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Identification of Unknown Inputs in Nonlinear Systems Based on Interval Methods

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Abstract—This paper considers the problem of estimating (identifying) the magnitude of unknown inputs in engineering systems described by nonlinear dynamic models with measurement noises, which is necessary to compensate for the effect of such inputs. The solution of the problem is based on minimal-order interval observers, and their explicit relations are obtained. Such an observer and a differentiator are used to derive relations for an interval estimate of the magnitude of unknown inputs. The theoretical results are illustrated by a practical example of estimating the magnitude of unknown inputs on an electric drive of a manipulator robot. The simulations in MATLAB demonstrate the correctness of the theoretical results. The above relations can be used to design fault-tolerant systems.

Keywords: nonlinear systems, unknown inputs, identification, observers, interval methods

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1. INTRODUCTION

During operation, engineering systems can be subjected to various unknown inputs. Different methods are used to compensate for them, and most of the methods employ information about the magnitudes of such inputs. Direct measurement of input magnitudes often encounters significant difficulties. In these cases, the problem can be solved based on known mathematical models of the systems and standard sensors, which provide the required information after appropriate processing.

There are various ways to process this information; approaches based on sliding mode observers are very popular, and the corresponding design methods were considered in [1–4] for different classes of systems. As a rule, certain constraints are imposed on the original system to construct sliding mode observers. In particular, it is required that the system be minimum phase or detectable, and a matching condition must hold. Furthermore, such observers are sensitive to perturbations.

An interesting method was proposed in [4]: the so-called interval observers are used to generate interval estimates of the desired functions. The design problem of such observers has been actively researched over the past 20 years; solutions for various classes of systems and examples of practical applications can be found in [5–9]. Note that in [4], the solution of the estimation problem was obtained by imposing the minimum-phase condition on the original linear system. Moreover, the author of [4] neglected measurement noises, which are always present in real systems.

In this paper, we pose and solve the following problem: using interval observers, it is required to estimate the magnitudes of unknown inputs in nonlinear continuous-time dynamic systems with measurement noises and without the minimum-phase condition. A similar problem was addressed in [9] for discrete-time nonlinear systems; the solution for continuous-time systems is significantly different. By analogy with [9], the solution provided below involves a reduced-order model of the original system. This approach simplifies the interval observer design, reduces the interval width, and allows one to account for measurement noises. In this case, the minimum-phase requirement is not used, and the transformations are limited to reducing the model to canonical form.

The objective of this work is to construct interval estimators for the magnitudes of unknown inputs in engineering systems described by nonlinear dynamic models with measurement noises.

Consider systems described by a nonlinear model of the form

$$\begin{aligned} \dot{x}(t) &= Fx(t) + Gu(t) + C\Psi(x(t), u(t)) + L\rho(t), \\ y(t) &= Hx(t) + w(t) \end{aligned} \tag{1.1}$$

with the following notation: $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, and $y(t) \in \mathbb{R}^l$ are the state, control, and output vectors, respectively; F, G, C, H, and L are constant matrices of dimensions $n \times n$, $n \times m$, $n \times q$, $l \times n$, and $n \times 1$, respectively; $w(t) \in \mathbb{R}^l$ is measurement noises, $\rho(t) \in \mathbb{R}$ is an unknown time-varying function describing system inputs. By assumption, $w(t)$ and $\rho(t)$ are bounded functions: $\underline{w} \leqslant w(t) \leqslant \overline{w}$ and $\underline{\rho} \leqslant \rho(t) \leqslant \overline{\rho}$ for all $t \geqslant 0$, where $\underline{w}$, $\overline{w}$, $\underline{\rho}$, and $\overline{\rho}$ are known. For vectors, the inequality sign $\leqslant$ is understood elementwise. Also, the initial state $x(0)$ is assumed to satisfy $\underline{x}(0) \leqslant x(0) \leqslant \overline{x}(0)$ for some $\underline{x}(0)$ and $\overline{x}(0)$. The nonlinear term $\Psi(x, u)$ has the form

$$\Psi(x, u) = \begin{pmatrix} \varphi_1(A_1 x, u) \\ \dots \\ \varphi_q(A_q x, u) \end{pmatrix},$$

where $A_1, \dots, A_q$ are known constant row matrices, and $\varphi_1, \dots, \varphi_q$ are arbitrary nonlinear functions.

It is required to construct an interval observer for estimating the function $\rho(t)$. The bounds $\underline{\rho}$ and $\overline{\rho}$ are treated as preliminary and will be refined during the problem solution. Similar to [3, 9], this problem is solved based on an input-insensitive reduced-order model of the original system. Such a model has a smaller dimension than the original system, which decreases computational complexity. By analogy with [4], we assume the matching condition $\operatorname{rank}(HL) = \operatorname{rank}(L)$: roughly speaking, the function $\rho(t)$ enters the equations for the state vector components that are measured by the system's sensors.

First, the linear case is considered: $C = 0$.

2. BUILDING THE REDUCED-ORDER MODEL

2.1. General Relations

The reduced-order model mentioned above has the form

$$\begin{aligned} \dot{x}_*(t) &= F_* x_*(t) + G_* u(t) + J_* H x(t) + L_* \rho(t), \\ y_*(t) &= H_* x_*(t), \end{aligned} \tag{2.1}$$

where $x_*(t) \in \mathbb{R}^k$ is the state vector; $y_*(t) \in \mathbb{R}$ is the output; and F_*, G_*, J_*, and L_* are matrices of compatible dimensions to be determined. The term $J_* H x(t)$ is introduced instead of the expected $J_* y(t)$ to account for measurement noises; it will appear as $J_* y(t)$ in the interval observer.

