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NONLINEAR SYSTEMS

Symmetric Multifrequency Oscillations of a Lagrangian System and Their Stabilization

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Abstract—This paper considers a Lagrangian system subjected to positional forces. By assumption, the frequencies of small oscillations for an equilibrium are incommensurable. The existence of multifrequency symmetric oscillations in the neighborhood of the equilibrium and their extension to global families of such oscillations is established; the families fill the entire domain of the phase space adjacent to the equilibrium. A nonlocal continuation of the above families in the system parameter is proved. In the conservative case, an autonomous control is found, and a symmetric multifrequency oscillation is stabilized in the large. In this case, feedback on potential energy is constructed.

Keywords: Lagrangian system, positional forces, reversibility, symmetric multifrequency oscillation, birth, global family, nonlocal continuation in parameter, conservative system, feedback, adaptive scheme, stabilization

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1. INTRODUCTION

Multifrequency oscillations are an important research area in nonlinear dynamics; see [1–4], other books, and numerous papers. Oscillations are studied using universal asymptotic methods and perturbation theory. The latter includes L.S. Pontryagin's theorem [5] on oscillation control by constructing a limit cycle. An example of a qualitatively reorganized phase portrait under a small perturbation can be found in the relaxation regimes (without a driving force) of a regenerative receiver, as investigated by B. Van der Pol [6]. In 1929, A.A. Andronov discovered that the physical embodiment of Poincaré curves is the stable self-oscillations, constructed by him and independently by Van der Pol. In 1881, A. Poincaré introduced the concept of a limit cycle for a system in the plane [7, Chap. VI]. The theory of self-oscillations was further developed within Andronov's school [8].

A qualitative reorganization is observed in the general case of a corrected conservative system [9]. In the constructed autonomous control system, the feedback is implemented by tracking potential energy at the current trajectory point.

In this paper, the idea from [9] is utilized to stabilize symmetric multifrequency oscillations. To stabilize an oscillation with a given energy level h^*, an adaptive scheme [10] was applied in [9]; according to this scheme, the gain $K(h^*)$ is chosen depending on the value of h^*. The computation of $K(h^*)$ in [9] is based on the known global family Σ_1 of single-frequency symmetric oscillations, constructed for all admissible energy levels h. The family Σ_1 is obtained by a global extension of the symmetric local Lyapunov family, whose existence is given by an analog [11, 12] of Lyapunov's center theorem [13] for reversible mechanical systems. For a symmetric multifrequency oscillation, the global family Σ is constructed below.

In this paper, we introduce the concept of a symmetric multifrequency oscillation of a Lagrangian system with n degrees of freedom, subjected to positional forces; prove a theorem on the birth of k-frequency oscillations ($1 \leqslant k \leqslant n$); provide their continuation in the phase space to the global family Σ_k ($1 \leqslant k \leqslant n$) of such oscillations; prove a nonlocal extension of Σ_k in the system parameter; and address other related issues. As a result, a global theory is developed to solve the stabilization problem in the large.

According to Lyapunov [13], the stability problem for a chosen solution can always be reduced to the study of the zero (trivial) solution. However, for a solution of an autonomous system, including a multifrequency oscillation, this leads to a special case in stability theory. For a periodic solution of an autonomous system, the Andronov–Vitt stability theorem [14] is applied.

In an autonomous system, a solution constitutes a family parameterized by β, i.e., the shift of the initial point along the trajectory. Therefore, two stabilization problems are distinguished here: stabilization of the entire family of solutions and stabilization of a chosen solution from this family. Usually, the second problem is solved using controls with an explicit dependence on time. Such formulations of control problems for single-frequency oscillations were given in [15–21].

When stabilizing the entire family of solutions, it is necessary that the controlled system trajectories be attracted to the family; this requires preserving the autonomy of the original system and using autonomous control. Here, known concepts are utilized: stability with respect to part of the variables [22], characterizing the deviation of trajectories of a family from other trajectories, orbital stability and orbital asymptotic stability, and attraction [23]. In the Andronov–Vitt theorem [14], attraction is ensured by the roots of the characteristic equation lying in the left half-plane.

Due to the action of control, the closed-loop system no longer admits the original solution. Therefore, in the controlled system, a solution close to the original one is stabilized. The closed-loop system is made close to the original system by using a control law with a small gain. This approach is characteristic of perturbation theory; it was pioneered by Pontryagin [5]. Attraction to the family (solution) is provided by dissipation, i.e., a velocity-linear force acting at each trajectory point. Simultaneously, the control law ensures the isolation of the stabilized oscillation. A textbook example of such stabilization is the Van der Pol equation [6]. Further studies on the application of Van der Pol-type dissipation were reviewed in [9]; also, see the bibliography in the paper. Only single-frequency oscillations were considered.

2. SYMMETRIC MULTIFREQUENCY OSCILLATIONS OF A LAGRANGIAN SYSTEM SUBJECTED TO POSITIONAL FORCES

Consider a smooth system governed by Lagrange's equations of the second kind and subjected to positional forces Q_s :

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_s} - \frac{\partial T}{\partial q_s} = Q_s(q_1, \ldots, q_n), \quad s = 1, \ldots, n. \tag{1}$$

The force $Q = (Q_1, \ldots, Q_n)$ is generally the sum of a potential force and a nonconservative positional force. The kinetic energy T is represented as a positive definite quadratic form of velocities; equations (1) describe the dynamics in an inertial frame. System (1) is invariant with respect to the transformation $(q, \dot{q}, t) \to (q, -\dot{q}, -t)$ and belongs to the class of reversible mechanical systems. The phase space of system (1) is symmetric with respect to the fixed set $M = \{q, \dot{q} : \dot{q} = 0\}$. An equilibrium of system (1) lies on M. Symmetric motions intersect the set M: at the intersection point, the velocity satisfies $\dot{q} = 0$. Nonsymmetric motions are realized in pairs; they are symmetric to each other with respect to M, and the directions of motion in a pair are mutually opposite. A symmetric motion intersecting the set M twice constitutes a single-frequency symmetric oscillation.

