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NONLINEAR SYSTEMS

A Hybrid–Robust Method for Estimating the Values of Linear Functionals under Essential Uncertainty and the Nonlinear Factor

Yu. G. Bulychev

JSC Concern Radioelectronic Technologies, Moscow, Russia
e-mail: profbulychev@yandex.ru

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Abstract—For secondary data processing tasks, a hybrid-robust method is developed to estimate various numerical characteristics (linear functionals) optimally under parametric uncertainty factors related to the information process, regular and irregular interferences, and the correlation function of measurement noise (including the case of nonlinear parameters). The method is fully or partially invariant with respect to these factors and yields the desired estimates without expanding the vector of estimated parameters or employing linearization procedures. A comparison with known estimation methods is presented, and the random and methodological errors, as well as the achieved computational effect, are analyzed. An illustrative example is provided.

Keywords: secondary data processing, essential uncertainty, vector linear functional, spectral coefficients, nonlinear factor, hybrid–robust method, invariance condition, constrained optimization, family of hypotheses, extended least squares method, decomposition

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1. INTRODUCTION

When solving a wide range of secondary data processing tasks based on indirect measurements (the issues related to signal detection, discrimination, and recognition having been settled), essential uncertainty often arises. It may be associated with no reliable knowledge regarding the distribution laws of measurement errors, the presence of anomalous single and group errors, etc.; for example, see [1–32]. Such situations are often encountered, for instance, when determining the motion of aircraft from trajectory measurements in the rocket and space sphere [3].

Data processing algorithms at these stages are based on classical probabilistic methods of maximum a posteriori probability density or maximum likelihood, usually in an extended version; for example, we refer to [1–4]. In this version, all unknown uncertainty parameters are included in the expanded state vector; however, in practice, the expansion procedure leads to a sharp increase in computational costs, the well-known effect of "smearing accuracy," and problems with the convergence and stability of the resulting estimates. According to an analysis, classical probabilistic estimation methods are often unsuitable under essential uncertainty; nevertheless, they can be effectively used to create hybrid-robust algorithms or study the potential capabilities of measuring systems at the stage of mathematical modeling.

There are other estimation methods applicable under uncertainty: the extended least squares method (ELS) [3], methods for minimizing different geometric and kinematic residuals [30], the method of invariants [31], various linear and nonlinear filtering methods (e.g., Kalman filtering) [1, 3, 10, 14, 25, 26], various adaptive methods with parameter tuning [9, 26], robust methods [1, 14, 27], randomized methods [20, 21], cluster methods [21, 32], and neural network methods [28], as well as numerous heuristic methods (for example, see [5, 8, 11, 13, 29]). Under several constraints on the observation conditions, these methods overcome uncertainty factors, e.g., successfully deal with anomalous errors. However, under essential uncertainty, these methods encounter various unforeseen limitations and stringent requirements for measurement processing algorithms, which significantly narrow the scope of their application. Moreover, they often ensure convergence on average (either over an ensemble of realizations or over a single sufficiently long realization), without any guarantee for the quality of the resulting estimate for a single sample of small or medium size. Most known estimation procedures involve weight processing, and the weights are calculated a priori based on a large amount of available statistical information. If such information is absent or unreliable, one should use posterior weights and vary them from one measurement sample to another.

As a rule, the methods considered above are intended to estimate spectral coefficients of information processes or find their smoothed values. However, in measurement practice, a more general problem is highly relevant, i.e., the optimal estimation of various numerical characteristics of information processes. This concerns the vector of linear functionals (VLF), whose coordinates can be, e.g., the derivatives of various orders of information processes used to predict the motion of aircraft and perform their identification in civil and military systems.

For relatively favorable observation conditions, VLF can also be estimated using classical probabilistic methods, e.g., maximum a posteriori probability density or maximum likelihood, the ordinary least squares (OLS) method and its extended analog (ELS), as well as their various modifications. In the presence of regular-structure interferences in observations (representable as a linear span of a given system of basis functions with unknown spectral coefficients), the method of generalized invariant-unbiased estimation (GIUE) [15, 16] is applicable for the optimal estimation of VLF. This method leads to the decomposition of computations and the inversion of matrices of smaller dimensions (compared to ELS), and uses information about zero mean and a given correlation function of the measurement noise as a priori data.

If observations contain not only regular but also irregular interferences (the later have no compact mathematical models in terms of the number of degrees of freedom), the modified GIUE method can be used for the spectral analysis of the components of the input observation [17]. This modification yields good results on samples of an appropriate size when a sufficient number of reduced time grids can be created so that some of them are not associated with irregular interference. However, for a large class of applied problems dealing with medium and small samples, the application of such a method becomes a challenge. Furthermore, the modified GIUE method is oriented towards the case of an a priori known correlation matrix of the input observation noise. In practice, this noise can be semi-defined, i.e., non-centered (with an unknown mean) and characterized by a parameterized correlation function specified up to its structure. Only some uncertainty region is indicated for its parameters (both linear and nonlinear). This uncertainty regarding the characteristics of the measurement noise, as well as the parameters of regular and irregular interferences, often significantly distorts the estimation results. Therefore, a more advanced method is required to estimate VLF under uncertainty.

Below, we develop such a hybrid–robust method (HRM) considering both a priori and a posteriori data without linearization or expansion of the vector of estimated parameters. This method retains all the advantages of GIUE [15, 16] and its modification [17] in terms of decomposition, employs only elementary operations of addition, multiplication, and matrix inversion, and constructs the resulting VLF estimate by a simple operation of combining a set of conditionally optimal partial estimates corresponding to the family of alternative hypotheses. The term "decomposition" means that it is possible to estimate independently (separately) any numerical characteristic (linear functional) for each component of the input observation representable as a finite linear combination of given basis functions with unknown spectral coefficients. Such components include irregular and regular interferences.

HRM is primarily intended for measuring systems with parallel computations to ensure the required speed for real-time processing of input observations. Such computations effectively overcome the nonlinearity problem by organizing many data processing channels for the family of alternative hypotheses regarding the nonlinear uncertainty factor.

2. BASIC CONCEPTS, MODELS, AND CONSTRAINTS

Let $x_{\mathrm{IP}}(t) = x_1(t)$ and $x_{\mathrm{RI}}(t) = x_2(t)$ denote an information process and a regular interference, respectively:

$$x_i(t) = \boldsymbol{\psi}_i^{\mathrm{T}}(t)\boldsymbol{\gamma}_i, \quad i \in \{1, 2\}, \quad t \in [0, T], \tag{2.1}$$

where $\boldsymbol{\psi}_i(t) = \left[\psi_{im}(t), m = \overline{1, V_i}\right]^{\mathrm{T}}$ is the vector of given basis functions, and $\boldsymbol{\gamma}_i = \left[\gamma_{im}, m = \overline{1, V_i}\right]^{\mathrm{T}}$ is the vector of unknown spectral coefficients.

