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Multi-Criteria Problems with Groups of Criteria Ordered by Importance (II). Decision Rules

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Abstract—For multi-criteria decision-making problems, by analogy with qualitative probability, the concepts of complete and partial qualitative importance are introduced as binary relations satisfying certain postulated properties. A new definition of a non-strict preference relation on the set of decision alternatives, generated by qualitative importance, is proposed, and its properties are examined. Analytical rules for pairwise comparison of alternatives in terms of preference are provided. The new preference relations are compared with those developed earlier for problems in which only individual criteria are initially ordered by importance. It is shown that when all criteria are completely ordered by importance, the two preference relations coincide.

Keywords: multi-criteria decision-making problems, ordering criteria by importance, criteria importance theory

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1. INTRODUCTION

This article is a continuation of the authors' work [1], which provided a survey of the literature on criteria importance theory (CIT) for problems in which groups of criteria are ordered by importance, demonstrated the relevance of the theory, and outlined directions for further research. Based on recent results obtained for single-criterion decision-making problems under uncertainty with qualitative probability and ordinal utility [2], extended preference and indifference relations are introduced in this article for multi-criteria problems with ordinal criteria on the set of vector outcomes (and thus on the set of decision alternatives). The properties of these relations are examined, and simple decision rules for pairwise comparison of alternatives in terms of preference are provided. The relationship between the new non-strict preference relations and the relations previously used for problems in which only individual criteria are ordered by importance is studied in detail.

2. MATHEMATICAL MODEL AND BASIC DEFINITIONS FROM THE CRITERIA IMPORTANCE THEORY

Consider the following mathematical model of the decision-making situation [1]:

$$\langle X, f, Z_0, R \rangle ,$$

here X is the set of solution options; $f = (f_1, \ldots, f_m)$ is a vector criterion consisting of $m \geqslant 2$ individual criteria f_i; $Z_0 \subseteq (-\infty, +\infty)$ is the range of criteria values (the "criteria scale"); R is the decision makers (DM) non-strict preference. Each criterion f_i is a function defined on X

and taking values from Z_0. Each option x from the set X is characterized by its vector estimate $y(x) = f(x) = (f_1(x), \ldots, f_m(x))$. The set of all vector estimates is $Z = Z_0^m$. Further, it is assumed that the criteria scale is ordinal: it is only known that preferences increase along their scale: the higher the estimate $z \in Z_0$, the more preferable it is.

The DM's preferences are modeled on Z using a non-strict preference relation R (yRz means that the vector estimate y is no less preferable than z). The relation R is a (partial) quasi-order, i.e., it is reflexive and transitive, and generates an indifference relation I and a (strict) preference relation P as follows: $yIz \Leftrightarrow yRz \wedge zRy$; $yPz \Leftrightarrow yRz \wedge \neg zRy$ (i.e., zRy is not true). The relation R is unknown and must be reconstructed based on information about the DM's preferences. In the absence of such information, only the strict preference relation $P^{\varnothing}$ (the Pareto relation) can be defined on the set of vector estimates Z: $yP^{\varnothing}z \Leftrightarrow (y_i \geqslant z_i, i = 1, \ldots, m, y \neq z)$. Let $R^{\varnothing}$ be the union of $P^{\varnothing}$ and the vector equality relation.

To extend the relation $P^{\varnothing}$, additional information Ω about the DM's preferences is required, namely, information about the importance of criteria, for example, a message like "One group of criteria is more important than another". Let us provide basic definitions of equality and superiority in importance for disjoint groups of criteria. For simplicity, we will further denote criteria by their numbers, i.e., for example, the criterion f_i will be denoted as i, and the group (set) of criteria with numbers $i_1, \ldots, i_q$ will be denoted as $A = \{i_1, \ldots, i_q\}$.

Let A and B be disjoint nonempty groups of criteria, and in a vector estimate y all components y_i with numbers from A are equal to each other and all components y_i with numbers from B are equal to each other (i.e., $y_i = y_j$ for $i, j \in A$ and for $i, j \in B$). Denote y^{AB} vector estimates with such a structure. Denote $y^{A\leftrightarrow B}$ the vector estimate obtained from y^{AB} by replacing each component y_j, $j \in B$, with (any) component y_i, $i \in A$, and replacing each component y_i, $i \in A$, with (any) component y_j, $j \in B$.

Definition 1. Criteria groups A and B are equally important ($A \sim B$) when all vector estimates y^{AB} and $y^{A\leftrightarrow B}$ are equal in preference (indifferent).

Definition 2. A group of criteria A is more important than a group of criteria B ($A \succ B$) when any vector estimate y^{AB} such that $y_i > y_j$ for $i \in A$ and $j \in B$ is preferable to $y^{A\leftrightarrow B}$.