Let $q = (q_1, \dots, q_n) = 0$ for the equilibrium. In system (1), the roots of the characteristic equation are split into pairs $\pm\lambda_s$. System (1) is studied under the assumption that all λ_s are pure imaginary, and the numbers $|\lambda_s| > 0$ are incommensurable. Then all motions in the linearized system near the equilibrium are described by quasi-periodic functions with k frequencies, $1 \leqslant k \leqslant n$. They represent symmetric oscillations (SOs). For $k = 1$, we have a single-frequency oscillation; for $k > 1$, a multifrequency oscillation arises.

An SO starts at a point $q^0 = (q_1^0, \dots, q_n^0) \in M$ at $t = 0$ and is described by a function $q(q^0, t)$, in which an arbitrary shift β of the initial point along the trajectory is set to zero. At the time instant t_s, the SO along the coordinate q_s will have the next zero velocity $\dot{q}_s$. In the case under study, the numbers t_s are supposed to be incommensurable. Therefore, at the time instants $t = t_s$ the velocities $\dot{q}_s$ on the SO satisfy the equalities

$$\dot{q}_s(q_1^0, \dots, q_n^0, t) = 0, \quad t = t_s; \quad s = 1, \dots, n. \tag{2}$$

The left-hand sides of equalities (2) contain the velocities $\dot{q}_s$ for the solution of system (1) as functions of the initial point q^0 and time t. Equalities (2) mean that for each variable q_s, there is a closed curve in the phase space. Conditions (2) are necessary and sufficient for the existence of an SO of system (1). They can hold in (2) for k velocities $\dot{q}_s$, where $s = 1, \dots, k$ and $k \leqslant n$. For $k = n$, system (1) admits an SO with the maximum possible number of frequencies; in the case $k < n$, at the initial point q^0 the accelerations satisfy $\ddot{q}_{k+1} = \ldots = \ddot{q}_n = 0$ and $t_{k+1} = \ldots = t_n$, so the system has an SO with $k < n$ frequencies: if $k = 1$, we have a single-frequency SO. Each coordinate q_s is given by a $2t_s$-periodic function of time t; therefore, the SO is described by a quasi-periodic function $(q(q^0, t), \dot{q}(q^0, t))$ with k frequencies. For $k = 1$, the function is periodic. For $k = n$, the phase-space trajectory belongs to the direct product of n closed curves, i.e., a symmetric n-torus; for $k < n$, we obtain symmetric k-tori.

In equalities (2), a separate time t_s is associated with each variable q_s. The *unified time* γ for all variables of the SO is introduced by the formula $t_s = \nu_s\gamma$. In time γ, all velocities vanish simultaneously: a multifrequency SO can be studied as an *imaginary* symmetric periodic motion with a separate time for each variable. Equalities (2) are written for fixed numbers $t_s = t_s^*$. If $t_s^* = \nu_s\gamma^*$, the number γ^* hence means the time instant when the trajectory intersects the fixed set M, as in the case of a single-frequency SO of period $2\gamma^*$. Therefore, the introduction of the unified time γ leads to the concept of a *common period* for all variables of a multifrequency SO. It equals $2\gamma^*$, and this number identifies the SO under consideration in the family. The number γ^* is denoted by τ and is called the SO parameter. It characterizes the transition from one SO to another and depends on the initial point q^0 for the SO.

The concepts of unified time and common period allow constructing a theory of multifrequency SOs similar to that of symmetric periodic motions. The latter motion represents an SO with $k = 1$; the half-period on it serves as the parameter τ.

Equalities (2) are written as

$$\dot{q}_s(q_1^0, \dots, q_n^0, \nu_s\tau) = 0, \quad s = 1, \dots, n. \tag{3}$$

The incommensurability condition of the numbers in the set $\{t_s^*\}$ translates to the set $\nu = \{\nu_1, \dots, \nu_n\}$: ν is called the *basis* (of frequencies) of an SO. When passing from one SO to another, the point $q^0 \in M$ changes. Equalities (2) are treated as a system of equations for finding the numbers $t_s = \nu_s\tau$. In equalities (3) with the parameter τ, the initial point q^0 is determined, while the basis ν will have to be obtained up to a factor $\delta(q^0)$ and thus remains preserved. Since the n equalities (3) relate the $n + 1$ variables $q_1^0, \dots, q_n^0, \tau$, they lead to a family of SOs of dimension n with the parameter τ. In the case of $k < n$ equalities in (3), the dimension of the SO family equals k.

When extending the SO family in the parameter τ, one changes the projection of the SO onto the fixed set M; on the interval $\Delta\tau$, the projection leads to the domain q^0 of initial points for the local SO family. This set is also obtained by proportionally changing the elements of the basis ν while preserving the incommensurability of all numbers ν_s in the basis. And the basis, defined up to a factor δ, remains invariable. This provides another local extension method for the SO. Note that both methods yield the same result.

3. THE BIRTH OF SYMMETRIC OSCILLATIONS FROM THE EQUILIBRIUM

Consider a k-frequency SO with an initial point $q^0 \in M$. It satisfies equalities (3), which relate the variables q^0 and τ. At the point q^0, the Jacobi matrix is computed:

$$\mathrm{J}(\mathrm{q}^0, \tau) = \left\| \frac{\partial \dot{q}_s(q^0, \tau)}{\partial q_j^0} \right\|. \tag{4}$$

Definition 1. A symmetric k-frequency oscillation $(1 \leqslant k \leqslant n)$ that satisfies the condition $\operatorname{rank} \mathrm{J}(\mathrm{q}^0, \tau) = n$ is said to be nondegenerate.