We also introduce an irregular interference $z(t)$ present in measurements, in one form or another; in practice, it cannot be analytically described in compact terms and has a small number of degrees of freedom. For example, the irregular interference may contain components of the tails of infinite functional series used to describe the information process and regular interference, as well as an unknown mean of the measurement noise. Furthermore, $z(t)$ may include various measurement errors, such as anomalous (single or group) errors of both natural and intentional origin. In addition, under various observation conditions, unexpected slow and/or fast-changing components of the irregular interference may appear, the nature of which is unknown in advance. Such an interference is extremely difficult to combat, as it does not match the expected conditions of the measurement experiment; it can be characterized by the sample clogging coefficient [17].

The equation of continuous-time observations has the form

$$h(t) = x_1(t) + x_2(t) + z(t) + \xi(t), \quad t \in [0, T], \tag{2.2}$$

where $\xi(t)$ is a centered semi-defined measurement noise characterized by zero mean and a parameterized (specified up to parameters) correlation function $k(t', t'', \mathbf{q})$, where $t', t'' \in [0, T]$ and $\mathbf{q} = \left[q_r, r = \overline{1, Q}\right]^{\mathrm{T}}$ is the vector of constant parameters.

When implementing known estimation methods (OLS, ELS, and GIUE), a nominal fixed value $\mathbf{q}_0$ is conventionally chosen for $\mathbf{q}$.

In the case of a non-centered noise $\widetilde{\xi}(t)$, it is also easy to use equation (2.2) by including the non-zero mean either in the regular interference (when described by a small set of basis functions) or directly in the irregular interference (when having an extended spectral representation). Constraints are imposed on the coordinates of the vector $\mathbf{q}$ in the form of a possible uncertainty region $\alpha_r \leqslant q_r \leqslant \beta_r$, where $\alpha_r > 0$ and $\beta_r > 0$; note that $\mathbf{q}_0$ belongs to this region.

We introduce the VLF (the set of numerical characteristics) $\boldsymbol{\Omega} = \left[\boldsymbol{\Omega}_1^{\mathrm{T}}, \boldsymbol{\Omega}_2^{\mathrm{T}}\right]^{\mathrm{T}} = \left[\varsigma_j, j = \overline{1, M_1 + M_2}\right]^{\mathrm{T}}$, where

$$\boldsymbol{\Omega}_i = \boldsymbol{\Omega}_i\{x_i(t)\} = \left[\omega_{ir}\{x_i(t)\}, r = \overline{1, M_i}\right]^{\mathrm{T}} = \left[\varsigma_{ir}, r = \overline{1, M_i}\right]^{\mathrm{T}}, \quad i \in \{1, 2\},$$

and ς_{ir} is the rth numerical characteristic for the component $x_i(t)$ from (2.1).

Consider the general case where different numerical characteristics (linear functionals) can be calculated separately for each component $x_1(t)$ or $x_2(t)$.

The problem is to find an optimal estimate $\boldsymbol{\Omega}_i^* = \mathbf{P}_i^{\boldsymbol{\Omega}}\mathbf{H}$ for the VLF $\boldsymbol{\Omega}_i$. Here, $\mathbf{P}_i^{\boldsymbol{\Omega}}$ is the linear estimation matrix, and $\mathbf{H}$ is the observation vector (a discrete sample for the function $h(t)$ on a time grid $\{t_1, \ldots, t_N\}$):

$$\mathbf{H} = \mathbf{X}_1 + \mathbf{X}_2 + \mathbf{Z} + \boldsymbol{\Xi}, \tag{2.3}$$

with $\mathbf{X}_i = \left[x_{in}, n = \overline{1, N}\right]^{\mathrm{T}}$, $x_{in} = x_i(t_n)$, $i \in \{1, 2\}$, $\mathbf{Z} = \left[z_n, n = \overline{1, N}\right]^{\mathrm{T}}$, $\boldsymbol{\Xi} = \left[\xi_n, n = \overline{1, N}\right]^{\mathrm{T}}$, $z_n = z(t_n)$, $\xi_n = \xi(t_n)$, and $t_n \in [0, T]$.

The irregular interference can be characterized by the clogging coefficient $\rho_{\mathbf{Z}} = 100 \times N_{\mathbf{Z}}/N$ [%], where $N_{\mathbf{Z}}$ is the number of all coordinates of the vector $\mathbf{Z}$ that can significantly affect the estimation results when using any optimal measurement processing algorithms neglecting this interference. By assumption, the sample size (2.3) at least exceeds the total number of degrees of freedom of the adopted mathematical models (i.e., $N \geqslant V_1 + V_2 + Q + N_{\mathbf{Z}}$, and often $N \gg V_1 + V_2 + Q + N_{\mathbf{Z}}$ in practice) and provides the necessary smoothing effect with respect to the noise $\boldsymbol{\Xi}$.

With the adopted models for the components $x_1(t)$ and $x_2(t)$, we represent the sample (2.3) as

$$\mathbf{H} = \boldsymbol{\Psi}_1\boldsymbol{\gamma}_1 + \boldsymbol{\Psi}_2\boldsymbol{\gamma}_2 + \mathbf{Z} + \boldsymbol{\Xi} = \boldsymbol{\Psi}\boldsymbol{\gamma} + \mathbf{Z} + \boldsymbol{\Xi}, \tag{2.4}$$

where: $\boldsymbol{\Psi}_i = \left[\psi_{imn}, n = \overline{1, N}, m = \overline{1, V_i}\right]$ is the basis matrix of the component $x_i(t)$, $\psi_{imn} = \psi_{im}(t_n)$; $\boldsymbol{\Psi} = [\boldsymbol{\Psi}_1; \boldsymbol{\Psi}_2]$ is the combined basis matrix for the two components of the observation equation (2.3); and $\boldsymbol{\gamma} = \left[\boldsymbol{\gamma}_1^{\mathrm{T}}, \boldsymbol{\gamma}_2^{\mathrm{T}}\right]^{\mathrm{T}} = \left[\gamma_j, j = \overline{1, V_1 + V_2}\right]^{\mathrm{T}}$.

The matrix $\mathbf{P}_i^{\boldsymbol{\Omega}} = [p_{imn}^{\boldsymbol{\Omega}}, m = \overline{1, M_i}, n = \overline{1, N}]$ shall satisfy a number of conditions.

The first condition (the invariance of the estimate with respect to $\mathbf{X}_j$) is

$$\mathbf{P}_i^{\boldsymbol{\Omega}}\mathbf{X}_j = \mathbf{P}_i^{\boldsymbol{\Omega}}\boldsymbol{\Psi}_j\boldsymbol{\gamma}_j = [\mathbf{0}]_{M_i \times 1}, \quad i \neq j. \tag{2.5}$$

The second condition (the unbiasedness of the estimate) is

$$\mathbf{P}_i^{\boldsymbol{\Omega}}\mathbf{X}_i = \mathbf{P}_i^{\boldsymbol{\Omega}}\boldsymbol{\Psi}_i\boldsymbol{\gamma}_i = \boldsymbol{\Omega}_i. \tag{2.6}$$

The third condition (the minimum trace of the correlation matrix $\mathbf{K}_i^{\boldsymbol{\Omega}} = \left[k_{ilp}^{\boldsymbol{\Omega}}, l, p = \overline{1, M_i}\right]$ of estimation errors) is

$$\operatorname{Sp} \mathbf{K}_i^{\boldsymbol{\Omega}} = \operatorname{Sp}\left[\mathbf{P}_i^{\boldsymbol{\Omega}}\mathbf{K}\left(\mathbf{P}_i^{\boldsymbol{\Omega}}\right)^{\mathrm{T}}\right] \to \min, \tag{2.7}$$

where $\mathbf{K} = \mathbf{K}(\mathbf{q}) = \left[k(t_n, t_m, \mathbf{q}), n, m = \overline{1, N}\right]$ stands for the correlation matrix of the noise $\boldsymbol{\Xi}$.