To design an interval observer, the matrix F_* is required to be stable and Metzler, i.e., have nonnegative off-diagonal elements. Therefore, F_* is specified in the diagonal Jordan form

$$F_* = \begin{pmatrix} \lambda_1 & 0 & 0 & \dots & 0 \\ 0 & \lambda_2 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \lambda_k \end{pmatrix},$$

which possesses these properties if the eigenvalues $\lambda_1, \dots, \lambda_k$ are chosen negative.

As is known [10], an observable system (2.1) can always be written with

$$F_* = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}$$

(the identification canonical form). Otherwise, it can be reduced to another system with an observable subsystem [10], and the latter's matrix can be written in the identification canonical form. One can always pass to the Jordan form from the identification form by introducing feedback with an appropriate gain matrix [11].

To build the reduced-order model, we assume the existence of a matrix Φ such that $x_*(t) = \Phi x(t)$ for all $t \geqslant 0$. According to [9], this matrix and those describing the model satisfy the equations

$$\Phi F = F_*\Phi + J_*H, \quad \Phi G = G_*, \quad \Phi L = L_*.$$

For the adopted form of F_*, the first equation above splits into k independent equations:

$$\Phi_i F = \lambda_i \Phi_i + J_{*i} H, \quad i = 1, \dots, k. \tag{2.2}$$

Each of them can be written as

$$(\Phi_i \quad -J_{*i}) \begin{pmatrix} F - \lambda_i I_n \\ H \end{pmatrix} = 0, \quad i = 1, \dots, k, \tag{2.3}$$

where I_n denotes an identity matrix of dimensions $n \times n$. To build the reduced-order model, an appropriate value $\lambda_i < 0$ is chosen so that the matrix Φ_i determined from (2.3) satisfies the two requirements

$$\Phi_i L = L_* \neq 0, \quad R_i H = \Phi_i \tag{2.4}$$

for some matrix R_i, with $H_* = 1$. To account for the second requirement, let us replace Φ_i in (2.2) with $R_i H$,

$$R_i H F = \lambda R_i H + J_{*i} H,$$

and rewrite the resulting equation as

$$(R_i \quad -J_{*i}) \begin{pmatrix} HF - \lambda_i H \\ H \end{pmatrix} = 0, \quad i = 1, \dots, k.$$

By analogy with the solution of (2.3), a value $\lambda_i < 0$ is chosen so that the matrix R_i determined from the above equation satisfies the condition $L_* = R_i H L \neq 0$. If this condition holds, $G_* = R_i H G$ is determined, and the linear model is successfully built. Otherwise, the problem has no solution by the current method.

2.2. Use of Virtual Sensors

The mutual independence of equations (2.2) restricts the solvability of the problem. Indeed, consider an example of an electric drive described by the equations

$$\begin{aligned}\dot{x}_1(t) &= k_1 x_2(t),\\ \dot{x}_2(t) &= k_2 x_2(t) + k_3 x_3(t) + k_7 \operatorname{sgn}(x_2(t)),\\ \dot{x}_3(t) &= k_4 x_2(t) + k_5 x_3(t) + k_6 u(t) + \rho(t).\end{aligned} \tag{2.5}$$

Here, the coefficients k_1–k_7 represent the parameters of the electric drive; and the unknown input $\rho(t)$ is due to changes in the armature winding resistance during overheating. The model under consideration is described by the matrices

$$F = \begin{pmatrix} 0 & k_1 & 0\\ 0 & k_2 & k_3\\ 0 & k_4 & k_5\end{pmatrix}, \quad G = \begin{pmatrix} 0\\ 0\\ k_6\end{pmatrix}, \quad C = \begin{pmatrix} 0\\ k_7\\ 0\end{pmatrix}, \quad L = \begin{pmatrix} 0\\ 0\\ 1\end{pmatrix}.$$

By assumption, $y(t) = x_3(t) + w(t)$, i.e., $H = (\,0 \;\; 0 \;\; 1\,)$. It is required to estimate the function $\rho(t)$.

Equation (2.3) takes the form

$$(\Phi \quad -J_*) \begin{pmatrix} -\lambda & k_1 & 0\\ 0 & k_2-\lambda & k_3\\ 0 & k_4 & k_5-\lambda\\ 0 & 0 & 1\end{pmatrix} = 0.$$

Since $k_2 < 0$, it has a unique solution $\lambda = k_2$ with the matrix $\Phi_1 = (0 \;\; 1 \;\; 0)$, which violates conditions (2.4). One recipe is to use a so-called virtual sensor, i.e., an observer estimating the unmeasured component of the state vector of the original system. In the case under study, the reduced-order model for constructing such a sensor is a special case of the general model (2.1) described by the equation

$$\dot{x}_{*1}(t) = k_2 x_{*1}(t) + k_3 H x(t) + k_7 \operatorname{sgn}(x_{*1}(t)),$$

where $x_{*1}(t) = \Phi_1 x(t)$. Note that since the measurement $y(t)$ is noisy, the readings of this sensor will include noises as well; this aspect will be considered below by interval methods. Taking the state $x_{*1}(t)$ as the readings of such a sensor $y_v(t) = H_v x(t)$ with $H_v = (0 \;\; 1 \;\; 0)$, we extend equation (2.2):

$$\Phi_i F = \lambda_i \Phi_i + J_{*i} H + J_{*v} H_v, \quad i = 1, \ldots, k;$$

in the case under consideration, it takes the form

$$(\Phi \quad -J_* \quad -J_{*v}) \begin{pmatrix} -\lambda & k_1 & 0\\ 0 & k_2-\lambda & k_3\\ 0 & k_4 & k_5-\lambda\\ 0 & 0 & 1\\ 0 & 1 & 0\end{pmatrix} = 0.$$