According to these definitions, the messages $A \sim B$ and $A \succ B$ define on the set Z the indifference relation $I^{A\sim B}$ and preference relation $P^{A\succ B}$, respectively:

$$\begin{aligned} yI^{A\sim B}z &\Leftrightarrow y = y^{AB}, \quad z = y^{A\leftrightarrow B}, \quad y_i \neq y_j, \quad i \in A, \quad j \in B; \\ yP^{A\succ B}z &\Leftrightarrow y = y^{AB}, \quad z = y^{A\leftrightarrow B}, \quad y_i > y_j, \quad i \in A, \quad j \in B. \end{aligned} \tag{1}$$

On the other hand, information Ω defines on the set 2^M of all subsets of the set of criteria (numbers) $M = \{1, \ldots, m\}$ the importance relation $\succsim^{\Omega}$:

If $A \sim B \in \Omega$ or $B \sim A \in \Omega$ then $A \sim^{\Omega} B$; if $A \succ B \in \Omega$ then $A \succ^{\Omega} B$.

Each importance statement $A \sim^{\Omega} B$ and $A \succ^{\Omega} B$ generates on the set of vector estimates Z according to (1) a corresponding relation R^{ω}, namely the preference relation P^{ω} or indifference relation I^{ω}. Based on this information, the non-strict preference relation R^{Ω} (a reflexive and transitive relation) is defined on the set Z as the transitive closure of the union of all relations R^{ω} and the relation $R^{\varnothing}$:

$$R^{\Omega} = \mathrm{T}\left[\left(\bigcup_{\omega \in \Omega} R^{\omega}\right) \bigcup R^{\varnothing}\right], \tag{2}$$

where T is the symbol of the transitive closure operation of a binary relation. According to (2) the relation $yR^{\Omega}z$ is true if and only if there exists a chain of the form

$$yR^{\omega_1}u^1, u^1R^{\omega_2}u^2, \ldots, u^{r-1}R^{\omega_r}z, \tag{3}$$

where u^k are vector estimates, and ω_k are messages from Ω (so that R^{ω_k} is P^{ω_k} or I^{ω_k} depending on the meaning of ω_k) or the symbol $\varnothing$. The relation of non-strict preference R^{Ω} generates the relations of preference P^{Ω} and indifference I^{Ω} in the manner indicated above.

3. QUALITATIVE IMPORTANCE OF CRITERIA

Qualitative importance is a binary relation $\succsim$ on the set 2^M of all subsets of the set of criteria (numbers) $M = \{1, \ldots, m\}$: the notation $A \succsim B$ means that the group of criteria A is no less important than the group of criteria B. This relation gives rise to the relations of strict superiority in importance $\succ$ and equal importance $\sim$. Qualitative importance is called *complete*, if it has the following properties (they are postulated as axioms):

C1. *Comparability*: $A \succsim B$ or $B \succsim A$ is true for any A and B.

C2. *Transitivity*: from $A \succsim B$ and $B \succsim C$ it follows $A \succsim C$.

C3. *Nontriviality*: $A \succ \varnothing$ for any $A \neq \varnothing$.

C4. *Additivity*: if $A \cap C = B \cap C = \varnothing$, then the relations $A \succsim B$ and $A \cup C \succsim B \cup C$ are either true or not true simultaneously.

Note that the validity of properties C1–C4 implies the validity of property

C5. *Reflexivity*: $A \succsim A$ for any A.

Qualitative importance is called *partial*, if it has properties C2–C5.

It should be noted that the postulated properties of qualitative importance are similar to the axioms of qualitative probability. (A detailed discussion of the properties of qualitative probability which are often unusual and far from obvious can be found in [3].) The difference lies only in the formulation of the axiom of nontriviality: qualitative probability requires only non-strict superiority for each event A, and therefore $A \sim \varnothing$ is allowed for $A \neq \varnothing$. This is explained by the fact that the axiom of nontriviality for probability covers cases of im-possible elementary events encountered in practice. For qualitative importance, statement $A \sim \varnothing$ means taking into account a criterion that has no significance at all. But if a set of criteria is formed in compliance with known requirements, which include non-redundancy and minimal dimension, then such a result should be considered as a violation of the specified requirements. This would mean adjusting the mathematical model of the problem situation by removing the corresponding criterion from the vector criterion. Therefore, for qualitative importance, it is natural to accept the requirement that $A \succ \varnothing$ for each $A \neq \varnothing$ (i.e., for every criterion).

Positive numbers $\beta_1, \ldots, \beta_m$ are called *qualitative (additive) importance values of criteria*, consistent with the qualitative importance $\succsim$, if they have the following property:

$$A \sim B \Rightarrow \sum_{i \in A} \beta_i = \sum_{i \in B} \beta_i; \qquad A \succ B \Rightarrow \sum_{i \in A} \beta_i > \sum_{i \in B} \beta_i.$$

If the importance values of criteria are normalized, i.e., equal to 1 in total, then they are called *qualitative importance coefficients of criteria* and are designated $\alpha_1, \ldots, \alpha_m$. The theory of qualitative probability indicates that for $m > 4$ qualitative importance values (and, of course, coefficients) may not exist [2, 3].

The complete or partial importance of criteria can be assessed by carrying out (with varying degrees of completeness) a procedure of pairwise comparisons of non-empty disjoint groups of criteria A and B, comparing by preference the corresponding vector estimates of the form y^{AB} and $y^{A \leftrightarrow B}$ and using the same considerations and procedure as when obtaining information on the qualitative importance of individual criteria. The procedure is described in detail in [3]. Having received a sufficient amount of information Ω, it is necessary to check it for consistency and, if necessary, correct it [1]. And, finally, construct $\succsim$ as an additive-transitive closure of the importance relation $\succsim^{\Omega}$, for example, using the matrix method specified in [1].