According to condition (2.5), when estimating the VLF for $x_i(t)$, the other component $x_j(t)$ is treated as an interference and vice versa. Conditions (2.6) and (2.7) are typical for any optimal linear estimation procedure [1–4].

In addition to the time grid $\{t_1, \ldots, t_N\}$, we also use a multidimensional node grid $\{\mathbf{q}_{(1)}, \ldots, \mathbf{q}_{(L)}\}$ covering the region $[\alpha_1, \beta_1] \times \ldots \times [\alpha_Q, \beta_Q]$ (e.g., uniformly) so that the cardinality L is sufficient for a qualitative description of the function $k(t', t'', \mathbf{q})$ over the entire domain of definition $[0, T] \times [0, T] \times [\alpha_1, \beta_1] \times \ldots \times [\alpha_Q, \beta_Q]$. The matrix $\mathbf{P}_i^{\boldsymbol{\Omega}}$ depends on $\mathbf{q}$; consequently, $\mathbf{P}_i^{\boldsymbol{\Omega}} = \mathbf{P}_i^{\boldsymbol{\Omega}}(\mathbf{q})$ and $\boldsymbol{\Omega}_i^* = \boldsymbol{\Omega}_i^*(\mathbf{q})$. Thus, an optimal value $\mathbf{q}^*$ for the parameter $\mathbf{q}$ shall be chosen according to an additional decision rule considering the presence of the component $\mathbf{Z}$ in (2.3) and the resulting set of partial estimates of the parameters $\boldsymbol{\gamma}_1$ and $\boldsymbol{\gamma}_2$ for the node family $\{\mathbf{q}_{(1)}, \ldots, \mathbf{q}_{(L)}\}$.

Taking into account (2.1)–(2.7), it is required to develop an HRM for estimating the values of the VLF $\mathbf{\Omega}_i$, $i \in \{1,2\}$,) under uncertainty factors regarding the vector parameters $\boldsymbol{\gamma}_1$, $\boldsymbol{\gamma}_2$, $\mathbf{Z}$, and $\mathbf{q}$. In such a general formulation, the estimation problem is nonlinear since some coordinates of the vector $\mathbf{q}$ may enter the function $k(t', t'', \mathbf{q})$ nonlinearly.

The HRM shall eliminate the above problems associated with nonlinearity and be oriented towards the elementary operations of addition, multiplication, and inversion of matrices in a linear setting. In contrast to GIUE [15, 16] and its known modification [17], a more general problem is considered here: VLF estimation is performed in addition to its spectral analysis; the measurement noise is semi-defined (there exists a parametric uncertainty regarding its correlation function due to the nonlinear factor); and, moreover, the irregular interference $\mathbf{Z}$ is present in the observation. Finally, the capability to separately estimate any coordinate (numerical characteristic) of the VLF, for both the information process and the regular interference, shall be provided.

3. FUNDAMENTAL FORMULAS

Let $\mathbf{P}_{ir}^{\mathbf{\Omega}} = \left[p_{irn}^{\mathbf{\Omega}}, n = \overline{1, N}\right]^{\mathrm{T}}$ denote the column vector representing the rth row of the matrix $\mathbf{P}_i^{\mathbf{\Omega}}$. This vector is used to find the estimate $\varsigma_{ir}^* = \left(\mathbf{P}_{ir}^{\mathbf{\Omega}}\right)^{\mathrm{T}} \mathbf{H}$ of the individual coordinate (numerical characteristic) ς_{ir} of the VLF $\mathbf{\Omega}_i$.

For a fixed value of $\mathbf{q}$, based on the above conditions (2.5)–(2.7), the optimal estimation vector $\mathbf{P}_{ir}^{\mathbf{\Omega}} = \mathbf{P}_{ir}^{\mathbf{\Omega}}(\mathbf{q})$ is obtained by minimizing the function

$$J\left(\mathbf{P}_{ir}^{\mathbf{\Omega}}, \boldsymbol{\varphi}_{ir}, \boldsymbol{\theta}_{ir}\right) = \left(\mathbf{P}_{ir}^{\mathbf{\Omega}}\right)^{\mathrm{T}} \mathbf{K}\mathbf{P}_{ir}^{\mathbf{\Omega}} + (\boldsymbol{\varphi}_{ir})^{\mathrm{T}} (\mathbf{\Psi}_j)^{\mathrm{T}} \mathbf{P}_{ir}^{\mathbf{\Omega}} + \left\{\omega_r^{\mathrm{T}}\{\boldsymbol{\psi}_i(t)\} - \left(\mathbf{P}_{ir}^{\mathbf{\Omega}}\right)^{\mathrm{T}} \mathbf{\Psi}_i\right\} \boldsymbol{\theta}_{ir}, \tag{3.1}$$

where: $[\boldsymbol{\varphi}_{ir}]_{V_j \times 1}$ and $[\boldsymbol{\theta}_{ir}]_{V_i \times 1}$ are vector Lagrange multipliers (from this point onwards, square brackets with subscripts indicate the dimensions of the corresponding matrices and vectors); $\omega_r\{\boldsymbol{\psi}_i(t)\} = \left[\omega_r\{\psi_{im}\}, m = \overline{1, M_i}\right]^{\mathrm{T}}$ is the value vector of the scalar functional $\omega_r\{\cdot\}$ on all basis functions of the component $x_i(t)$ from (2.1). (By the natural assumption, $i \neq j$.)

The function (3.1) corresponds to the equivalent conditions formulated previously:

$$\omega_r^{\mathrm{T}}\{\boldsymbol{\psi}_i(t)\} - \left(\mathbf{P}_{ir}^{\mathbf{\Omega}}\right)^{\mathrm{T}} \mathbf{\Psi}_i = [\mathbf{0}]_{1 \times V_i} \tag{3.2}$$

(the first condition),

$$\mathbf{\Psi}_j^{\mathrm{T}} \mathbf{P}_{ir}^{\mathbf{\Omega}} = [\mathbf{0}]_{V_j \times 1} \tag{3.3}$$

(the second condition), and

$$\min_{\mathbf{P}_{ir}^{\mathbf{\Omega}}} \left(\sigma_{ir}^{\mathbf{\Omega}}\right)^2 = \min_{\mathbf{P}_{ir}^{\mathbf{\Omega}}} \left(\mathbf{P}_{ir}^{\mathbf{\Omega}}\right)^{\mathrm{T}} \mathbf{K}\mathbf{P}_{ir}^{\mathbf{\Omega}} \tag{3.4}$$

(the third condition), where $\left(\sigma_{ir}^{\mathbf{\Omega}}\right)^2$ means the variance of the estimation error of the numerical characteristic ς_{ir}.