For a nondegenerate SO, the following local continuation properties hold.

1°. A nondegenerate SO locally continues in the parameter τ.

Let (3) be valid. For this nondegenerate SO, we form the equations in the deviations Δq^0 and $\Delta\tau$ in the linear approximation to obtain the linear system

$$\frac{\partial \dot{q}_s(q^0, \tau)}{\partial q_1^0} \Delta q_1^0 + \ldots \frac{\partial \dot{q}_s(q^0, \tau)}{\partial q_n^0} \Delta q_n^0 + \frac{\partial \dot{q}_s(q^0, \tau)}{\partial \tau} \Delta\tau = 0, \quad s = 1, \ldots, n. \tag{5}$$

In this system of equations, the principal matrix coincides with $\mathrm{J}(\mathrm{q}^0, \tau)$, and the right-hand side is represented by the acceleration vector with a minus sign, computed on M. For a nondegenerate SO, the inequality $\det \mathrm{J}(\mathrm{q}^0, \tau) \neq 0$ holds: the acceleration vector vanishes only at the equilibrium. Therefore, the system is consistent, and the SO locally extends in $\Delta\tau$, both toward increasing and decreasing $\Delta\tau$. The factor $\delta(q^0)$ is also incremented simultaneously.

2°. A nondegenerate SO locally continues in the system parameter μ.

For a system with the parameter μ, system (5) contains this parameter and, in addition, a term with the increment $\Delta\mu$. For a nondegenerate SO, the Jacobi matrix is nonsingular; therefore, the extension is carried out in τ and μ. Restricting the analysis only to the increment in μ, we obtain property 2°.

In the neighborhood of the equilibrium, system (1) is written as

$$\ddot{q}_s + \sum_{j=1}^{n} a_{sj} q_j + \tilde{Q}_s(q, \dot{q}) = 0, \quad \tilde{Q}_s(q, \dot{q}) = \tilde{Q}_s(q, -\dot{q}), \quad s = 1, \ldots, n, \tag{6}$$

where $||a_{sj}||$ is a constant matrix, and $\tilde{Q}_s$ are nonlinear functions.

The following result is true.

Theorem 1 (on the birth of SOs). *The neighborhood of the equilibrium of system* (6) *with incommensurable frequencies ν_s, $s = 1, \ldots, n$, of the linear system is filled with families of nondegenerate k-frequency SOs, $1 \leqslant k \leqslant n$. They are parameterized by the distance to the equilibrium.*

Proof. Let us perform the change of variables $(q, \dot{q}) \to (\varepsilon\xi, \varepsilon\dot{\xi})$. Then, a quasilinear system is obtained in the variables ξ_s, and its linear approximation coincides with that of system (6).

For $\varepsilon = 0$, the resulting system admits a k-frequency nondegenerate SO, $1 \leqslant k \leqslant n$. For this system, $\mathrm{J}(0, 0) = -||a_{sj}||_1^n$, $\operatorname{rank} \mathrm{J}(0, 0) = n$, and the basis for k-frequency oscillations contains k

numbers from the set $\nu(0) = \{\nu_s(0)\}$, $\nu_s(0) = |\lambda_s|$. Moreover, the system containing a k-frequency SO is described by k variables, and the SO basis includes k numbers. For the whole system, SOs of the linear system continue in the parameter ε: the quasilinear system possesses families of nondegenerate SOs in ε with all frequencies: $1 \leqslant k \leqslant n$. When considering the SOs with all possible frequencies k, $1 \leqslant k \leqslant n$, the neighborhood of the equilibrium of system (6) is completely filled with families of nondegenerate SOs parameterized by the distance ε to the equilibrium. The proof of Theorem 1 is complete.

Remark 1. For $k = 1$, we obtain [11, 12] the reversible analog of Lyapunov's center theorem [13].

Remark 2. Under the hypotheses of Theorem 1, the equilibrium of the system is stable.

4. THE GLOBAL FAMILY OF NONDEGENERATE SYMMETRIC OSCILLATIONS IN THE PHASE SPACE

A nondegenerate SO possesses the local continuation property in the family parameter τ. Theorem 2 below formulates a result on the global extension of SOs in the parameter τ.

Definition 2. A family of nondegenerate SOs parameterized by τ is called a global family of SOs if the parameter τ takes all possible values for this family.

Theorem 2 (on the global family of SOs in the phase space). *Let a Lagrangian system subjected to positional forces admit a nondegenerate k-frequency SO,* $1 \leqslant k \leqslant n$. *Then the SO always extends in the family parameter* τ *to a global family* Σ_k *of nondegenerate SOs, and the parameter* τ *changes monotonically on this family. Also, the condition* $\operatorname{rank} J(q^0, \tau) = n$ *holds on the global family, being violated on its boundary.*

Proof. During the local extension of a nondegenerate k-frequency SO in the parameter τ, its initial point $q^0(\tau)$ will be a function of τ. And the domain $\Upsilon_k(q^0, \tau)$ of points with the condition $\operatorname{rank} J(q^0, \tau) = n$ is formed on the fixed set M. We apply the local continuation property of a nondegenerate SO to the points of $\Upsilon_k(q^0, \tau)$. Then two situations arise. In the first situation, no new points appear in the domain $\Upsilon_k(q^0, \tau)$. Then the global family Σ_k is successfully constructed. In the second situation, the domain $\Upsilon_k(q^0, \tau)$ has been increased, which is accompanied by a monotonic change of the parameter τ. Again, two situations arise here.