4. PREFERENCE AND INDIFFERENCE RELATIONS

Let us first extend the definitions of equality and superiority in importance to the general case where criteria groups A and B may intersect or be empty. Consider a vector estimate $y \in Z$ such that $y_i = a$ for $i \in A \cup B$, i.e., all of its components with numbers from A and B are equal to $a \in Z_0$. Let $b \in Z_0$ and $y' = (y \parallel y_i = b, i \in A) \in Z$ be the vector estimate obtained from y by replacing all of its components with numbers from A with b, and $y'' = (y \parallel y_j = b, j \in B) \in Z$ be the vector estimate obtained from y by replacing all of its components with numbers from B with b.

Definition 3. Criteria groups A and B are equally important ($A \sim B$), when all vector estimates y' and y'' are equally preferable.

Definition 4. A non-empty group of criteria A is more important than a group of criteria B ($A \succ B$), when any vector estimate y' with $b > a$ is preferable to a vector estimate y''.

Note that according to Definition 3, $A \sim A$ is true, and according to Definition 4, if $A \supset B$, then $A \succ B$; in particular, if $A \neq \varnothing$, then $A \succ \varnothing$.

We will introduce the definition of the relations $R^{\succsim}$, and thus the relations $P^{\succsim}$ and $I^{\succsim}$ on the set of vector estimates $Z = Z_0^m$ based on a given qualitative importance $\succsim$ (complete or only partial) axiomatically, i.e., we will specify a system of axioms that these relations must satisfy. First outline the class of binary relations.

Axiom 1. The relation $R^{\succsim}$ is reflexive.

Axiom 2. The relation $R^{\succsim}$ is transitive.

If these axioms are satisfied for $R^{\succsim}$, then the relation $P^{\succsim}$ is strict partial order, and the relation $I^{\succsim}$ is equivalence.

Axiom 3 (Pareto axiom). If the first vector estimate is not less than the second vector estimate component-wise, and at least one component of the first is greater than the corresponding component of the second, then the first vector estimate is preferable to the second: if $y_i \geqslant z_i$, $i = 1, \dots, m$, $y \neq z$, then $yP^{\succsim}z$.

Consider two groups of criteria A, B and two vector estimates $y, z \in Z$ such that $y_i = z_j = a$, $i \in A$, $j \in B$, i.e., all their components with numbers from A and B are equal to $a \in Z_0$. Let $b \in Z_0$ and $y' = (y \parallel y_i = b, i \in A) \in Z$ and $z' = (z \parallel z_j = b, j \in B) \in Z$. We will say that vector estimates y and z *are changed (respectively improved, worsened) in the same way* according to groups of criteria A and B if they are replaced by y' and z' for arbitrary b and a (respectively for $b > a$, for $b < a$).

Axiom 4 (preference conservation). Let two vector estimates be improved in the same way according to two equally important groups of criteria A and B; if these vector estimates are of equal preference, then they will remain so; if the first vector es-timate is preferable to the second, then it will remain preferable to the second: for $A \sim B$ and $a < b$, if $yI^{\succsim}z$ then $y'I^{\succsim}z'$, and if $yP^{\succsim}z$ then $y'P^{\succsim}z'$.

Axiom 5 (preference strengthening). If two vector estimates, the first of which is no less preferable than the second, are improved in the same way according to two groups of criteria, the first of which is more important than the second, then the first of the obtained vector estimates will be preferable to the second: for $A \succ B$ and $a < b$, if $yR^{\succsim}z$ then $y'P^{\succ}z'$.

Axioms 4 and 5 express the following simple idea, which is most clearly formulated for the case of two criteria. Suppose it is possible to change the preference of each of the two vector estimates by changing the initial identical values of only one of the two criteria by the same amount. If the criteria are equally important, then it makes no difference which criterion is changed. If the first criterion is more important than the second, then it is preferable to improve the value of the more important criterion and worsen the value of the less important criterion. In an even more general form, the idea is well known: if two problems are being solved (for example, the assault of two strongholds) and the completeness of their solutions is equal, then it is reasonable to devote additional indivisible resources (reserves) to solving the more important problem. If the problems are equally important, then both methods of using resources are equally preferable.

The system of Axioms 1–5 provides a recursive (inductive) definition of relations $R^{\succsim}$, $P^{\succsim}$, $I^{\succsim}$, starting with relations $yI^{\succsim}y$, which are true for any $y \in Z$.