Omitting straightforward but cumbersome calculations, we obtain the following solution of the constrained optimization problem (3.1)–(3.4) (by analogy with [17]):

$$\mathbf{P}_{ir}^{\mathbf{\Omega}} = \mathbf{\Lambda}_j \mathbf{K}^{-1} \mathbf{\Psi}_i \left(\mathbf{\Psi}_i^{\mathrm{T}} \mathbf{\Lambda}_j \mathbf{K}^{-1} \mathbf{\Psi}_i\right)^{-1} \omega_r\{\boldsymbol{\psi}_i(t)\}, \tag{3.5}$$

$$\mathbf{\Lambda}_j = \mathbf{E} - \mathbf{K}^{-1} \mathbf{\Psi}_j \left[(\mathbf{\Psi}_j)^{\mathrm{T}} \mathbf{K}^{-1} \mathbf{\Psi}_j\right]^{-1} (\mathbf{\Psi}_j)^{\mathrm{T}}, \tag{3.6}$$

where $\mathbf{E}$ is an identity matrix of dimensions $N \times N$.

In view of (3.5), the desired matrix $\mathbf{P}_i^{\boldsymbol{\Omega}} = \mathbf{P}_i^{\boldsymbol{\Omega}}(\mathbf{q})$ for the linear estimation of the VLF is

$$\mathbf{P}_i^{\boldsymbol{\Omega}} = \left[\boldsymbol{\Lambda}_j \mathbf{K}^{-1} \boldsymbol{\Psi}_i \left(\boldsymbol{\Psi}_i^{\mathrm{T}} \boldsymbol{\Lambda}_j \mathbf{K}^{-1} \boldsymbol{\Psi}_i\right)^{-1} \boldsymbol{\Omega}\{\boldsymbol{\psi}_i(t)\}\right]^{\mathrm{T}}, \tag{3.7}$$

where $\boldsymbol{\Omega}\{\boldsymbol{\psi}_i(t)\} = \left[\omega_r\{\psi_{im}(t)\}, m = \overline{1, V_i}, r = \overline{1, M_i}\right] = \left[\varsigma_{irm}^{\boldsymbol{\psi}}, m = \overline{1, V_i}, r = \overline{1, M_i}\right]$.

We emphasize that all components in (3.5)–(3.7) depend on the parameter $\mathbf{q}$.

In the special case where $M_i = V_i$ and $\boldsymbol{\Omega}_i = \boldsymbol{\gamma}_i$ (i.e., the VLF consists of the spectral coefficients of the component $x_i(t)$), according to (3.7), the linear estimation matrix is given by

$$\mathbf{P}_i^{\boldsymbol{\gamma}} = \left[\boldsymbol{\Lambda}_j \mathbf{K}^{-1} \boldsymbol{\Psi}_i \left(\boldsymbol{\Psi}_i^{\mathrm{T}} \boldsymbol{\Lambda}_j \mathbf{K}^{-1} \boldsymbol{\Psi}_i\right)^{-1}\right]^{\mathrm{T}}. \tag{3.8}$$

For fixed values of i and $\mathbf{q}$, we construct the optimal estimate for the VLF:

$$\boldsymbol{\Omega}_i^*(\mathbf{q}) = \mathbf{P}_i^{\boldsymbol{\Omega}}(\mathbf{q})\mathbf{H}. \tag{3.9}$$

It corresponds to the following correlation matrix of estimation errors:

$$\mathbf{K}_i^{\boldsymbol{\Omega}}(\mathbf{q}) = \mathbf{P}_i^{\boldsymbol{\Omega}}(\mathbf{q})\mathbf{K}(\mathbf{q}) \left[\mathbf{P}_i^{\boldsymbol{\Omega}}(\mathbf{q})\right]^{\mathrm{T}}. \tag{3.10}$$

In the special case, using the matrix $\mathbf{P}_i^{\boldsymbol{\gamma}}(\mathbf{q})$, we find the optimal estimate

$$\boldsymbol{\gamma}_i^*(\mathbf{q}) = \mathbf{P}_i^{\boldsymbol{\gamma}}(\mathbf{q})\mathbf{H} \tag{3.11}$$

and the correlation matrix

$$\mathbf{K}_i^{\boldsymbol{\gamma}}(\mathbf{q}) = \mathbf{P}_i^{\boldsymbol{\gamma}}(\mathbf{q})\mathbf{K}(\mathbf{q})[\mathbf{P}_i^{\boldsymbol{\gamma}}(\mathbf{q})]^{\mathrm{T}}. \tag{3.12}$$

Note that for $\mathbf{Z} \neq \mathbf{0}$, the above estimates are biased.

For a fixed $\mathbf{q}$, taking the two estimates $\boldsymbol{\gamma}_1^*(\mathbf{q})$ and $\boldsymbol{\gamma}_2^*(\mathbf{q})$ into account, it is possible to find the residual

$$\boldsymbol{\Delta}(\mathbf{q}) = \mathbf{H} - \mathbf{H}^*(\mathbf{q}) = \mathbf{H} - \boldsymbol{\Psi}\boldsymbol{\gamma}^*(\mathbf{q}), \tag{3.13}$$

where $\boldsymbol{\gamma}^*(\mathbf{q}) = \left[[\boldsymbol{\gamma}_1^*(\mathbf{q})]^{\mathrm{T}}, [\boldsymbol{\gamma}_2^*(\mathbf{q})]^{\mathrm{T}}\right]^{\mathrm{T}} = \left[\gamma_j^*(\mathbf{q}), j = \overline{1, V_1 + V_2}\right]^{\mathrm{T}}$ is the estimate for the vector $\boldsymbol{\gamma}(\mathbf{q}) = \left[\boldsymbol{\gamma}_1^{\mathrm{T}}(\mathbf{q}), \boldsymbol{\gamma}_2^{\mathrm{T}}(\mathbf{q})\right]^{\mathrm{T}} = \left[\gamma_j(\mathbf{q}), j = \overline{1, V_1 + V_2}\right]^{\mathrm{T}}$.

The value of $\mathbf{q}$ can be chosen arbitrarily. Therefore, to overcome the nonlinear factor and partially compensate for the impact of the irregular interference $\mathbf{Z}$ on the estimation results, let us proceed as follows. First, we form a family of hypotheses $\left\{\Gamma_{(l)}\right\}_{l=1}^{L}$, each corresponding to some most preferable fixed node $\mathbf{q}_{(l)}$ from the uncertainty region $[\alpha_1, \beta_1] \times \ldots \times [\alpha_Q, \beta_Q]$. Consequently, for each $\mathbf{q}_{(l)}$, it is possible to find the partial estimate $\boldsymbol{\gamma}_{(l)}^* = \boldsymbol{\gamma}^*\left(\mathbf{q}_{(l)}\right)$ and the partial residual

$$\boldsymbol{\Delta}_{(l)}^* = \boldsymbol{\Delta}\left(\boldsymbol{\gamma}_{(l)}^*\right) = \mathbf{H} - \mathbf{H}_{(l)}^* = \mathbf{H} - \boldsymbol{\Psi}\boldsymbol{\gamma}_{(l)}^*.$$