As a result, at some iteration, the global family Σ_k is successfully constructed, or the construction process continues with an infinite approach to the boundary of the initial domain $\Upsilon_{\sigma k}(q^0, \tau)$ for the global family Σ_k. At each point of this domain, we have $\operatorname{rank} J(q^0, \tau) = n$. The monotonic change of SOs of the family Σ_k in the parameter τ follows from the construction procedure. The proof of Theorem 2 is complete.

Remark 3. The proof of Theorem 2 is valid for k-frequency SOs, $1 \leqslant k \leqslant n$. For a single-frequency oscillation ($k = 1$), the result was described in [24].

Remark 4. For the analytic system (6) with incommensurable numbers $\nu_1, \ldots, \nu_n$, a rigorous transformation to a system with k-dimensional manifolds ($1 \leqslant k \leqslant n$) was constructed in [25], including its convergence conditions [25, Theorem 2]. System (6) possesses reversibility; as a result, the phase space contains n manifolds, and a single-frequency SO is realized on each manifold.

5. THE GLOBAL EXTENSION OF A NONDEGENERATE SYMMETRIC OSCILLATION IN THE PARAMETER SPACE OF THE SYSTEM

A nondegenerate SO possesses the local continuation property in the system parameter μ. Thus, in perturbation theory, a nondegenerate perturbed SO always exists together with an SO of system (1). The boundary of existence of a nondegenerate SO in the parameter μ is an open issue in perturbation theory. In applications, the boundary is usually found by direct computation.

Theorem 3 formulates a result on the global extensibility of a nondegenerate SO in the parameter μ. The parameter $\mu = (\mu_1, \ldots, \mu_m)$ is generally a vector.

Theorem 3. *Let a Lagrangian system with the parameter μ, subjected to positional forces, admit an SO with the parameter μ, nondegenerate for $\mu = \mu^*$. Then this SO globally extends to the entire domain of μ. On the global family of nondegenerate SOs in the parameter μ, the condition* $\operatorname{rank} J(q^0, \tau) = n$ *holds. The basis ν is preserved on this family. On the boundaries of the global family, the condition* $\operatorname{rank} J(q^0, \tau) = n$ *is violated.*

Proof. An SO with the parameter μ, nondegenerate at $\mu = \mu^*$, possesses property 2°. It is applied for the global extension in μ exactly as property 1° for the global extension of a nondegenerate SO in the parameter τ. The extension in μ is carried out until exhausting the entire domain of μ with $\det \mathrm{J}(\mathrm{q}^0(\mu), \tau) \neq 0$.

Remark 5. Theorem 3 concerns the parameter μ on which the SO depends. Another possible situation is demonstrated in Example 2 below.

Theorems 2 and 3 applied to a nondegenerate SO yield the following conclusion.

General postulate. If a Lagrangian system subjected to positional forces admits a nondegenerate SO, then it globally extends in the parametric phase space.

Remark 6. This paper employs Pontryagin's approach [26] to autonomous ordinary differential equations: parameters and phase variables are treated equally.

6. THE OSCILLATION DOMAIN AND STABILIZATION OF THE SYMMETRIC OSCILLATIONS OF SYSTEM (6)

Consider system (6) with incommensurable frequencies $\nu_1, \ldots, \nu_n$ in the linear system. In the neighborhood of the equilibrium, k-frequency SOs ($1 \leqslant k \leqslant n$) are born (see Theorem 1); they fill the entire neighborhood of the equilibrium. The continuation of SOs in the parameter τ (Theorem 2) leads in the phase space to an oscillation domain Ω, adjacent to the equilibrium and filled with global families of k-frequency SOs, $1 \leqslant k \leqslant n$. They are denoted by Σ_k: the number of the families equals the binomial coefficient C_n^k (in combinatorics, the number of ways to choose k out of n objects). The family Σ_k fills the integral manifold $\tilde{\Sigma}_k$. Each SO in the family Σ_k is identified by the value of the parameter τ.

In this paper, we develop the approach of [9], originally proposed for a single-frequency SO, for multifrequency SOs. In the stabilization problem of an SO for system (6), an autonomous controlled system, ε-close to (6), is constructed; in this system, an isolated oscillation (IO) is separated. In the perturbed system, the manifold $\tilde{\Sigma}_k$ is associated with the manifold $\tilde{\Sigma}_k(\varepsilon)$: $\tilde{\Sigma}_k = \tilde{\Sigma}_k(0)$. When separating the IO, we simultaneously ensure its orbital asymptotic stabilization. A local attraction problem is usually posed: trajectories from the neighborhood of the IO are attracted to it. However, due to the structure of the domain Ω, it is possible to pose the problem of attracting all trajectories from $\tilde{\Sigma}_k(\varepsilon)$, as well as from the neighborhood of $\tilde{\Sigma}_k(\varepsilon)$ [9]. This is the case, e.g., in the Van der Pol equation [6] ($k = 1$). In a conservative system with one degree of freedom, all trajectories from the oscillation domain are attracted [9].

In a conservative system, the kinetic energy on the set M is zero. When passing from one SO of the family to another, the potential energy $\Pi(q)$ computed on M changes. One can therefore take the value of the function $\Pi(q)$ on M or the constant h of the total mechanical energy E as the SO parameter instead of τ. Due to the monotonic change of the parameter τ on the global family, the energy h changes monotonically as well. By specifying the value h, we obtain all SOs with this energy level. On the graph of the function $\Pi(q)$, the value $h = 0$ corresponds to the equilibrium; for $h > 0$, the born k-frequency SOs extend to the global families Σ_k unboundedly together with the energy h.

In a conservative system, the SOs in the family Σ_k are parameterized by the constant h and form integral k-dimensional manifolds $\tilde{\Sigma}_k$, $1 \leqslant k \leqslant n$. Hence, the SOs of all dimensions k correspond to a given value of h. For this reason, a control law independent of the number of frequencies in the SO is designed. For the energy level $h = h^*$, attraction to the IO on $\tilde{\Sigma}_k(\varepsilon)$ with all numbers k: $1 \leqslant k \leqslant n$ is investigated accordingly.