Example 1. Assume that $Z_0 = \{1, 2, 3, 4\}$ and the following consistent information about the importance of $m = 3$ criteria is provided: $1 \succ 2$, $2 \sim 3$, $\{2, 3\} \succ 1$. Using the additive-transitive closure of these relations the following complete qualitative importance is obtained:

$$\{1,2,3\} \succ \{1,2\} \sim \{1,3\} \succ \{2,3\} \succ 1 \succ 2 \sim 3 \succ \varnothing.$$

Start with $y = (2, 2, 2)$. Since $\{2, 3\} \succ 1$, then $(2, 3, 3)P^{\succsim}(3, 2, 2)$ according to Axiom 5. And since $1 \succ 2$, then $(4, 3, 3)P^{\succsim}(3, 4, 2)$. It is convenient to represent these conclusions as a double chain (the arrows are drawn from the more preferable vector estimates to the less preferable ones), because it is necessary to change the vector estimates in each pair in the same way:

$$\begin{array}{ccccc} (2,2,2) \leftarrow & (2,3,3) & \leftarrow & (4,3,3) \\ & \downarrow_{23\succ 1} & & \downarrow_{1\succ 2} \\ (2,2,2) \leftarrow & (3,2,2) & \leftarrow & (3,4,2)\,. \end{array}$$

Therefore $(4, 3, 3)P^{\succsim}(3, 4, 2)$. *Double chains* like the one presented can be called *explanatory*: they clearly show why one of the vector estimates is preferable to the other (or why they are indifferent).

If the groups A and B are not empty and do not intersect, then for the vector estimates involved in the formulations of Axioms 4 and 5, $z' = y'^{A\leftrightarrow B}$ is satisfied in case $y = z$. Consequently, in the general case, the relation$R^{\succsim}$ includes the relation R^{Ω} (see (2)).

Remark. The deterioration of vector estimates can also be taken into account in the following analogs of Axioms 4 and 5:

Axiom 4*: for $A \sim B$ and $a > b$, if $yI^{\succsim}z$ then $y'I^{\succsim}z'$, and if $yP^{\succsim}z$ then $y'P^{\succsim}z'$.

Axiom 5*: for $A \prec B$ and $a > b$, if $yR^{\succsim}z$ then $y'P^{\succsim}z'$.

It can be shown that the system of Axioms 1, 2, 3, 4*, 5* is equivalent to the system of Axioms 1–5 in the sense that it defines the same relations $R^{\succsim}$, $P^{\succsim}$, $I^{\succsim}$. Double chains using Axioms 4* and 5* are *decreasing*. Thus, Axioms 4* and 5* can be used instead of Axioms 4 and 5. The combined use of Axioms 4, 5, 4*, and 5* does not extend the relations $R^{\succsim}$, $P^{\succsim}$, $I^{\succsim}$.

The decision rules presented below are based on the development of ideas from the theory of qualitative importance of criteria in multi-criteria decision-making problems [1]. Introduce the set of components of vector estimates y and z, ordered in non-decreasing order:

$$W(y,z) = \{y_1\} \cup \cdots \cup \{y_m\} \cup \{z_1\} \cup \cdots \cup \{z_m\} = \{w_1, \ldots, w_q\},$$

where $q \leqslant 2m$ and depend on y and z, $w_1 < \cdots < w_q$. Define $2(q-1)$ sets of component numbers:

$$M_k(y) = \{i \in M \mid y_i \geqslant w_k\}, \quad M_k(z) = \{i \in M \mid z_i \geqslant w_k\}, \quad k = 2, 3, \ldots, q.$$
$$L_k(y) = \{i \in M \mid y_i \leqslant w_k\}, \quad L_k(z) = \{i \in M \mid z_i \leqslant w_k\}, \quad k = q-1, q-2, \ldots, 1.$$

The following two theorems are reformulations of the corresponding theorems from [2] as applied to multicriteria problems.

Theorem 1. *The relation $yR^{\succsim}z$ is true if and only if all of the following relations are true*

$$M_k(y) \succsim M_k(z), \quad k = 2, 3, \ldots, q. \tag{4}$$

Furthermore, if in (4) *all* $\succsim$ *are* $\sim$, *then* $yI^{\succsim}z$; *otherwise* (*when* (4) *contains* $\succ$) $yP^{\succsim}z$ *is true.*

Theorem 2. *The relation $yR^{\succsim}z$ is true if and only if all of the following relations are true*

$$L_k\left(y\right) \precsim L_k\left(z\right), \quad k = q-1, q-2, \ldots, 1. \tag{5}$$

Furthermore, if in (5) *all* $\precsim$ *are* $\sim$, *then* $yI^{\succsim}z$; *otherwise* (*when* (5) *contains* $\prec$) $yP^{\succsim}z$ *is true.*

Note that if we have a set $Z_0 = \{1, \ldots, q\}$, then it can be used in the role of $W(y, z)$.

Example 2. Under the conditions of Example 1 compare the vector estimates $y = (4, 3, 3)$ and $z = (3, 4, 2)$ by preference using Theorem 1. We have $W(y, z) = \{2, 3, 4\}$, $q = 3$ and

$$k = 2, \;\; w_2 = 3, \;\; M_2(y) = \{1, 2, 3\} \succ \{1, 2\} = M_2(z);$$
$$k = 3, \;\; w_3 = 4, \;\; M_3(y) = \{1\} \succ \{2\} = M_3(z).$$

Therefore $yP^{\succsim}z$ is true. Now compare the vector estimates $u = (1, 2, 3)$ and $v = (3, 1, 1)$ using Theorem 2. We have $W(u, v) = \{1, 2, 3\}$, $q = 3$ and

$$k = 2, \;\; w_2 = 2, \;\; L_2(u) = \{1, 2\} \succ \{2, 3\} = L_2(v);$$
$$k = 1, \;\; w_1 = 1, \;\; L_1(u) = \{1\} \prec \{2, 3\} = L_1(v).$$

Therefore, the vector estimates u and v are not comparable in preference.