Now it has another solution $\lambda = k_5 < 0$ with the matrices $\Phi_2 = (0 \;\; 0 \;\; 1)$ and $(J_* \;\; J_{*v}) = (0 \;\; 1)$; the first satisfies conditions (2.3) with $R = 1$ and $L_* = 1$. The model with the virtual sensor is described by the equations

$$\begin{aligned}\dot{x}_{*1}(t) &= k_2 x_{*1}(t) + k_3 H x(t) + k_7 \operatorname{sgn}(x_{*1}(t)),\\ \dot{x}_{*2}(t) &= k_5 x_{*2}(t) + k_4 x_{*1}(t) + k_6 u(t) + \rho(t),\\ y_*(t) &= x_{*2}(t) = y(t).\end{aligned} \tag{2.6}$$

In general, there can be several virtual sensors.

2.3. Obtaining the Interval Estimate

We begin with the case where a virtual sensor is not required. Then, for the chosen value λ, model (2.1) becomes one-dimensional:

$$\begin{aligned} &\dot{x}_*(t) = \lambda x_*(t) + G_* u(t) + J_* H x(t) + L_* \rho(t), \\ &y_*(t) = x_*(t). \end{aligned} \tag{2.7}$$

Obviously, if the measurement noises are absent or can be neglected, then, taking the equalities $y_*(t) = x_*(t) = Ry(t)$ and $J_* H x(t) = J_* y(t)$ into account, the input $\rho(t)$ can be estimated as

$$\hat{\rho}(t) = L_*^{-1}(\dot{y}_*(t) - (\lambda y_*(t) + G_* u(t) + J_* y(t))). \tag{2.8}$$

To compute the derivative $\dot{y}_*(t)$, we apply the differentiator proposed in [12]:

$$\begin{aligned} &\dot{z}_1(t) = z_0(t), \\ &z_0(t) = z_2(t) - \alpha_1 |z_1(t) - y_*(t)|^{1/2} \operatorname{sgn}(z_1(t) - y_*(t)), \\ &\dot{z}_2(t) = -\alpha_2 \operatorname{sgn}(z_1(t) - y_*(t)), \end{aligned}$$

where α_1 and α_2 are positive constants chosen according to the recommendations in [12]. As shown therein, $z_2(t)$ provides an exact estimate of $\dot{y}_*(t)$ in finite time under no unknown inputs in $y_*(t)$. If the function $y_*(t)$ is subjected to noises of intensity $w_y(t)$, then $|\dot{y}_*(t) - z_2(t)| \leqslant |w_y(t)|^{1/2}$.

If the function $\rho(t)$ represents a vector, the described approach should be applied to each of its components. By analogy with [4], the estimate $\hat{\rho}(t)$ can be used to estimate the full state vector $x(t)$ of the original system based on the equation

$$\dot{\hat{x}}(t) = F\hat{x}(t) + Gu(t) + L\hat{\rho}(t) + K(y(t) - H\hat{x}(t)),$$

where the gain K is chosen to ensure the stability of the matrix $F - KH$.

Note that for linear systems, a similar problem was considered in [4]; in contrast to the simple relations (2.7) and (2.8), it was solved using complex analytical transformations of the original system and interval observers, which are unnecessary here.

If the original system is nonlinear, then an additional term $\Psi_{**}(y_*, y, u)$ is incorporated into (2.7) (see the details below), and the estimate (2.8) takes the form

$$\hat{\rho}(t) = L_*^{-1}(\dot{y}_*(t) - (\lambda y_*(t) + G_* u(t) + J_* y(t) + \Psi_{**}(y_*, y, u))). \tag{2.9}$$

With a virtual sensor described by the equation

$$\begin{aligned} &\dot{x}_v(t) = \lambda_v x_v(t) + G_v u(t) + J_v y(t), \\ &y_v(t) = x_v(t) \end{aligned} \tag{2.10}$$

(by assumption, without $\rho(t)$), the expression (2.8) gets complicated:

$$\hat{\rho}(t) = L_*^{-1}(\dot{y}_*(t) - (\lambda y_*(t) + G_* u(t) + J_* y(t) + J_{*v} y_v(t))). \tag{2.11}$$

The presence of measurement noise makes the estimates (2.8) and (2.9) approximate. In this case, the use of an interval observer allows finding guaranteed limits (bounds) of the estimate, by analogy with confidence intervals in mathematical statistics. By assumption, the lower $\underline{\rho}$ and upper $\overline{\rho}$ bounds of the function $\rho(t)$ introduced previously are a first approximation, which will be refined later.

3. INTERVAL OBSERVER AND ITS USE

We begin with the case where a virtual sensor is not required. By analogy with [5, 9], the interval observer based on model (2.1) takes the form

$$\begin{aligned}\dot{\underline{y}}_*(t) &= \lambda \underline{y}_*(t) + G_*u(t) + J_*y(t) + J_*^-\underline{w} - J_*^+\overline{w} + L_*^+\underline{\rho} - L_*^-\overline{\rho},\\ \dot{\overline{y}}_*(t) &= \lambda \overline{y}_*(t) + G_*u(t) + J_*y(t) + J_*^-\overline{w} - J_*^+\underline{w} + L_*^+\overline{\rho} - L_*^-\underline{\rho},\end{aligned} \tag{3.1}$$

where $Q^+ = \max\{0, Q\}$ and $Q^- = Q^+ - Q$ for an arbitrary matrix Q; obviously, Q^+ and Q^- are nonnegative.