In the case $k = 1$, the family Σ_1 is described by a system with one degree of freedom. Therefore, the stabilization problem naturally decomposes as follows [9]: first, the attraction problem on $\tilde{\Sigma}_1(\varepsilon)$ is solved; and in the system of order $(n-1)$, local attraction to the trivial solution (the IO on $\tilde{\Sigma}_1(\varepsilon)$) is ensured, for which $h = h^*$.

For SOs with $k > 1$ frequencies, $\tilde{\Sigma}_k$ has dimension k. Local attraction to the trivial solution is ensured in the system of order $(n-k)$. Regardless of the number of frequencies k, the trivial solution in the problem is associated with the IO corresponding to the energy level $h = h^*$ in the controlled system.

In a conservative system with n degrees of freedom, the controlled system with k frequencies has C_n^k stabilizable IOs. The sum over all k gives the number of all stabilizable IOs.

7. STABILIZATION OF THE SYMMETRIC OSCILLATIONS OF A CONSERVATIVE SYSTEM

Consider the controlled (corrected) system

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_s} - \frac{\partial L}{\partial q_s} = \varepsilon[1 - K\Pi_\varphi]\frac{\partial T}{\partial \dot{q}_s}, \quad L = T - \Pi, \quad s = 1, \ldots, n, \tag{7}$$

given by Lagrange's equations of the second kind. Here, T and Π are the kinetic and potential energies, respectively. (When computed on the SO family, the latter is denoted by Π_φ.) By assumption, for $\varepsilon = 0$, system (7) admits the set Σ_σ containing all global families of SOs born from the equilibrium with incommensurable frequencies $\nu_1, \ldots, \nu_n$. We choose $0 < \varepsilon < \varepsilon_0$, where ε_0 is some finite number for Σ_σ; h^* is the energy level corresponding to the separated oscillation: the equilibrium is associated with the value $h = 0$. In system (7), the coefficient K is set equal to the value $K(h^*)$ of the function $K(h)$ for the stabilized oscillation. The dependence $K(h)$ itself will be defined below.

The control system (7) was applied in [9] for the adaptive stabilization scheme of a symmetric periodic motion ($k = 1$). The control law on the right-hand side of equality (7) is universal for stabilizing SOs with $k = 1$. It is represented by a nonlinear force linear in velocity. For $\varepsilon = 0$, system (7) belongs to the class of reversible mechanical systems. The feedback is implemented by tracking the value of the potential energy $\Pi(q)$ at the current point of the closed-loop system trajectory.

In the stabilization problem, one should understand that a unique family Σ_k is realized in the set of global families Σ_σ in the system with $\varepsilon = 0$. Moreover, a unique SO is realized on the family Σ_k, which corresponds to the parameter τ (the energy h in the conservative system). Therefore, we pose the problem of stabilizing a unique SO corresponding to the separated energy level $h = h^*$. For this SO, the variables are $q_{k+1} = \ldots = q_n = 0$.

The following computations are valid for all global families Σ_k with k frequencies, $1 \leqslant k \leqslant n$, regardless of the number of frequencies k of the oscillations.

For system (7) with $\varepsilon = 0$, we determine the total mechanical energy

$$E \equiv T(q, \dot{q}) + \Pi(q). \tag{8}$$

It takes a constant value h on the solutions of the conservative system. System (7) satisfies the equality

$$\frac{dE}{dt} = \varepsilon[1 - K(h^*)\Pi_\varphi]T(q, \dot{q}). \tag{9}$$

For $\varepsilon = 0$, we have $E = h(\text{const}) > 0$. On the SO, the coordinate is $q = \varphi$. The SOs form a global family Σ_k belonging to the invariant integral manifold $\tilde{\Sigma}_k$. We stabilize the SO with the parameter $\tau = \tau^*$ $(h = h^*)$.

The energy E is the sum of two terms: $E = E_\varphi + E_\varepsilon$, where E_φ is computed on the solutions in $\tilde{\Sigma}_k(\varepsilon)$, and E_ε on the solutions in the neighborhood of $\tilde{\Sigma}_k(\varepsilon)$. The law for E_ε involves the kinetic energy of motion in the ε-neighborhood of $\tilde{\Sigma}_k(\varepsilon)$. It is a quadratic form of the velocities $\dot{q}_{k+1}, \ldots, \dot{q}_n$. Thus, the motion in the neighborhood of $\tilde{\Sigma}_k(\varepsilon)$ is tracked by the potential energy of the SO on Σ_k.

For $\varepsilon > 0$, we write the expressions

$$\Pi(q) = \Pi(\varphi) + h\varepsilon\rho(\varepsilon, h, q), \quad T = E - \Pi(q) = h - \Pi(\varphi) + h\varepsilon\sigma(\varepsilon, h, q)$$

$(\Pi(\varphi) = \Pi_\varphi)$ with functions ρ and σ of order 1. As a result, the right-hand side of equality (9) has the principal term

$$f = \varepsilon[(1 - K(h^*)\Pi_\varphi)](h - \Pi(\varphi)),$$

which is computed on the SO; it does not contain t explicitly.

For SOs, the *unified* time γ has been introduced for all variables q_s (Section 2). And the multifrequency SO in the function E is treated here as a single-frequency SO with a *common period* equal to 2τ. The time γ is determined up to a factor. In the unified time, the rate of change of the variable q_s acquires its individual factor ν_s such that the numbers in the set $\{\nu_s\}$ are incommensurable. Finally, the potential energy in the function f is computed on the SO $q = \varphi$: f also changes as a function of γ.