In view of (2.3) and (3.11), we transform this residual to

$$\begin{aligned} \boldsymbol{\Delta}_{(l)}^* &= \mathbf{H} - \left(\boldsymbol{\Psi}_1\boldsymbol{\gamma}_{1(l)}^* + \boldsymbol{\Psi}_2\boldsymbol{\gamma}_{2(l)}^*\right) = \mathbf{Z} + \boldsymbol{\Xi} - \left(\boldsymbol{\Psi}_1\mathbf{P}_{1(l)} + \boldsymbol{\Psi}_2\mathbf{P}_{2(l)}\right)(\mathbf{Z} + \boldsymbol{\Xi}) \\ &= \left[\mathbf{E} - (\boldsymbol{\Psi}_1\mathbf{P}_{1(l)} + \boldsymbol{\Psi}_2\mathbf{P}_{2(l)})\right](\mathbf{Z} + \boldsymbol{\Xi}) = \vartheta_{(l)}(\mathbf{Z} + \boldsymbol{\Xi}), \end{aligned} \tag{3.14}$$

where $[\vartheta_{(l)}]_{N\times N}$ is the matrix of influence coefficients of the component $\mathbf{Z} + \boldsymbol{\Xi}$.

The mean of this residual is

$$M\left\{\boldsymbol{\Delta}^*_{(l)}\right\} = \vartheta_{(l)}\mathbf{Z}. \tag{3.15}$$

Note that for all nodes of the multidimensional grid, the linear estimation matrices $\mathbf{P}^{\boldsymbol{\Omega}}_{ir}(\mathbf{q}_{(l)})$ are constructed using only the a priori statistical information and the adopted mathematical models. The set of such matrices yields the families of partial estimates $\boldsymbol{\Omega}^*_{1(l)} = \mathbf{P}^{\boldsymbol{\Omega}}_{1(l)}\mathbf{H}$, $\boldsymbol{\Omega}^*_{2(l)} = \mathbf{P}^{\boldsymbol{\Omega}}_{2(l)}\mathbf{H}$, $\boldsymbol{\Omega}^*_{(l)} = \left[\left(\boldsymbol{\Omega}^*_{1(l)}\right)^{\mathrm{T}}, \left(\boldsymbol{\Omega}^*_{2(l)}\right)^{\mathrm{T}}\right]^{\mathrm{T}}$ and $\boldsymbol{\gamma}^*_{1(l)} = \mathbf{P}^{\boldsymbol{\gamma}}_{1(l)}\mathbf{H}$, $\boldsymbol{\gamma}^*_{2(l)} = \mathbf{P}^{\boldsymbol{\gamma}}_{2(l)}\mathbf{H}$, $\boldsymbol{\gamma}^*_{(l)} = \left[\left(\boldsymbol{\gamma}^*_{1(l)}\right)^{\mathrm{T}}, \left(\boldsymbol{\gamma}^*_{2(l)}\right)^{\mathrm{T}}\right]^{\mathrm{T}}$, where $l = \overline{1, L}$.

In turn, the resulting estimates $\boldsymbol{\gamma}^*$ and $\boldsymbol{\Omega}^*$ based on the partial estimates are found using HRM with some operator $\mathfrak{I}_{\mathrm{opt}}$ considering all available a priori and a posteriori information:

$$\mathfrak{I}_{\mathrm{opt}} : \left(\left\{\boldsymbol{\gamma}^*_{(l)}\right\}_{l=1}^{L}, \left\{\boldsymbol{\Omega}^*_{(l)}\right\}_{l=1}^{L}\right) \to \left(\boldsymbol{\gamma}^*, \boldsymbol{\Omega}^*\right). \tag{3.16}$$

For example, $\mathfrak{I}_{\mathrm{opt}}$ can be the minimizer of the quadratic form $F_{(l)} = \left(\boldsymbol{\Delta}^*_{(l)}\right)^{\mathrm{T}} \mathbf{W}_{(l)}\boldsymbol{\Delta}^*_{(l)}$ (*extremum search algorithm*):

$$\boldsymbol{\gamma}^* = \boldsymbol{\gamma}^*_{(l^*)}, \quad \boldsymbol{\Omega}^* = \boldsymbol{\Omega}^*_{(l^*)}, \quad l^* = \arg\min_{l} F_{(l)}, \tag{3.17}$$

where l^* is the optimal number of the node $\mathbf{q}_{(l^*)}$, $l^* \in \{1, \ldots, L\}$.

In (3.17), $\mathbf{W}_{(l)}$ denotes the normalized weight matrix

$$\mathbf{W}_{(l)} = \mathbf{W}(\mathbf{q}_{(l)}) = \left\|\mathbf{K}^{-1}(\mathbf{q}_{(l)})\right\|^{-1}\mathbf{K}^{-1}(\mathbf{q}_{(l)}), \quad \|\mathbf{W}_{(l)}\| = 1 \quad \forall l = \overline{1, L},$$

where $\|\cdot\|$ is any norm used in measurement processing.

Despite its optimality, the algorithm (3.17) has a major drawback: it requires estimates of all coordinates of the expanded vector $\boldsymbol{\gamma}^*$ necessary to form the residual $\boldsymbol{\Delta}^*_{(l)} = \mathbf{H} - \boldsymbol{\Psi}\boldsymbol{\gamma}^*_{(l)}$. Furthermore, the quadratic form considered as a function of the argument l shall have a unique global minimum. If there are several competing nodes $\mathbf{q}_{(l_1)}, \ldots, \mathbf{q}_{(l_d)}$, where $l_1, \ldots, l_d \in \{1, \ldots, L\}$ and $d \leqslant L$, with approximately equal local minima, then the following parameter (expressed as a percentage) is introduced to select such nodes:

$$\delta_{F(l)} = 100(F_{(l)} - F_{\min})(F_{\max} - F_{\min})^{-1},$$

where $F_{\min} = \min\limits_{l} F_{(l)}$ and $F_{\max} = \max\limits_{l} F_{(l)}$.

If we set a numerical threshold δ_{F0} (also in percentage terms), then the competing nodes are formed using the criterion $\delta_{F(l)} \leqslant \delta_{F0}$. For such nodes, it is reasonable to apply the operator $\overline{\mathfrak{I}}_{\mathrm{opt}}$ that averages the competing partial estimates instead of the operator $\mathfrak{I}_{\mathrm{opt}}$ (*averaging algorithm*):

$$\overline{\boldsymbol{\gamma}}^* = d^{-1}\sum_{k=1}^{d}\boldsymbol{\gamma}^*_{(l_k)}, \quad \overline{\boldsymbol{\Omega}}^* = d^{-1}\sum_{k=1}^{d}\boldsymbol{\Omega}^*_{(l_k)}. \tag{3.18}$$

To eliminate the main drawback of the optimal algorithm (3.17), one should use the quasi-optimal robust majority estimation operator $\overline{\overline{\mathfrak{I}}}_{\mathrm{opt}}$ [33], e.g., based on the sample median method (with a variational series for even or odd cases, the so-called *majority algorithm*).