Similar to [5, 9], it can be shown that the condition $\overline{x}(0) \geqslant x(0) \geqslant \underline{x}(0)$ implies $\overline{y}_*(t) \geqslant y_*(t) \geqslant \underline{y}_*(t)$ for all $t \geqslant 0$. By analogy with [4], in this case, there exists a variable $0 \leqslant \beta(t) \leqslant 1$ such that

$$y_*(t) = \underline{y}_*(t) + \beta(t)(\overline{y}_*(t) - \underline{y}_*(t)) = \overline{y}_*(t) + (\beta(t) - 1)(\overline{y}_*(t) - \underline{y}_*(t)) \tag{3.2}$$

for all $t \geqslant 0$. Replacing $y_*(t)$ with $Ry(t)$ yields

$$\beta(t) = \frac{Ry(t) - \underline{y}_*(t)}{\overline{y}_*(t) - \underline{y}_*(t)}, \tag{3.3}$$

where all variables can be determined. By assumption, the function $\beta(t)$ is differentiable.

Theorem 1. *The refined upper $\overline{\overline{\rho}}(t)$ and lower $\underline{\underline{\rho}}(t)$ bounds for $\rho(t)$ are given by*

$$\overline{\overline{\rho}}(t) = L_*^{-1}(\dot{\beta}(t)(\overline{y}_*(t) - \underline{y}_*(t)) + L_*^+\underline{\rho} - L_*^-\overline{\rho} + \beta(t)(|L_*|(\overline{\rho} - \underline{\rho}) + |J_*|(\overline{w} - \underline{w}))) \tag{3.4}$$

and

$$\underline{\underline{\rho}}(t) = L_*^{-1}(\dot{\beta}(t)(\overline{y}_*(t) - \underline{y}_*(t)) + L_*^+\overline{\rho} - L_*^-\underline{\rho} + (\beta(t) - 1)(|L_*|(\overline{\rho} - \underline{\rho}) + |J_*|(\overline{w} - \underline{w}))), \tag{3.5}$$

respectively.

Proof. We compute $\dot{y}_*(t)$ based on the first part of equality (3.2) and equation (3.1):

$$\begin{aligned}\dot{y}_*(t) &= \dot{\underline{y}}_*(t) + \dot{\beta}(t)(\overline{y}_*(t) - \underline{y}_*(t)) + \beta(t)\frac{d}{dt}(\overline{y}_*(t) - \underline{y}_*(t))\\ &= \lambda \underline{y}_*(t) + G_*u(t) + J_*y(t) + J_*^-\underline{w} - J_*^+\overline{w} + L_*^+\underline{\rho} - L_*^-\overline{\rho}\\ &\quad + \dot{\beta}(t)(\overline{y}_*(t) - \underline{y}_*(t)) + \lambda\beta(t)(\overline{y}_*(t) - \underline{y}_*(t))\\ &\quad + \beta(t)(|L_*|(\overline{\rho} - \underline{\rho}) + |J_*|(\overline{w} - \underline{w})),\end{aligned}$$

where the matrix $|Q| = Q^+ + Q^-$ is composed of the absolute values of the elements of the matrix Q. Let the variable $\dot{y}_*(t)$ in this expression be replaced by the right-hand side of equality (2.7):

$$\begin{aligned}&\lambda y_*(t) + G_*u(t) + J_*Hx(t) + L_*\rho(t)\\ &= \lambda \underline{y}_*(t) + G_*u(t) + J_*y(t) + J_*^-\underline{w} - J_*^+\overline{w} + L_*^+\underline{\rho} - L_*^-\overline{\rho}\\ &\quad + \dot{\beta}(t)(\overline{y}_*(t) - \underline{y}_*(t)) + \lambda\beta(t)(\overline{y}_*(t) - \underline{y}_*(t))\\ &\quad + \beta(t)(|L_*|(\overline{\rho} - \underline{\rho}) + |J_*|(\overline{w} - \underline{w})).\end{aligned}$$

After straightforward transformations, we have

$$\begin{aligned}L_*\rho(t) &= \lambda(\underline{y}_*(t) - y_*(t)) + J_*w(t) + J_*^-\underline{w} - J_*^+\overline{w} + L_*^+\underline{\rho} - L_*^-\overline{\rho}\\ &\quad + \dot{\beta}(t)(\overline{y}_*(t) - \underline{y}_*(t)) + \lambda\beta(t)(\overline{y}_*(t) - \underline{y}_*(t))\\ &\quad + \beta(t)(|L_*|(\overline{\rho} - \underline{\rho}) + |J_*|(\overline{w} - \underline{w}))\\ &= \dot{\beta}(t)(\overline{y}_*(t) - \underline{y}_*(t)) + \beta(t)(|L_*|(\overline{\rho} - \underline{\rho}) + |J_*|(\overline{w} - \underline{w}))\\ &\quad + J_*w(t) + J_*^-\underline{w} - J_*^+\overline{w} + L_*^+\underline{\rho} - L_*^-\overline{\rho}.\end{aligned}$$

Since

$$J_*w(t) + J_*^-\underline{w} - J_*^+\overline{w} = J_*^+w(t) - J_*^-w(t) + J_*^-\underline{w} - J_*^+\overline{w}$$
$$= J_*^+(w(t) - \overline{w}) + J_*^-(\underline{w} - w(t)) \leqslant 0,$$

the bound (3.4) is obtained by removing the term $J_*w(t) + J_*^-\underline{w} - J_*^+\overline{w}$ from the last expression.