Let the criteria $f_1, \ldots, f_m$ have a quantitative "physical" scale, which is interval [4] or of a higher level (for example, monetary income), and $\alpha_1, \ldots, \alpha_m$ are qualitative importance coefficients consistent with the qualitative importance $\succsim$. Construct a weighted sum

$$\sum_{i=1}^{m} \alpha_i \varphi\left(f_i\left(x\right)\right) = \sum_{i=1}^{m} \alpha_i \varphi\left(y_i\right),$$

where φ is an increasing function (in generalized criteria it is called a normalizing function). The value of the weighted sum is briefly denoted as $y(\alpha)$.

Theorem 3. *If $yR^{\succsim}z$ then $y(\alpha) \geqslant z(\alpha)$. Furthermore, if $yP^{\succsim}z$ then $y(\alpha) > z(\alpha)$, and if $yI^{\succsim}z$ then $y(\alpha) = z(\alpha)$.*

The proofs of this and the following theorems are given in the Appendix. Theorem 3 shows the possibility of using, if necessary, additive value functions (in particular, generalized criteria) for further analysis of a multi-criteria problem solved using the CIT.

5. PROBLEMS WITH INDIVIDUAL CRITERIA ORDERED BY IMPORTANCE

Consider problems in which only individual criteria are comparable in importance, i.e., the set Ω contain only messages of the form $i \succ j$ and $i \sim j$. A binary relation $\dot{\succsim}$ on the set of criteria (numbers) M that satisfies the axioms of reflexivity and transitivity is referred to as *ordinal importance*. If all criteria are ordered, then the importance is complete; otherwise, it is partial.

Assume that $A, B \in M$, in the formulation of the Axioms 1–5. Then the system of Axioms 1–5 on the basis of a given ordinal importance $\dot{\succsim}$ (complete or only partial) defines the relation $R^{\dot{\succsim}}$ on the set of vector estimates Z. It turns out that this relation coincides with $R^{\succsim}$ in the case when the information Ω concerns the importance of only individual criteria, i.e., when the qualitative importance $\succsim$ is an extension (continuation) of the ordinal importance $\dot{\succsim}$ on 2^M.

Theorem 4. $R^{\succsim} = R^{\dot{\succsim}}$ *in the case of ordinal importance.*

The following example shows that the relations $R^{\succsim}$ and $R^{\dot{\succsim}}$ for the qualitative importance $\succsim$ and ordinal importance $\dot{\succsim}$ generated by information Ω can be wider than the relation R^{Ω}.

Example 3. Let $m = 4, Z_0 = \{1, 2, 3, 4\}$ and $\Omega = \{1 \succ 3, 2 \succ 3, 2 \succ 4\}$. It is easy to verify, that transitive closure does not change the relation $\succsim^{\Omega}$. We will not write out the partial relation of qualitative importance $\succsim$: the validity of the relations that arise during the analysis for the pairs of sets under consideration (taking into account the properties of C2 and C4) will be obvious. For $y = (3, 4, 1, 2)$ and $z = (2, 1, 4, 3)$ we have $W(y, z) = Z_0,\ q = 4$. Taking into account Theorem 1, we obtain:

$$k = 2,\ \ w_2 = 2:\ M_2(y) = \{1, 2, 4\} \succ \{1, 3, 4\} = M_2(z);$$
$$k = 3,\ \ w_3 = 3:\ M_3(y) = \{1, 2\} \succ \{3, 4\} = M_3(z);$$
$$k = 4,\ \ w_4 = 4:\ M_4(y) = \{2\} \succ \{3\} = M_4(z),$$

so $yP^{\succsim}z$ is true. According to Theorem 4, $yP^{\dot{\succsim}}z$ is also true. However, it is impossible to construct an explanatory chain (3), so y and z are not comparable with respect to R^{Ω}.

It turns out, however, that in the case where information Ω concerns the importance of only individual criteria and on its basis all criteria can be ordered by importance, the relations $R^{\succsim}$ and R^{Ω} coincide.

Theorem 5. *$R^{\succsim} = R^{\Omega}$ in the case of complete ordinal importance.*

Example 4. Assume that additional information $4 \succ 1$ became known in Example 3. Then all criteria are ordered by importance: $2 \succ 4 \succ 1 \succ 3$. Now the explanatory chain (3) appears:

$$y = (3, 4, 1, 2)P^{1\succ 3}(1, 4, 3, 2)P^{2\succ 3}(1, 3, 4, 2)P^{2\succ 4}(1, 2, 4, 3)P^{2\succ 1}(2, 1, 4, 3) = z,$$

so $yP^{\Omega}z$ is true. The same result is obtained by adding the statement $4 \sim 1$ to Ω, which also allows us to completely non-strictly order all criteria by importance: $2 \succ 4 \sim 1 \succ 3$.

Thus, Theorem 5 shows that in the case of complete ordinal importance, the non-strict preference relation introduced earlier in the CIT is not extended, and therefore any suitable previously developed decision rules from this theory can be used.