For instance, for the vector $\boldsymbol{\gamma}$, all like-named coordinates $\gamma^*_{j(l)}, \ldots, \gamma^*_{j(L)}$ of the vectors $\left\{\boldsymbol{\gamma}^*_{(l)} = \left[\gamma^*_{j(l)}, j = \overline{1, V_1 + V_2}\right]^{\mathrm{T}}\right\}_{l=1}^{L}$ for each fixed $j \in \{1, \ldots, V_1 + V_2\}$ are arranged into the variational series

$\gamma^*_{j[1]}, \gamma^*_{j[2]}, \ldots, \gamma^*_{j[L]}$, where $\gamma^*_{j[1]} < \gamma^*_{j[2]} < \ldots < \gamma^*_{j[L]}$. For the vectors $\mathbf{\Omega}^*_{(l)} = \left[\left(\mathbf{\Omega}^*_{1(l)}\right)^{\mathrm{T}}, \left(\mathbf{\Omega}^*_{2(l)}\right)^{\mathrm{T}} \right]^{\mathrm{T}} = \left[\varsigma^*_{j(l)}, j = \overline{1, M_1 + M_2}\right]^{\mathrm{T}}$, the series $\varsigma^*_{j[1]}$, $\varsigma^*_{j[2]}, \ldots, \varsigma^*_{j[L]}$ are constructed similarly.

The optimal resulting estimate of each coordinate γ_j of the vector $\boldsymbol{\gamma} = \left[\boldsymbol{\gamma}_1^{\mathrm{T}}, \boldsymbol{\gamma}_2^{\mathrm{T}}\right]^{\mathrm{T}} = \left[\gamma_j, j = \overline{1, V_1 + V_2}\right]^{\mathrm{T}}$ and each coordinate ς_j of the vector $\mathbf{\Omega} = \left[\mathbf{\Omega}_1^{\mathrm{T}}, \mathbf{\Omega}_2^{\mathrm{T}}\right]^{\mathrm{T}} = \left[\varsigma_j, j = \overline{1, M_1 + M_2}\right]^{\mathrm{T}}$ is given by the following rule:

for even L,

$$\overline{\overline{\gamma}}{}^*_j = 2^{-1}\left(\gamma^*_{j[L/2]} + \gamma^*_{j[L/2+1]}\right), \quad \overline{\overline{\varsigma}}{}^*_j = 2^{-1}\left(\varsigma^*_{j[L/2]} + \varsigma^*_{j[L/2+1]}\right) \tag{3.19}$$

(i.e., the estimate corresponds to the average of the two central terms of the variational series);

for odd L,

$$\overline{\overline{\gamma}}{}^*_j = \gamma^*_{j[(L-1)/2-1]}, \quad \overline{\overline{\varsigma}}{}^*_j = \varsigma^*_{j[(L-1)/2-1]} \tag{3.20}$$

(i.e., the estimate coincides with the central term of the variational series).

For cases (3.19) and (3.20), the majority algorithm is based on discarding the tails of this series and is highly stable to uncertainty factors. Its implementation does not require estimating all coordinates of the vector $\boldsymbol{\gamma}$, constructing the residuals $\mathbf{\Delta}^*_{(l)}$, or minimizing the quadratic form $F_{(l)} = \left(\mathbf{\Delta}^*_{(l)}\right)^{\mathrm{T}} \mathbf{W}_{(l)} \mathbf{\Delta}^*_{(l)}$ with respect to the argument l.

The optimal resulting estimate of the VLF $\mathbf{\Omega}_i$ for the component $x_i(t)$ is found as

$$\mathbf{\Omega}^*_{i(l^*)} = \mathbf{P}^{\mathbf{\Omega}}_{i(l^*)}\mathbf{H}. \tag{3.21}$$

This estimate is associated with the following correlation matrix of estimation errors:

$$\mathbf{K}^{\mathbf{\Omega}}_{i(l^*)} = \mathbf{P}^{\mathbf{\Omega}}_{i(l^*)}\mathbf{K}_{(l^*)} \left(\mathbf{P}^{\mathbf{\Omega}}_{i(l^*)}\right)^{\mathrm{T}}. \tag{3.22}$$

It can be shown that the construction procedure of the estimate (3.21) for the VLF $\mathbf{\Omega}_i$ primarily relates to finding the optimal resulting estimate (for the vector $\boldsymbol{\gamma}$)

$$\boldsymbol{\gamma}^*_{i(l^*)} = \mathbf{P}^{\boldsymbol{\gamma}}_{i(l^*)}\mathbf{H} \tag{3.23}$$

with the correlation matrix of estimation errors

$$\mathbf{K}^{\boldsymbol{\gamma}}_{i(l^*)} = \mathbf{P}^{\boldsymbol{\gamma}}_{i(l^*)}\mathbf{K}_{(l^*)} \left(\mathbf{P}^{\boldsymbol{\gamma}}_{i(l^*)}\right)^{\mathrm{T}}. \tag{3.24}$$

This follows from the fact that

$$\mathbf{\Omega}_i = \mathbf{\Omega}\{x_i(t)\} = \mathbf{\Omega}\left\{\boldsymbol{\psi}_i^{\mathrm{T}}(t)\boldsymbol{\gamma}_i\right\} = \mathbf{\Omega}\left\{\boldsymbol{\psi}_i^{\mathrm{T}}(t)\right\}\boldsymbol{\gamma}_i. \tag{3.25}$$

In view of (3.21), we have

$$\mathbf{\Omega}^*_{i(l^*)} = \mathbf{\Omega}\left\{\boldsymbol{\psi}_i^{\mathrm{T}}(t)\right\}\boldsymbol{\gamma}^*_{i(l^*)} = \mathbf{\Omega}\left\{\boldsymbol{\psi}_i^{\mathrm{T}}(t)\right\}\mathbf{P}^{\boldsymbol{\gamma}}_{i(l^*)}\mathbf{H}. \tag{3.26}$$

Thus, for comparing different VLF estimation methods, it suffices to analyze the case where $M_i = V_i$ and $\mathbf{\Omega}_i = \boldsymbol{\gamma}_i$, i.e., to consider the problem of estimating the spectral coefficients of the component $x_i(t)$.

Note that for $\mathbf{Z} \neq \mathbf{0}$, the resulting estimates are also biased; however, due to (3.16)–(3.20), it becomes possible to minimize this bias for each fixed sample $\mathbf{H}$.

Formulas (3.1)–(3.26) are the foundation of HRM.

The next section is devoted to the applicability conditions of HRM and its comparative analysis with OLS and ELS.

4. MATHEMATICAL ASPECTS OF STUDYING THE EFFICIENCY OF THE PROPOSED METHOD

Consider the issue of a possible bias of the VLF estimate relative to the true value, caused by the irregular interference $\mathbf{Z} \neq \mathbf{0}$ and uncertainty regarding the parameter $\mathbf{q}$. Let $\widetilde{\mathbf{q}} = \mathbf{q} + \boldsymbol{\Delta}\mathbf{q}$, where $\boldsymbol{\Delta}\mathbf{q}$ will be treated as a perturbation. Let us determine its impact on the estimation result. Now, the matrix $\mathbf{P}_i^{\boldsymbol{\Omega}}(\mathbf{q})$ is replaced by

$$\mathbf{P}_i^{\boldsymbol{\Omega}}(\widetilde{\mathbf{q}}) = \mathbf{P}_i^{\boldsymbol{\Omega}}(\mathbf{q} + \boldsymbol{\Delta}\mathbf{q}) = \mathbf{P}_i^{\boldsymbol{\Omega}}(\mathbf{q}) + \boldsymbol{\Delta}\mathbf{P}_i^{\boldsymbol{\Omega}}(\widetilde{\mathbf{q}}),$$

where $\boldsymbol{\Delta}\mathbf{P}_i^{\boldsymbol{\Omega}}(\widetilde{\mathbf{q}})$ is the perturbation matrix with the condition $\boldsymbol{\Delta}\mathbf{P}_i^{\boldsymbol{\Omega}}(\mathbf{q}) = \mathbf{0}$.