Similar operations with the second part of equality (3.2) give

$$\begin{aligned}\dot{y}_*(t) &= \dot{\overline{y}}_*(t) + \dot{\beta}(t)(\overline{y}_*(t) - \underline{y}_*(t)) + (\beta(t) - 1)\tfrac{d}{dt}(\overline{y}_*(t) - \underline{y}_*(t)) \\ &= \lambda\overline{y}_*(t) + G_*u(t) + J_*y(t) + J_*^-\overline{w} - J_*^+\underline{w} + L_*^+\overline{\rho} - L_*^-\underline{\rho} \\ &\quad + \dot{\beta}(t)(\overline{y}_*(t) - \underline{y}_*(t)) + \lambda\beta(t)(\overline{y}_*(t) - \underline{y}_*(t)) \\ &\quad - \lambda(\overline{y}_*(t) - \underline{y}_*(t)) + (\beta(t) - 1)(|L_*|(\overline{\rho} - \underline{\rho}) + |J_*|(\overline{w} - \underline{w}));\end{aligned}$$

the replacement of $\dot{y}_*(t)$ and analogous transformations lead to

$$\begin{aligned}L_*\rho(t) &= \dot{\beta}(t)(\overline{y}_*(t) - \underline{y}_*(t)) + (\beta(t) - 1)(|L_*|(\overline{\rho} - \underline{\rho}) + |J_*|(\overline{w} - \underline{w})) \\ &\quad + J_*w(t) + J_*^-\overline{w} - J_*^+\underline{w} + L_*^+\overline{\rho} - L_*^-\underline{\rho}.\end{aligned}$$

Since $J_*w(t) + J_*^-\overline{w} - J_*^+\underline{w} \geqslant 0$, we remove this term to obtain the bound (3.5). Clearly,

$$\underline{\underline{\rho}}(t) \leqslant \rho(t) \leqslant \overline{\overline{\rho}}(t),$$

and the proof of Theorem 1 is complete.

To compute the derivative $\dot{\beta}(t)$, the above differentiator can be used:

$$\begin{aligned}\dot{z}_1(t) &= z_0(t), \\ z_0(t) &= z_2(t) - \gamma_1|z_1(t) - \beta(t)|^{1/2}\operatorname{sgn}(z_1(t) - \beta(t)), \\ \dot{z}_2(t) &= -\gamma_2\operatorname{sgn}(z_1(t) - \beta(t)),\end{aligned} \tag{3.6}$$

where γ_1 and γ_2 are positive constants; the variable $z_2(t)$ provides the exact estimate of the function $\dot{\beta}(t)$ in finite time under no unknown inputs in $\beta(t)$.

If a virtual sensor (2.10) is required, it must be implemented in interval form as well:

$$\begin{aligned}\dot{\underline{x}}_v(t) &= \lambda_v\underline{x}_v(t) + G_vu(t) + J_vy(t) + J_v^-\underline{w} - J_v^+\overline{w}, \\ \dot{\overline{x}}_v(t) &= \lambda_v\overline{x}_v(t) + G_vu(t) + J_vy(t) + J_v^-\overline{w} - J_v^+\underline{w}, \\ \underline{y}_v(t) &= \underline{x}_v(t), \\ \overline{y}_v(t) &= \overline{x}_v(t).\end{aligned}$$

The interval observer (3.1) is supplemented with the corresponding elements:

$$\begin{aligned}\dot{\underline{y}}_*(t) &= \lambda\underline{y}_*(t) + G_*u(t) + J_*y(t) + J_*^-\underline{w} - J_*^+\overline{w} + L_*^+\underline{\rho} - L_*^-\overline{\rho} + J_{*v}^+\underline{y}_v(t) - J_{*v}^-\overline{y}_v(t), \\ \dot{\overline{y}}_*(t) &= \lambda\overline{y}_*(t) + G_*u(t) + J_*y(t) + J_*^-\overline{w} - J_*^+\underline{w} + L_*^+\overline{\rho} - L_*^-\underline{\rho} + J_{*v}^+\overline{y}_v(t) - J_{*v}^-\underline{y}_v(t),\end{aligned}$$

and the upper bound $\overline{\overline{\rho}}(t)$ is refined to

$$\begin{aligned}\overline{\overline{\rho}}(t) &= L_*^{-1}(\dot{\beta}(t)(\overline{y}_*(t) - \underline{y}_*(t)) + L_*^+\underline{\rho} - L_*^-\overline{\rho} \\ &\quad + \beta(t)(|L_*|(\overline{\rho} - \underline{\rho}) + |J_*|(\overline{w} - \underline{w}) + |J_{*v}|(\overline{y}_v(t) - \underline{y}_v(t)))).\end{aligned}$$

The lower bound is refined by analogy.

4. NONLINEAR SYSTEMS

In the nonlinear case, the reduced-order model is built using the linear model obtained previously: for the matrix Φ, it is necessary to verify the possibility of transforming the argument of the nonlinear term $\Psi(x,u)$ based on the following operations [3]. For simplicity, we restrict the considerations to the case of no virtual sensor; if such a sensor is needed and contains a nonlinear component, the model can be built by analogy.

