Curves: Theory, Models, and Applications*, Jaber, M.Y., Ed., Boca Raton: CRC Press, 2011.
7. Qi, Z., *Learning curve*, in *The ECPH Encyclopedia of Psychology*, Kan, Z., Ed., Singapore: Springer, 2024, pp. 826–828.

This paper was recommended for publication by A.A. Galyaev, a member of the Editorial Board