The residual due to the presence of $\boldsymbol{\Delta}\mathbf{q} \neq \mathbf{0}$ is

$$\begin{aligned}\boldsymbol{\Delta\Omega}_i(\widetilde{\mathbf{q}}) = \boldsymbol{\Omega}_i^*(\widetilde{\mathbf{q}}) - \boldsymbol{\Omega}_i^*(\mathbf{q}) &= \mathbf{P}_i^{\boldsymbol{\Omega}}(\mathbf{q})\mathbf{H} + \boldsymbol{\Delta}\mathbf{P}_i^{\boldsymbol{\Omega}}(\widetilde{\mathbf{q}})\mathbf{H} - \mathbf{P}_i^{\boldsymbol{\Omega}}(\mathbf{q})\mathbf{H} \\ &= \boldsymbol{\Delta}\mathbf{P}_i^{\boldsymbol{\Omega}}(\widetilde{\mathbf{q}})\left(\mathbf{X}_i + \mathbf{X}_j + \mathbf{Z} + \boldsymbol{\Xi}\right).\end{aligned} \tag{4.1}$$

This error has the mean

$$M\left\{\boldsymbol{\Delta\Omega}_i(\widetilde{\mathbf{q}})\right\} = \boldsymbol{\Delta}\mathbf{P}_i^{\boldsymbol{\Omega}}(\widetilde{\mathbf{q}})\left(\mathbf{X}_1 + \mathbf{X}_2 + \mathbf{Z}\right). \tag{4.2}$$

Choosing any norm $\|\cdot\|$, we can utilize the inequality

$$\Delta_i^{\boldsymbol{\Omega}} \leqslant \Delta_i^{\mathbf{P}}\eta_i, \tag{4.3}$$

where $\Delta_i^{\boldsymbol{\Omega}} = \max\limits_{\widetilde{\mathbf{q}}}\left\|M\left\{\boldsymbol{\Delta\Omega}_i(\widetilde{\mathbf{q}})\right\}\right\|$, $\Delta_i^{\mathbf{P}} = \max\limits_{\widetilde{\mathbf{q}}}\left\|\boldsymbol{\Delta}\mathbf{P}_i^{\boldsymbol{\Omega}}(\widetilde{\mathbf{q}})\right\|$, and $\eta_i = \max\limits_{\boldsymbol{\gamma}_1,\boldsymbol{\gamma}_2,\mathbf{Z}}\left\|\boldsymbol{\Psi}_1\boldsymbol{\gamma}_1 + \boldsymbol{\Psi}_2\boldsymbol{\gamma}_2 + \mathbf{Z}\right\|$.

When calculating $\Delta_i^{\boldsymbol{\Omega}}$ and $\Delta_i^{\mathbf{P}}$, the maximum is found over the entire uncertainty region $[\alpha_1, \beta_1] \times \ldots \times [\alpha_Q, \beta_Q]$; and when calculating η_i, over the region of possible values of the vectors $\boldsymbol{\gamma}_1$, $\boldsymbol{\gamma}_2$, and $\mathbf{Z}$.

Consider three estimates as follows: $\boldsymbol{\gamma}_{\mathrm{OLS1}}^*$ is the separate estimate for $\boldsymbol{\gamma}_1$ based on OLS (with neglecting the regular and irregular interferences); $\boldsymbol{\gamma}_{\mathrm{ELS}}^*$ is the combined estimate for $\boldsymbol{\gamma} = \left[\boldsymbol{\gamma}_1^{\mathrm{T}}, \boldsymbol{\gamma}_2^{\mathrm{T}}\right]^{\mathrm{T}}$ based on ELS (with neglecting the irregular interference), and $\boldsymbol{\gamma}_{\mathrm{HRM}i}^*$ is the separate estimate for $\boldsymbol{\gamma}_i$ based on the developed HRM. (For brevity, the parameter $\mathbf{q}$ is omitted below.) Resting on [3], we will comprehensively assess these methods in terms of their correctness in measurement experiment design in the absence and presence of the regular and irregular interferences.

For OLS, which involves no expansion, let us determine the impact of the combined interference $\mathbf{Y}_2 = \mathbf{X}_2 + \mathbf{Z} = \boldsymbol{\Psi}_2\boldsymbol{\gamma}_2 + \mathbf{Z}$ on the estimate $\boldsymbol{\gamma}_{\mathrm{OLS1}}^*$ of the vector $\boldsymbol{\gamma}_1$ of the spectral coefficients of the information process. The expectation operator $M\{\cdot\}$ can be used to show that

$$M\left\{\boldsymbol{\gamma}_{\mathrm{OLS1}}^*\right\} = \boldsymbol{\gamma}_1 + \mathbf{C}_1^{-1}\boldsymbol{\Psi}_1^{\mathrm{T}}\mathbf{K}^{-1}\mathbf{Y}_2 = \boldsymbol{\gamma}_1 + \mathbf{C}_1^{-1}\boldsymbol{\Psi}_1^{\mathrm{T}}\mathbf{K}^{-1}\left(\boldsymbol{\Psi}_2\boldsymbol{\gamma}_2 + \mathbf{Z}\right), \tag{4.4}$$

where $\mathbf{C}_1 = \boldsymbol{\Psi}_1^{\mathrm{T}}\mathbf{K}^{-1}\boldsymbol{\Psi}_1$.

The second term in (4.4) corresponds to the bias of the estimate $\boldsymbol{\gamma}_{\mathrm{OLS1}}^*$ relative to the true value $\boldsymbol{\gamma}_1$ when neglecting the contribution of the combined interference $\mathbf{Y}_2$. The scatter of $\boldsymbol{\gamma}_{\mathrm{OLS1}}^*$ relative to $\boldsymbol{\gamma}_1$ (without considering $\mathbf{Y}_2$) is given by

$$\begin{aligned}\widetilde{\mathbf{K}}_{\mathrm{OLS1}}^{\boldsymbol{\gamma}} = M\left\{\left(\boldsymbol{\gamma}_{\mathrm{OLS1}}^* - \boldsymbol{\gamma}_1\right)\left(\boldsymbol{\gamma}_{\mathrm{OLS1}}^* - \boldsymbol{\gamma}_1\right)^{\mathrm{T}}\right\} &= \mathbf{C}_1^{-1} + \mathbf{B}_1\mathbf{Y}_2\mathbf{Y}_2^{\mathrm{T}}\mathbf{B}_1^{\mathrm{T}} \\ &= \mathbf{C}_1^{-1} + \mathbf{B}_1(\boldsymbol{\Psi}_2\boldsymbol{\gamma}_2 + \mathbf{Z})(\boldsymbol{\Psi}_2\boldsymbol{\gamma}_2 + \mathbf{Z})^{\mathrm{T}}\mathbf{B}_1^{\mathrm{T}},\end{aligned} \tag{4.5}$$

where $\mathbf{B}_1 = \mathbf{C}_1^{-1}\boldsymbol{\Psi}_1^{\mathrm{T}}\mathbf{K}^{-1}$.