6. CONCLUSION

This article introduces the concepts of complete and partial qualitative importance and axiomatically defines the non-strict preference relation on a set of decision options based on these concepts. In general, this relation is broader than a similar relation previously introduced in the criteria importance theory (CIT).

The presented fairly simple decision rules allow us to analyze multi-criteria problems with the information on the relative importance of not only individual criteria, but also groups of criteria with an ordinal scale.

It is shown that in the case of a complete ordering of individual criteria by importance, the non-strict preference relation introduced earlier in the CIT is not expanded, and any suitable previously developed decision rules from this theory can be used.

The obtained results form the basis for further development of the CIT for problems with groups of criteria ordered by importance, having more sophisticated scales than the ordinal one.

The presented results are useful for the initial analysis of political, social, and other decision-making problems in which preference information (criteria importance and their scales) is initially qualitative (non-numerical).

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APPENDIX

A.1. PROOF OF THEOREM 3

If $yR^{\succsim}z$ is true, then there exists a double chain in which the first node consists of two equal vector estimates $y^0 = z^0$, and the last node consists of y and z.

Consider two arbitrarily chosen adjacent nodes of this double chain, where the vector estimates s and t are involved in the first node, and the relation $A \succsim B$ is used to obtain vector estimates u and v in the second node. So, we have $\sum_{i\in A}\alpha_i \geqslant \sum_{i\in B}\alpha_i$ or $\sum_{i\in A\setminus B}\alpha_i - \sum_{i\in B\setminus A}\alpha_i \geqslant 0$, where the inequalities are strict in case $A \succ B$, and become equalities when $A \sim B$. The components of the vector estimates u and v with numbers from A and B are represented by some same number c, so that:

$$u_i = \begin{cases} c+l, & i\in A,\\ c, & i\in B\setminus A,\\ s_i, & i\notin A\cup B; \end{cases} \qquad v_i = \begin{cases} c+l, & i\in B,\\ c, & i\in A\setminus B,\\ t_i, & i\notin A\cup B. \end{cases}$$

The following equalities are true (note that if the summation is performed over empty set of indices, then it is equal to zero by definition):

$$\begin{aligned}
[u(\alpha)-v(\alpha)]-[s(\alpha)-t(\alpha)] &= \left[\sum_{i\in M}\alpha_i\varphi(u_i)-\sum_{i\in M}\alpha_i\varphi(v_i)\right]-\left[\sum_{i\in M}\alpha_i\varphi(s_i)-\sum_{i\in M}\alpha_i\varphi(t_i)\right]\\
&= \left[\left(\sum_{i\in A\setminus B}\alpha_i\varphi(c+l)+\sum_{i\in A\cap B}\alpha_i\varphi(c+l)+\sum_{i\in B\setminus A}\alpha_i\varphi(c)+\sum_{i\in M\setminus(A\cup B)}\alpha_i\varphi(s_i)\right)\right.\\
&\quad \left.-\left(\sum_{i\in B\setminus A}\alpha_i\varphi(c+l)+\sum_{i\in A\cap B}\alpha_i\varphi(c+l)+\sum_{i\in A\setminus B}\alpha_i\varphi(c)+\sum_{i\in M\setminus(A\cup B)}\alpha_i\varphi(t_i)\right)\right]\\
&\quad -\left[\left(\sum_{i\in A\setminus B}\alpha_i\varphi(c)+\sum_{i\in A\cap B}\alpha_i\varphi(c)+\sum_{i\in B\setminus A}\alpha_i\varphi(c)+\sum_{i\in M\setminus(A\cup B)}\alpha_i\varphi(s_i)\right)\right.\\
&\quad \left.-\left(\sum_{i\in B\setminus A}\alpha_i\varphi(c)+\sum_{i\in A\cap B}\alpha_i\varphi(c)+\sum_{i\in A\setminus B}\alpha_i\varphi(c)+\sum_{i\in M\setminus(A\cup B)}\alpha_i\varphi(t_i)\right)\right]\\
&= \sum_{i\in A\setminus B}\alpha_i\varphi(c+l)+\sum_{i\in B\setminus A}\alpha_i\varphi(c)-\sum_{i\in B\setminus A}\alpha_i\varphi(c+l)-\sum_{i\in A\setminus B}\alpha_i\varphi(c)\\
&= (\varphi(c+l)-\varphi(c))\left(\sum_{i\in A\setminus B}\alpha_i-\sum_{i\in B\setminus A}\alpha_i\right)=\delta\geqslant 0,
\end{aligned}$$

therefore

$$u(\alpha)-v(\alpha)\geqslant s(\alpha)-t(\alpha)+\delta.$$

Since $y^0(\alpha) - z^0(\alpha) = 0$ in the first node of the double chain, then the difference $u(\alpha) - v(\alpha)$ does not decrease and remains non-negative when moving along the chain to the right. Therefore $y(\alpha) \geqslant z(\alpha)$. Furthermore, if $yP^{\succsim}z$, then a relation of the form $A \succ B$ will be encountered at least once in the chain and therefore $y(\alpha) > z(\alpha)$ will be true; otherwise, if $yI^{\succsim}z$, then all relations will be of the type $A \sim B$ and the equality $y(\alpha) = z(\alpha)$ will be true. Theorem 3 is proved.