The matrix $C_* = \Phi C$ is computed, and the serial numbers $j_i, \ldots, j_s$ of its nonzero columns are determined. Next, it is necessary to check the condition

$$\operatorname{rank}\begin{pmatrix}\Phi\\ H\end{pmatrix} = \operatorname{rank}\begin{pmatrix}\Phi\\ H\\ A_j\end{pmatrix}, \quad j = j_i, \ldots, j_s. \tag{4.1}$$

If this condition holds, then the nonlinear component is sought in the form

$$\Psi_*(y_*, Hx, u) = \begin{pmatrix}\varphi_{j_1}(A_{*j_1}\xi, u)\\ \ldots\\ \varphi_{j_s}(A_{*j_s}\xi, u)\end{pmatrix},$$

where $\xi = \begin{pmatrix} y_*\\ Hx\end{pmatrix}$, the matrices $A_{*j_1}, \ldots, A_{*j_s}$ are determined from the linear equations

$$A_j = A_{*j}\begin{pmatrix}\Phi\\ H\end{pmatrix}, \quad j = j_i, \ldots, j_s,$$

and the component $C_*\Psi_*(y_*, Hx, u)$ is added to the linear model (2.1). If condition (4.1) fails, it is necessary to find another solution of equation (2.3) and repeat the above procedure with a new matrix Φ of the same or increased dimension k. If (4.1) does not hold for all k, a virtual sensor should be added to satisfy condition (4.1).

The nonlinear model is described by the equation

$$\begin{aligned}\dot{x}_*(t) &= x_*(t) + J_* Hx(t) + G_* u(t) + C_*\Psi_*(y_*(t), Hx(t), u(t)) + L_*\rho(t),\\ y_*(t) &= x_*(t).\end{aligned}$$

If measurement noises are absent or negligible, then the estimate is given by the expression (2.9). Otherwise, an interval observer is designed under the following assumption for $\Psi_{**}(y_*, Hx, u) = C_*\Psi_*(y_*, Hx, u)$.

Assumption. If $\underline{w} \leqslant w(t) \leqslant \overline{w}$ for $\underline{w}, \overline{w} \in \mathbb{R}^l$, then

$$\underline{\Psi}_{**}(y_*, \underline{y}, \overline{y}, u) \leqslant \Psi_{**}(y_*, Hx, u) \leqslant \overline{\Psi}_{**}(y_*, \underline{y}, \overline{y}, u) \tag{4.2}$$

for some functions $\underline{\Psi}_{**}(y_*, \underline{y}, \overline{y}, u)$ and $\overline{\Psi}_{**}(y_*, \underline{y}, \overline{y}, u)$ and all u, where $\underline{y} = y - \overline{w}$ and $\overline{y} = y - \underline{w}$. The assumption itself is valid for arbitrary functions, but according to [13, 14], the procedure for computing the bounds $\underline{\Psi}_{**}(y_*, \underline{y}, \overline{y}, u)$ and $\overline{\Psi}_{**}(y_*, \underline{y}, \overline{y}, u)$ is known only for a continuous function $\Psi_{**}(y_*, Hx, u)$; it involves methods of interval arithmetic. In special cases, however, condition (4.2) can be established heuristically; see an illustrative example below.

The interval observer takes the form

$$\begin{aligned}\dot{\underline{y}}_*(t) &= \lambda\underline{y}_*(t) + G_{*i}u(t) + J_* y(t) + J_*^-\underline{w} - J_*^+\overline{w}\\ &\quad + L_*^+\underline{\rho} - L_*^-\overline{\rho} + \underline{\Psi}_{**}(y_*(t), \underline{y}(t), \overline{y}(t), u(t)),\\ \dot{\overline{y}}_*(t) &= \lambda\overline{y}_*(t) + G_{*i}u(t) + J_* y(t) + J_*^-\overline{w} - J_*^+\underline{w}\\ &\quad + L_*^+\overline{\rho} - L_{*i}^-\underline{\rho} + \overline{\Psi}_{**}(y_*(t), \underline{y}(t), \overline{y}(t), u(t)).\end{aligned}$$

Theorem 2. *The refined upper* $\overline{\overline{\rho}}(t)$ *and lower* $\underline{\underline{\rho}}(t)$ *bounds without using a virtual sensor are given by*

$$
\begin{aligned}
\overline{\overline{\rho}}(t) &= L_*^{-1}\Big(\dot{\beta}(t)(\overline{y}_*(t)-\underline{y}_*(t)) + L_*^{+}\underline{\rho} - L_*^{-}\overline{\rho} \\
&\quad + \beta(t)(|L_*|(\overline{\rho}-\underline{\rho}) + |J_*|(\overline{w}-\underline{w}) + \overline{\Psi}_{**}(y_*(t),\underline{y}(t),\overline{y}(t),u(t)) \\
&\quad - \underline{\Psi}_{**}(y_*(t),\underline{y}(t),\overline{y}(t),u(t)))\Big), \\
\underline{\underline{\rho}}(t) &= L_*^{-1}\Big(\dot{\beta}(t)(\overline{y}_*(t)-\underline{y}_*(t)) + L_*^{+}\overline{\rho} - L_*^{-}\underline{\rho} + (\beta(t)-1)(|L_*|(\overline{\rho}-\underline{\rho}) \\
&\quad + |J_*|(\overline{w}-\underline{w}) + \overline{\Psi}_{**}(y_*(t),\underline{y}(t),\overline{y}(t),u(t)) \\
&\quad - \underline{\Psi}_{**}(y_*(t),\underline{y}(t),\overline{y}(t),u(t)))\Big).
\end{aligned}
\tag{4.3}
$$

Proof. In view of the nonlinear term, the estimate (3.4) takes the form

$$
\begin{aligned}
\overline{\overline{\rho}}(t) &= L_*^{-1}(\dot{\beta}(t)(\overline{y}_*(t)-\underline{y}_*(t)) + L_*^{+}\underline{\rho} - L_*^{-}\overline{\rho} + \beta(t)(|L_*|(\overline{\rho}-\underline{\rho}) + |J_*|(\overline{w}-\underline{w}) \\
&\quad + \overline{\Psi}_{**}(y_*(t),\underline{y}(t),\overline{y}(t),u(t)) - \underline{\Psi}_{**}(y_*(t),\underline{y}(t),\overline{y}(t),u(t)))) \\
&\quad + \underline{\Psi}_{**}(y_*(t),\underline{y}(t),\overline{y}(t),u(t)) - \Psi_{**}(y_*,Hx,u).
\end{aligned}
$$