In the unified time γ, the rate of change of the energy is proportional to its rate in the time t. Therefore, the increment $\Delta E_\varphi(\varepsilon, h)$ of the energy in γ is computed from the right-hand side of (9) as an integral over the period 2τ in the variable γ :

$$\Delta E_\varphi(\varepsilon, h) = \varepsilon 2\tau(h)[I(h) + \varepsilon h\Phi(\varepsilon, h)]. \tag{10}$$

This formula is valid on the entire global family Σ_k. The family Σ_k consists of SOs, and each value h is associated with only one SO. If $\Delta E_\varphi(\varepsilon, h^*) = 0$ at the point $h = h^*$, then the value h^* corresponds to a k-frequency oscillation. The oscillation will be nonsymmetric since $\varepsilon = 0$ for a symmetric one. For the isolated root h^*, the oscillation will be isolated. Thus, we obtain an isolated oscillation (IO), representing a cycle in the case $k = 1$. The IO is a one-parameter family in the shift of the initial point for the oscillation.

The function $I(h)$ has the form

$$I(h) \equiv \int_0^{2\tau^*} [1 - K(h^*)\Pi_\varphi](h - \Pi(\varphi))d\gamma \tag{11}$$

and gives the amplitude (bifurcation) equation $I(h) = 0$ for the multifrequency SO. As in the case of symmetric periodic motion in [9], a simple root h^* of equation (11) leads at the point $h = h^*$ of the family of k-frequency SOs to a bifurcation and the birth of a k-frequency IO.

In the integral (11), the function $\Pi(\varphi)$ on the SO family is computed through the coordinate $q = \varphi(h, \gamma)$, given in the unified time γ for the SO. By replacing the variable γ in the integral (11) with $\tilde{\gamma} = (\tau^*/\tau(h))\gamma$, where $\tau^* = \tau(h^*)$ for the stabilized oscillation with the energy level h^*, we make the coordinate q directly dependent on $\tilde{\gamma}$.

In the case $k = 1$, the family of nonlinear oscillations is reduced to an isochronous family in [9].

The potential energy on the SO changes as a function of $\tilde{\gamma}$; therefore, in view of the equality $\Pi(\varphi(h,0)) = h$, it can be written as $\Pi(\varphi(h,\tilde{\gamma})) = hz(\tilde{\gamma})$, $0 \leqslant z \leqslant 1$. Consequently, the integral (11) becomes

$$I(h) \equiv 2h \int_0^1 (1 - K(h^*)hz(\tilde{\gamma}))(1 - z(\tilde{\gamma}))d\tilde{\gamma}, \tag{12}$$

being represented as a quadratic function without a constant term:

$$I(h) = 2h(\alpha - \beta h), \quad \alpha = \int_0^1 (1 - z)d\tilde{\gamma}, \quad \beta = -K(h^*) \int_0^1 z(1 - z)d\tilde{\gamma}. \tag{13}$$

Due to the expression (13), the equation $I(h) = 0$ clearly admits a unique nonzero root $h^* = \alpha/\beta$. The root h^* corresponds to the multifrequency IO of the controlled system (7). This oscillation is ε-close to the SO with the energy level $h = h^*$; however, the oscillation is not symmetric.

Note that each number $K(h^*)$ is uniquely associated with the root $h = h^*$ and vice versa. Thus, the following function is defined:

$$K(h) = \frac{b}{h}, \quad b = \text{const}, \ b > 0.$$

The kinetic energy on the fixed set M is zero; therefore, the potential energies of the systems under study on the SO are distinguished by the constant b.

The root of the equation $I(h) = 0$ is computed for the one-parameter (in h) family Σ_k of trajectories in the ε-corrected system. The root h^* corresponds to a fixed point of the mapping defined on the manifold $\tilde{\Sigma}_k(\varepsilon)$, and its isolation indicates that the existence of the root is derived via the linear (principal) terms in ε. Hence, $\Delta E_\varphi(\varepsilon, h^*) = 0$, and we arrive at the equality $\Phi(\varepsilon, h^*) = 0$.

Lemma 1. *The equation*

$$I(h) + \varepsilon h \Phi(\varepsilon, h) = 0 \tag{14}$$

has a unique root for $0 < \varepsilon < \varepsilon_0$, where ε_0 is a number independent of h^*.

Proof. The number ε_0 is found using exactly the same procedure as the one described for a single-frequency oscillation in [9, Lemma 1].

Next, it is necessary to study the corrected system (7) for $0 < \varepsilon < \varepsilon_0$. Consider the integral manifold $\tilde{\Sigma}_k$ containing the global family of k-frequency SOs, $1 \leqslant k \leqslant n$. It is obtained by the continuation, in the parameter h, of the small oscillations with incommensurable frequencies.

Theorem 4. *Let a conservative system be governed by Lagrange's equations of the second kind and admit the equilibrium with incommensurable frequencies of small oscillations. Then the system has an integral manifold $\tilde{\Sigma}_\sigma$ containing the set Σ_σ of global families Σ_k of k-frequency oscillations that fill the manifolds $\tilde{\Sigma}_k$, $1 \leqslant k \leqslant n$. For the corrected system* (7), *there always exists a number $\varepsilon_0 > 0$ such that for $0 < \varepsilon < \varepsilon_0$, system* (7) *possesses k-frequency IOs, $1 \leqslant k \leqslant n$, ε-close to the oscillation of the conservative system with the energy level $h = h^*$. The number of k-frequency IOs is C_n^k. Each IO attracts trajectories from $\tilde{\Sigma}_k(\varepsilon)$ and trajectories from the ε-neighborhood of $\tilde{\Sigma}_k(\varepsilon)$. On the solutions of the corrected system, the energy $E = T + \Pi$ of the conservative system changes according to the law* (9).