To prove Theorems 4 and 5, we will need one auxiliary statement that characterizes the structure of pairs of criteria groups related by qualitative importance in the case of ordinal importance under consideration.

Lemma 1. *If the qualitative importance $\succsim$ is obtained by extending the ordinal importance $\dot{\succsim}$ on 2^M, then the groups of criteria A and B in any relation $A \succsim B$ can be divided into separate criteria as follows:*

$$\begin{gathered} A = i_1 \cup i_2 \cup \dots \cup i_{n_A},\ i_p \neq i_r \text{ for } p \neq r; \\ B = j_1 \cup j_2 \cup \dots \cup j_{n_B},\ j_p \neq j_r \text{ for } p \neq r; \\ n_A \geqslant n_B; \quad i_p \dot{\succsim} j_p, \quad p = 1, \dots, n_B. \end{gathered} \tag{A.1}$$

A.2. PROOF OF LEMMA 1

We will use the mathematical induction method. At the first step of constructing the relation $\succsim$ it contains all pairs $(i, j) \in \dot{\succsim}$ and $(i, \varnothing)$, $i \in M$, for which the conditions of Lemma 1 are satisfied. At each subsequent step we will add one new relation to the obtained relation, applying the rules of additivity or transitivity to the existing relations.

Suppose that conditions of Lemma 1 are satisfied for all pairs $A \succsim B$ at the step k of constructing the relation $\succsim$. Let the following additivity rule be applied at the step $(k+1)$: if $A \succsim B$ and $A \cap C = B \cap C = \varnothing$ then $A \cup C \succsim B \cup C$. Partition (A.1) for the relation $A \cup C \succsim B \cup C$ can be obtained from the existing partition (A.1) for the relation $A \succsim B$ by simply adding pairs $i_p \dot{\succsim} i_p$ for all $i_p \in C$.

Now let the following additivity rule be applied at the step $(k+1)$: if $A \succsim B$ and $A \supseteq C$, $B \supseteq C$ then $A \backslash C \succsim B \backslash C$. Partition (A.1) for the relation $A \backslash C \succsim B \backslash C$ can be obtained from the existing partition (A.1) for the relation $A \succsim B$ by first transforming it so that it contains pairs $i_p \dot{\succsim} i_p$ for all $i_p \in C$, and then excluding all these pairs. Such a transformation can always be carried out. Suppose that there are two pairs $i \dot{\succsim} j$ and $t \dot{\succsim} i$ for some $i \in C$ in the partition (A.1) for $A \succsim B$. Then, taking into account the transitivity of $\dot{\succsim}$, they can be replaced by the pairs $i \dot{\succsim} i$ and $t \dot{\succsim} j$. And this should be done with each number $i \in C$.

Let the transitivity rule be applied at the step $(k+1)$: if $A \succsim B$ and $B \succsim C$ then $A \succsim C$. Partition (A.1) for the relation $A \succsim C$ can be obtained from the existing partitions (A.1) for the relations $A \succsim B$ and $B \succsim C$. We have:

$$\begin{gathered} A = i_1 \cup i_2 \cup \dots \cup i_{n_A},\ i_p \neq i_r \text{ for } p \neq r; \\ B = j_1 \cup j_2 \cup \dots \cup j_{n_B},\ j_p \neq j_r \text{ for } p \neq r; \\ C = t_1 \cup t_2 \cup \dots \cup t_{n_C},\ t_p \neq t_r \text{ for } p \neq r; \\ n_A \geqslant n_B \geqslant n_C; \quad i_p \dot{\succsim} j_p, \quad p = 1, \dots, n_B; \quad j_p \dot{\succsim} t_p, \quad p = 1, \dots, n_C. \end{gathered}$$

Using the transitivity of the relation $\dot{\succsim}$ the required partition (A.1) for the relation $A \succsim C$ is obtained: $i_p \dot{\succsim} t_p,\ p = 1, \dots, n_C$. Lemma 1 is proved.

A.3. PROOF OF THEOREM 4

Since $\dot{\succsim} \subseteq \succsim$ on 2^M, we have $R^{\dot{\succsim}} \subseteq R^{\succsim}$. Therefore, it suffices to prove that $R^{\dot{\succsim}} \supseteq R^{\succsim}$. Let $yR^{\succsim}z$ be true for some vector estimates $y, z \in Z$. Then according to Theorem 1 the relations (4) are satisfied. Introduce vector estimates y^k and $z^k, k = 1, \dots, q-1$:

$$y_i^k = \begin{cases} y_i, & y_i \leqslant w_k, \\ w_k, & y_i > w_k; \end{cases} \qquad z_i^k = \begin{cases} z_i, & z_i \leqslant w_k, \\ w_k, & z_i > w_k. \end{cases}$$

Construct a double chain starting with vector estimates $y^1 = z^1 = (w_1, \dots, w_1)$ and using only statements $i \dot{\succsim} j$. According to Lemma 1 there exists a partition (A.1) for the relation $M_2(y) \succsim M_2(z)$ from (4). Denote $n = |M_2(z)|$ for convenience. Let $u^1 = y^1$, $v^1 = z^1$. Improve the vector estimates u^p and v^p, $p = 1, \dots, n$ in the same way from $a = w_1$ to $b = w_2$ according to criteria i_p and j_p for each pair $i_p \dot{\succsim} j_p,\ p = 1, \dots, n$, from (A.1) one by one. According to Axioms 4 and 5 for each such improvement we obtain $u^{p+1}R^{\dot{\succsim}}v^{p+1}$, $p = 1, \ldots, n$. Note that $y^2 R^{\varnothing} u^{n+1}$ and $z^2 = v^{n+1}$. According to Axiom 2 (transitivity), we obtain $y^2 R^{\dot{\succsim}} z^2$.