Due to (4.2), the difference $\underline{\Psi}_{**}(y_*(t),\underline{y}(t),\overline{y}(t),u(t)) - \Psi_{**}(y_*,Hx,u)$ is negative; then, by removing it from the last expression, we obtain the first of formulas (4.3). Similarly, the estimate (3.5) takes the form

$$
\begin{aligned}
\underline{\underline{\rho}}(t) &= L_*^{-1}(\dot{\beta}(t)(\overline{y}_*(t)-\underline{y}_*(t)) + L_*^{+}\overline{\rho} - L_*^{-}\underline{\rho} + (\beta(t)-1)(|L_*|(\overline{\rho}-\underline{\rho}) + |J_*|(\overline{w}-\underline{w}) \\
&\quad + \overline{\Psi}_{**}(y_*(t),\underline{y}(t),\overline{y}(t),u(t)) - \underline{\Psi}_{**}(y_*(t),\underline{y}(t),\overline{y}(t),u(t)))) \\
&\quad + \overline{\Psi}_{**}(y_*(t),\underline{y}(t),\overline{y}(t),u(t)) - \Psi_{**}(y_*,Hx,u).
\end{aligned}
$$

According to (4.2), $\overline{\Psi}_{**}(y_*(t),\underline{y}(t),\overline{y}(t),u(t)) - \Psi_{**}(y_*,Hx,u) \geqslant 0$, and, by removing this difference, we obtain the second formula in (4.3). The proof of Theorem 2 is complete.

The nonlinearity entering the virtual sensor can be considered by analogy, i.e., using a corresponding assumption of the form (4.2).

5. EXAMPLE

We continue the analysis of system (2.5), for which model (2.6) has been constructed in the form

$$
\begin{aligned}
\dot{x}_{*1}(t) &= k_2x_{*1}(t) + k_3Hx(t) + k_7\,\mathrm{sgn}(x_{*1}(t)), \\
\dot{x}_{*2}(t) &= k_5x_{*2}(t) + k_4x_{*1}(t) + k_6u(t) + \rho(t), \\
y_*(t) &= x_{*2}(t) = y(t).
\end{aligned}
\tag{5.1}
$$

Since $k_4 < 0$, $k_3 > 0$, and $k_7 < 0$, we have $k_3^{+} = k_3$, $k_3^{-} = 0$, $k_4^{+} = 0$, $k_4^{-} = |k_4|$, $k_7^{+} = 0$, $k_7^{-} = |k_7|$, and $L_* = 1$.

In the absence of measurement noises, the exact estimate of the input $\rho(t)$ is given by the expression (2.11):

$$\hat{\rho}(t) = \dot{y}(t) - (k_5y(t) + k_6u(t) + k_4x_{*1}(t)),$$

and the virtual sensor is given by the first equation in (5.1) with $Hx(t) = y(t)$. In the presence of noises, it is necessary to construct the interval observer (3.1); together with the virtual sensor, it takes the form

$$
\begin{aligned}
\dot{\underline{x}}_{*1}(t) &= k_2\underline{x}_{*1}(t) + k_3y(t) - k_3\overline{w} - |k_7|\,\mathrm{sgn}(\overline{x}_{*1}(t)), \\
\dot{\overline{x}}_{*1}(t) &= k_2\overline{x}_{*1}(t) + k_3y(t) - k_3\underline{w} - |k_7|\,\mathrm{sgn}(\underline{x}_{*1}(t)), \\
\dot{\underline{y}}_*(t) &= k_5\underline{y}_*(t) + k_6u(t) - |k_4|\overline{x}_{*1}(t) + \underline{\rho}, \\
\dot{\overline{y}}_*(t) &= k_5\overline{y}_*(t) + k_6u(t) - |k_4|\underline{x}_{*1}(t) + \overline{\rho}.
\end{aligned}
\tag{5.2}
$$

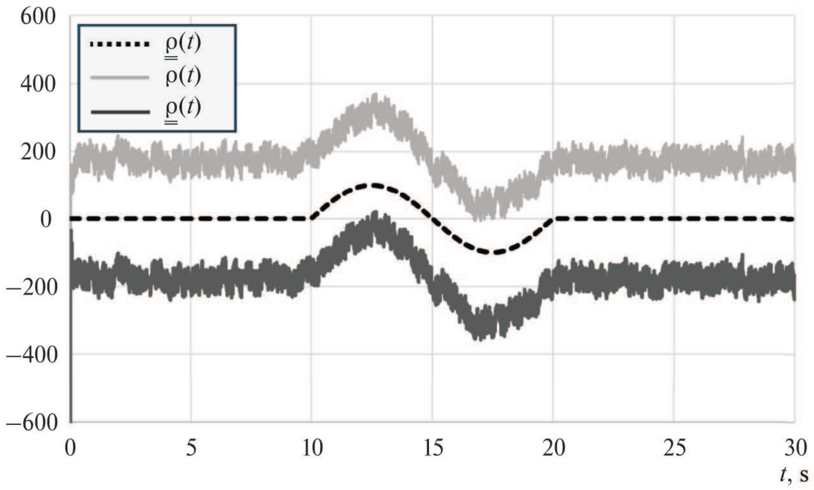

Fig. 1. The unknown input and its lower and upper bounds as functions of time.