Proof. The existence of the global family Σ_σ under the incommensurability of frequencies for the equilibrium follows from Theorems 1 and 2. The boundary value ε_0 is established by Lemma 1. Formula (10) is valid for k-frequency SOs, $1 \leqslant k \leqslant n$. Based on the above analysis of formula (10), there exists an isolated k-frequency oscillation ε-close to the oscillation of the conservative system with the energy level h^*. In system (7), the entire domain of initial points for solutions different from the equilibrium belongs to $\tilde{\Sigma}_\sigma$.

Formula (9) holds separately for E_φ and E_ε. The rate of change of E_φ coincides in sign with that of the energy E_ε: different kinetic energies are used. As a result,

$$\Delta E_\varphi(\varepsilon, h) \to \Delta E_\varphi(\varepsilon, h^*) = 0, \quad \Delta E_\varepsilon(\varepsilon, h) \to \Delta E_\varepsilon(\varepsilon, h^*) = 0$$

simultaneously as $h \to h^*$. The first limit means that in the controlled system (7), for points from $\tilde{\Sigma}_k(\varepsilon)$, all trajectories tend to the IO corresponding to the SO with the energy value h^* from Σ_k. According to the second limit, trajectories from the neighborhood of $\tilde{\Sigma}_k(\varepsilon)$ tend to $\tilde{\Sigma}_k(\varepsilon)$ with a zero limit value of the corresponding kinetic energy. Therefore, these trajectories also tend to the IO.

The number of k-frequency IOs coincides with C_n^k, the number of combinations of integral manifolds where k-frequency SOs are realized.

The proof of Theorem 4 is complete.

Remark 7. Theorem 4 concerns k-frequency SOs, $1 \leqslant k \leqslant n$. For the particular case $k = 1$, it was established in [9], with weaker conditions on the roots of the characteristic equation.

Example 1. Signal source recognition by extracting frequencies. Consider two linear oscillators with frequencies ω_x and ω_y. In the system, they possess families of symmetric two-frequency oscillations, degenerating into two families of single-frequency oscillations. To stabilize the IO of the controlled system, we apply the closed-loop system

$$\begin{aligned} &\dot{x} + \omega_x^2 x = \varepsilon(1 - K(h^*)\Pi)\dot{x}, \\ &\dot{y} + \omega_y^2 y = \varepsilon(1 - K(h^*)\Pi)\dot{y}, \quad \Pi = (\omega_x^2 x^2 + \omega_y^2 y^2)/2, \end{aligned} \tag{15}$$

where the coefficient $K(h^*)$ is computed from the energy value $h = h^*$ of the entire system. All IOs are stabilized according to a single scenario. System (15) contains two invariant manifolds where the individual oscillators are stabilized.

The signal is captured by energy. In system (15), the oscillation frequencies are extracted from the received signal, and the sources are recognized by the frequencies. The recognition capabilities grow when increasing the number of oscillators in the system.

The symmetric oscillation of a harmonic oscillator with a frequency ω is given by the formula $r = A\cos(\omega t)$, where $A^2 = h/2$. In the case of a single oscillator, the coefficient is $K(h) = 2/h$.

In (15), we have the frequency basis $\nu = (\omega_x, \omega_y)$, and the unified time is $\gamma = t$. Hence, the system of two oscillators behaves as a single harmonic oscillator with the frequency ω. The formula $K(h) = 2/h$ remains valid for system (15). Therefore, in system (15), we set

$$K(h^*) = 2/h^* = 1/(A_x^2\omega_x^2 + A_y^2\omega_x^2),$$

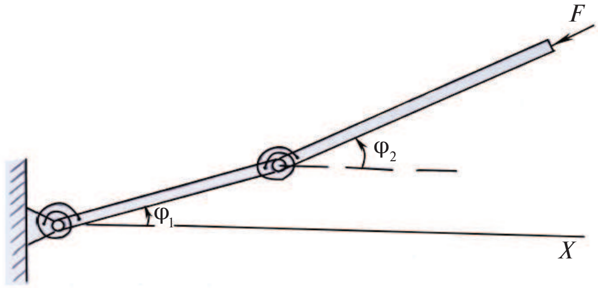
where A_x and A_y are the amplitudes of the oscillators with the energy levels h_x^* and h_y^*, respectively.

Example 2. Let the above system of two oscillators (Example (15)) be the linear approximation of the smooth system

$$\begin{aligned} &\dot{x} + \omega_x^2 x = X(x, y), \\ &\dot{y} + \omega_y^2 y = Y(x, y), \quad X(0,0) = Y(0,0) = 0. \end{aligned} \tag{16}$$

To study the neighborhood of zero in detail, we increase the observation scale for the system: $x = \varepsilon\xi$, $y = \varepsilon\eta$. The resulting quasilinear system has the form

$$\begin{aligned} &\ddot{\xi} + \omega_x^2 \xi = \varepsilon X(\varepsilon, \xi, \eta), \\ &\ddot{\eta} + \omega_y^2 \eta = \varepsilon Y(\varepsilon, \xi, \eta), \quad X(\varepsilon, 0, 0) = Y(\varepsilon, 0, 0) = 0. \end{aligned} \tag{17}$$



The model of an elastic rod under a follower force.

For $\varepsilon = 0$, system (17) admits single-frequency and two-frequency SOs: the latter have the basis (ω_x, ω_y). By Theorem 1, for $\varepsilon > 0$, the neighborhood of the equilibrium is filled with families of SOs with one and two frequencies. According to Theorem 2, each local SO family extends in system (16) to a global SO family and fills the oscillation domain in the system.

Example 3. The model of an elastic rod under a follower force [27].

The oscillations of beams, ropes, and cables are often studied based on adequate discrete rod models. Consider a particular case from [27]: the mechanical system consists of two identical rods of mass m and length l, connected by a hinge and a linear spring of stiffness c_2 (see the figure).