Next, similarly use the relation $M_3(y) \succsim M_3(z)$ from (4) to construct a double chain starting with vector estimates y^2 and z^2. Since $M_2(y) \supseteq M_3(y)$ and $M_2(z) \supseteq M_3(z)$ the improvements of the vector estimates in the same way from $a = w_2$ to $b = w_3$ according to the criteria from $M_3(y)$ and $M_3(z)$ are possible in this case. As a result, we similarly obtain $y^3 R^{\dot{\succsim}} z^3$. Continue the described procedure until at step $k = q$, using the relation $M_q(y) \succsim M_q(z)$ from (4), we obtain $y^q R^{\dot{\succsim}} z^q$, which means $yR^{\dot{\succsim}}z$. Theorem 4 is proved.

A.4. PROOF OF THEOREM 5

In the general case $R^{\succsim} \supseteq R^{\Omega}$, therefore it is sufficient to prove that $R^{\succsim} \subseteq R^{\Omega}$ in the case of complete ordinal importance. We will use the well-known analytical decision rule to check $yR^{\Omega}z$in the case of complete ordinal importance. In this case, there exist ordinal coefficients of the criteria importance—positive numbers $\alpha_1^o, \ldots, \alpha_m^o$, equal in sum to one, for which the following conditions are satisfied $i \dot{\sim} j \Rightarrow \alpha_i^o = \alpha_j^o$, $i \dot{\succ} j \Rightarrow \alpha_i^o > \alpha_j^o$. Introduce the numbers

$$\alpha_{ik} = \begin{cases} \alpha_i^o, & y_i \geqslant w_k \\ 0, & y_i < w_k \end{cases}, \qquad i \in M, \; k = 2, \ldots, q.$$

Arrange all components of the vector $\alpha^k(y) = (\alpha_{1k}, \alpha_{2k}, \ldots, \alpha_{mk})$ in non-increasing order, and denote the resulting vector as $\alpha^k_{\searrow}(y)$. The following decision rule holds [3]:

$$yR^{\Omega}z \Leftrightarrow \alpha^k_{\searrow}(y) \geqq \alpha^k_{\searrow}(z), \quad k = 2, \ldots, q, \tag{A.2}$$

here $\geqq$ is the sign of vector non-strict inequality for n-dimensional vectors ($n \geqslant 2$): $a \geqq b \Leftrightarrow a_i \geqslant b_i$, $i = 1, \ldots, n$.

Let $yR^{\succsim}z$ be true for some vector estimates $y, z \in Z$. According to Theorem 1 the relations (4) are satisfied. For arbitrary $k = 2, \ldots, q$ we will prove that the relation $M_k(y) \succsim M_k(z)$ implies $\alpha^k_{\searrow}(y) \geqq \alpha^k_{\searrow}(z)$. Note that the numbers $\alpha_{ik}(y)$ are not equal to zero for $i \in M_k(y)$. Therefore, the vector $\alpha^k(y)$ consists of nonzero components with indices from $M_k(y)$. According to Lemma 1 the relation $M_k(y) \succsim M_k(z)$ has a partition A.1. Therefore, there also exists a corresponding pairwise partition of the components of the vectors $\alpha^k(y)$ and $\alpha^k(z)$:

$$\alpha_{i_1 k}(y) \geqslant \alpha_{j_1 k}(z), \; \alpha_{i_2 k}(y) \geqslant \alpha_{j_2 k}(z), \; \ldots, \; \alpha_{i_m k}(y) \geqslant \alpha_{j_m k}(z).$$

So, we obtain $\alpha^k_{\searrow}(y) \geqq \alpha^k_{\searrow}(z)$. Then, according to (A.2) $yR^{\Omega}z$ is true. Theorem 5 is proved.

REFERENCES

1. Nelyubin, A.P. and Podinovski, V.V., Multicriteria problems with importance-ordered criteria groups, *Autom. Remote Control*, 2022, no. 7, pp. 1108–1122.

2. Nelyubin, A.P. and Podinovski, V.V., Theoretical Basis and Methods of Decision Analysis under Uncertainty with Qualitative Estimates of Probabilities and Preferences, *Autom. Remote Control*, 2026, no. 7, pp. 657–667.

3. Podinovski, V.V., *Mnogokriterial'nye zadachi prinyatiya reshenii: teoriya i metody analiza: uchebnik dlya vuzov* (Multicriteria Decision-Making Problems: Theory and Methods of Analysis: A Textbook for Universities), Moscow: Urait, 2025.

4. Pfanzagl, J., *Theory of measurement*, Wiley, 1968.

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