Let us introduce the error $\underline{e}_1(t) = x_{*1}(t) - \underline{x}_{*1}(t)$. Obviously, $\underline{x}(0) \leqslant x(0) \leqslant \overline{x}(0)$ and $x_{*1}(t) = \Phi x(t)$ imply $\underline{x}_{*1}(0) \leqslant x_{*1}(0) \leqslant \overline{x}_{*1}(0)$ and $\underline{e}_1(0) \geqslant 0$. Further, if $\underline{x}_{*1}(t) \leqslant x_{*1}(t) \leqslant \overline{x}_{*1}(t)$, then $\operatorname{sgn}(\underline{x}_{*1}(t)) \leqslant \operatorname{sgn}(x_{*1}(t)) \leqslant \operatorname{sgn}(\overline{x}_{*1}(t))$, and since $k_7 < 0$, we have

$$-|k_7| \operatorname{sgn}(\underline{x}_{*1}(t)) \geqslant k_7 \operatorname{sgn}(x_{*1}(t)) \geqslant -|k_7| \operatorname{sgn}(\overline{x}_{*1}(t)).$$

In other words, $k_7 \operatorname{sgn}(x_{*1}(t))$ satisfies an assumption of the form (4.2) with the functions

$$\underline{\Psi}_{**}(\underline{x}_{*1}, \overline{x}_{*1}) = -|k_7| \operatorname{sgn}(\overline{x}_{*1}), \quad \overline{\Psi}_{**}(\underline{x}_{*1}, \overline{x}_{*1}) = -|k_7| \operatorname{sgn}(\underline{x}_{*1}).$$

The error $\underline{e}_1(t)$ satisfies the equation

$$\dot{\underline{e}}_1(t) = k_2 \underline{e}_1(t) - k_3 w(t) + k_3 \overline{w} + k_7 \operatorname{sgn}(x_{*1}(t)) + |k_7| \operatorname{sgn}(\overline{x}_{*1}(t)).$$

In view of $k_2 < 0$, $\underline{e}_1(0) \geqslant 0$, and $-k_3 w(t) + k_3 \overline{w} + k_7 \operatorname{sgn}(x_{*1}(t)) + k_7 \operatorname{sgn}(x_{*1}(t)) \geqslant 0$, according to [13], by induction we obtain $\underline{e}_1(t) \geqslant 0$, i.e., $x_{*1}(t) \geqslant \underline{x}_{*1}(t)$ for all $t \geqslant 0$. Similar operations can be used to prove the inequality $\overline{x}_{*1}(t) \geqslant x_{*1}(t)$ for all $t \geqslant 0$.

By analogy, based on the above relations and the third and fourth equations in (5.2), it can be shown that $\overline{y}_*(t) \geqslant y_*(t) \geqslant \underline{y}_*(t)$ for all $t \geqslant 0$; and the function $\beta(t)$ can be determined in the form (3.3).

The refined lower $\underline{\underline{\rho}}$ and upper $\overline{\overline{\rho}}$ bounds for $\rho(t)$ take the form

$$\begin{aligned}
\overline{\overline{\rho}} &= \dot{\beta}(t)(\overline{y}_*(t) - \underline{y}_*(t)) + \underline{\rho} + \beta(t)(\overline{\rho} - \underline{\rho} + |k_3|(\overline{w} - \underline{w}) \\
&\quad + |k_4|(\overline{x}_{*1}(t) - \underline{x}_{*1}(t)) - |k_7|(\operatorname{sgn}(\underline{x}_{*1}) - \operatorname{sgn}(\overline{x}_{*1}))), \\
\underline{\underline{\rho}} &= \dot{\beta}(t)(\overline{y}_*(t) - \underline{y}_*(t)) + \overline{\rho} + (\beta(t) - 1)(\overline{\rho} - \underline{\rho} + |k_3|(\overline{w} - \underline{w}) \\
&\quad + |k_4|(\overline{x}_{*1}(t) - \underline{x}_{*1}(t)) - |k_7|(\operatorname{sgn}(\underline{x}_{*1}) - \operatorname{sgn}(\overline{x}_{*1}))).
\end{aligned}$$

The derivative $\dot{\beta}(t)$ is computed by formula (3.6) with $Ry = y$.

The following model parameters were used for simulation: $k_1 = 0.01$, $k_2 = -10$, $k_3 = 100$, $k_4 = -25$, $k_5 = -100$, $k_6 = 1000$, and $k_7 = -0.02$; the measurement noise was represented by a random variable with the uniform distribution on the interval $[-0.3, 0.3]$, i.e., $\underline{w} = -0.3$ and $\overline{w} = 0.3$; the unknown input was specified as $\rho(t) = 100 \sin(\pi t/5)$ for $10 \leqslant t \leqslant 20$, and $\rho(t) = 0$ otherwise, hence $\underline{\rho} = -100$ and $\overline{\rho} = 100$. The simulation results with $u(t) = 2\sin(t/5) + 2$ are shown in the figure. Due to the presence of measurement noises, as well as the large value of the coefficient k_3, the refined bounds $\underline{\underline{\rho}}$ and $\overline{\overline{\rho}}$ are strongly noisy; and their average values are explained by the accepted preliminary values of the bounds $\underline{\rho}$ and $\overline{\rho}$.

6. CONCLUSIONS

In this paper, we have estimated (identified) the magnitudes of unknown inputs in systems described by nonlinear models with measurement noises. In contrast to known solution methods using sliding mode observers, the approach developed above expands the class of systems for which the identification procedure can be performed. The reason is that the design method for sliding mode observers restricts the systems for which unknown inputs need to be identified. Note also that unknown inputs can be interpreted as faults in system dynamics; therefore, the proposed methods are applicable to design fault-tolerant systems by compensating for the estimated fault effect by a control system.

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