The first rod is attached by a hinge and a spring of stiffness c_1 to a fixed point O. A constant force F acts on the end of the second rod, directed along its axis. The system is located on a smooth horizontal plane. The deviations of the rods from the x-axis are described by the angles φ_1 and φ_2, respectively, measured from the straight line x; for the equilibrium, $\varphi_1 = \varphi_2 = 0$.

The kinetic energy T and the generalized forces Q_1 and Q_2 are

$$\begin{aligned} &T = \frac{ml^2}{6}[4\dot{\varphi}_1^2 + 3\dot{\varphi}_1\dot{\varphi}_2\cos(\varphi_2 - \varphi_1) + \dot{\varphi}_2^2], \\ &Q_1 = c_1\varphi_1 + c_2(\varphi_2 - \varphi_1) - Fl\sin(\varphi_2 - \varphi_1), \quad Q_2 = c_2(\varphi_2 - \varphi_1). \end{aligned} \tag{18}$$

We have a Lagrangian system with two degrees of freedom subjected to positional forces. For $F = 0$, the system is conservative.

In the equilibrium, the forces are $Q_1 = Q_2 = 0$: the system admits a unique (trivial) equilibrium. We compile the characteristic equation for the equilibrium:

$$7\lambda^4 + \frac{6a}{ml^2}\lambda^2 + \frac{36}{m^2l^4}c_1c_2 = 0, \quad a = 2c_1 + 16c_2 - 5Fl. \tag{19}$$

Under the condition $a^2 > 28c_1c_2$, the roots of this equation are pure imaginary, $\lambda = \pm\omega_{1,2}$, and the squared frequencies are given by

$$\omega_{1,2}^2 = \frac{3}{7ml^2}(a \pm \sqrt{a^2 - 28c_1c_2}).$$

The boundary value of the follower force F for the stability of the rectilinear rod configuration in the linear approximation is thus found. How will the oscillations of the rod respond to finite deviation angles?

According to Theorem 1, under incommensurable frequencies, the neighborhood of the equilibrium is filled with families of nondegenerate SOs with one and two frequencies; by Theorem 2, they extend to global families of nondegenerate SOs. Thus, only the oscillation domain filled with nondegenerate SOs is adjacent to the equilibrium. The equilibrium of the system is stable in the large.

Theorem 2 is applied to the system with a fixed parameter F that satisfies the frequency incommensurability condition. In Theorem 3, the nonlocal extension of a multifrequency oscillation in the parameter is carried out while preserving the basis. For $F > 0$, the numbers $\omega_{1,2}$ depend on F; the basis changes, so Theorem 3 is not applicable. The found nondegenerate SOs for countable values of the follower force F are not realized.

Note that for this model, the qualitative conclusions do not depend on the number of rods, and the oscillation amplitude is limited only by the spring stiffness.

For $F = 0$, symmetric oscillations exist; they are stabilized (Theorem 4) in the adaptive scheme (7).

8. CONCLUSIONS

A Lagrangian system subjected to positional forces is invariant with respect to the transformation $(q, \dot{q}, t) \to (q, -\dot{q}, -t)$ (i.e., belongs to the class of reversible mechanical systems). The phase space of the system is symmetric with respect to the set $M = \{q, \dot{q} : \dot{q} = 0\}$. An equilibrium admitting small oscillations with incommensurable frequencies is analyzed. The neighborhood of this equilibrium is filled with symmetric k-frequency oscillations (SOs), $1 \leqslant k \leqslant n$. (Here, n is the number of degrees of freedom of the system.) There are no other motions in the neighborhood. All SOs extend to global families of SOs and form an oscillation domain Ω adjacent to the equilibrium. The SO families globally extend in the system parameter. These results have been established in the paper and characterize the subject of the stabilization problem.

In an autonomous system, solutions form a family in the shift of the initial point along the trajectory. A distinction is made between the problems of stabilizing the entire family of solutions and an individual solution from the family. The second problem is solved using time-varying control. The family of solutions is stabilized by constructing a corrected system, i.e., a controlled system with an autonomous control law that acts with a small gain.

The results presented above are motivated by the problem of stabilizing multifrequency oscillations of a reversible mechanical system. This problem has not been posed previously. The subject of stabilization itself has been previously known only in the part related to Poincaré perturbation theory.

In this paper, we have described a global theory of symmetric multifrequency oscillations (symmetric conditionally periodic solutions, symmetric quasi-periodic motions, and symmetric conditionally periodic orbits) for the first time in the literature. The basic concepts have been introduced; necessary and sufficient conditions for the existence of a symmetric oscillation have been found; and local continuation properties of SOs have been established. Theorems on the birth of SOs and their extension to a global SO family, and a theorem on the global extension of SOs in the system parameter have been proved. Based on new original ideas, the theory has an accessible presentation without special knowledge requirements. Thus, a complete theory of symmetric multifrequency oscillations for a Lagrangian system subjected to positional forces has been elaborated.

This theory has been employed to pose and solve the stabilization problem (in the large) of multifrequency oscillations of a conservative system. The feedback control has been applied here; the idea was proposed by the author [9] and tested for single-frequency oscillations before. The control system (7) is independent of the number of frequencies in the SO. In the adaptive scheme, an isolated oscillation (IO) of the controlled system is stabilized. For the case of k frequencies, an IO is found close to the SO with the desired energy level h^*: the SO belongs to the global family Σ_k of k-frequency SOs. Stabilization is ensured by attracting trajectories to the IO.

The autonomous control used is universal: it applies to any Lagrangian system and stabilizes k-frequency oscillations, $1 \leqslant k \leqslant n$, where n is the number of degrees of freedom of the system. A corrected conservative system has been constructed, in which the current potential energy on the trajectory of the multifrequency SO is tracked via feedback.

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