In turn, the scatter of the estimate $\boldsymbol{\gamma}_{\mathrm{OLS1}}^*$ relative to the mean (4.4) (for OLS with neglecting $\mathbf{Y}_2$) is calculated as follows:

$$\mathbf{K}_{\mathrm{OLS1}}^{\boldsymbol{\gamma}} = M\left\{\left(\boldsymbol{\gamma}_{\mathrm{OLS1}}^* - M\left\{\boldsymbol{\gamma}_{\mathrm{OLS1}}^*\right\}\right)\left(\boldsymbol{\gamma}_{\mathrm{OLS1}}^* - M\left\{\boldsymbol{\gamma}_{\mathrm{OLS1}}^*\right\}\right)^{\mathrm{T}}\right\} = \mathbf{C}_1^{-1}. \tag{4.6}$$

In (4.6), $\mathbf{K}_{\mathrm{OLS1}}^{\boldsymbol{\gamma}}$ denotes the correlation matrix of estimation errors for the spectral coefficients of the information process.

Next, we employ ELS to generate the combined estimate $\boldsymbol{\gamma}_{\mathrm{ELS}}^{*}$ for the vector $\boldsymbol{\gamma} = \left[\boldsymbol{\gamma}_1^{\mathrm{T}}, \boldsymbol{\gamma}_2^{\mathrm{T}}\right]^{\mathrm{T}}$ of the spectral coefficients of the information process and regular interference (with neglecting the irregular interference $\mathbf{Z}$):

$$\boldsymbol{\gamma}_{\mathrm{ELS}}^{*} = \mathbf{C}^{-1}\boldsymbol{\Psi}^{\mathrm{T}}\mathbf{K}^{-1}\mathbf{H}, \tag{4.7}$$

where $\mathbf{C} = \boldsymbol{\Psi}^{\mathrm{T}}\mathbf{K}^{-1}\boldsymbol{\Psi}$.

By analogy with (4.4), the mean of such an estimate is

$$M\left\{\boldsymbol{\gamma}_{\mathrm{ELS}}^{*}\right\} = \boldsymbol{\gamma} + \mathbf{C}^{-1}\boldsymbol{\Psi}^{\mathrm{T}}\mathbf{K}^{-1}\mathbf{Z}, \tag{4.8}$$

where the second term characterizes its bias due to the presence of the irregular interference.

In this case, the correlation matrix of ELS estimation errors has the form

$$\mathbf{K}_{\mathrm{ELS}}^{\boldsymbol{\gamma}} = \mathbf{C}^{-1} = \left(\boldsymbol{\Psi}^{\mathrm{T}}\mathbf{K}^{-1}\boldsymbol{\Psi}\right)^{-1}; \tag{4.9}$$

similar to (4.5), the scatter matrix of the estimate $\boldsymbol{\gamma}_{\mathrm{ELS}}^{*}$ relative to the true value $\boldsymbol{\gamma}$ is found as

$$\widetilde{\mathbf{K}}_{\mathrm{ELS}}^{\boldsymbol{\gamma}} = \mathbf{C}^{-1} + \mathbf{B}\mathbf{Z}\mathbf{Z}^{\mathrm{T}}\mathbf{B}^{\mathrm{T}}, \tag{4.10}$$

where $\mathbf{B} = \mathbf{C}^{-1}\boldsymbol{\Psi}^{\mathrm{T}}\mathbf{K}^{-1}$.

Since ELS and HRM are linear optimal estimates, they inherit the same potential capabilities in terms of accuracy. The matrix $\widetilde{\mathbf{K}}_{\mathrm{ELS}}^{\boldsymbol{\gamma}}$ contains four blocks, i.e.,

$$\widetilde{\mathbf{K}}_{\mathrm{ELS}}^{\boldsymbol{\gamma}} = \begin{bmatrix} \mathbf{S}_{11} & \mathbf{S}_{12} \\ \mathbf{S}_{21} & \mathbf{S}_{22} \end{bmatrix}, \tag{4.11}$$

and the conditions $\mathbf{S}_{11} = \mathbf{K}_1^{\boldsymbol{\gamma}} = \mathbf{P}_1^{\boldsymbol{\gamma}}\mathbf{K}\left[\mathbf{P}_1^{\boldsymbol{\gamma}}\right]^{\mathrm{T}}$ and $\mathbf{S}_{22} = \mathbf{K}_2^{\boldsymbol{\gamma}} = \mathbf{P}_2^{\boldsymbol{\gamma}}\mathbf{K}\left[\mathbf{P}_2^{\boldsymbol{\gamma}}\right]^{\mathrm{T}}$ hold accordingly.

Now, let the combined interference $\mathbf{Y}_2 = \mathbf{X}_2 + \mathbf{Z}$ (the sum of the regular and irregular interferences) admit the finite-analytic representation $\mathbf{Y}_2 = \boldsymbol{\Psi}_{\mathrm{RI+II}}\boldsymbol{\gamma}_{\mathrm{RI+II}}$, where $\boldsymbol{\Psi}_{\mathrm{RI+II}}$ is a given basis matrix of the regular and irregular interferences, and $\boldsymbol{\gamma}_{\mathrm{RI+II}}$ is the vector of unknown spectral coefficients. When the vectors $\boldsymbol{\gamma}_1$ and $\boldsymbol{\gamma}_{\mathrm{RI+II}}$ are estimated simultaneously by ELS, one can find the correlation matrix of joint estimation errors:

$$\mathbf{K}_{\mathrm{ELS1}}^{\boldsymbol{\gamma}} = \mathbf{C}_1^{-1} + \mathbf{B}_1\boldsymbol{\Psi}_{\mathrm{RI+II}}\mathbf{D}_1\left(\mathbf{B}_1\boldsymbol{\Psi}_{\mathrm{RI+II}}\right)^{\mathrm{T}}, \tag{4.12}$$

where

$$\mathbf{D}_1 = \left(\boldsymbol{\Psi}_{\mathrm{RI+II}}^{\mathrm{T}}\mathbf{K}^{-1}\boldsymbol{\Psi}_{\mathrm{RI+II}} - \boldsymbol{\Psi}_{\mathrm{RI+II}}^{\mathrm{T}}\mathbf{K}^{-1}\boldsymbol{\Psi}_1\mathbf{C}_1^{-1}\boldsymbol{\Psi}_1^{\mathrm{T}}\mathbf{K}^{-1}\boldsymbol{\Psi}_{\mathrm{RI+II}}\right)^{-1}.$$

For practice, the following question is of interest: is it necessary to expand the vector of estimated parameters considering the combined interference $\mathbf{Y}_2 = \mathbf{X}_2 + \mathbf{Z}$? Let the traces $\widetilde{\chi} = Sp\left[\mathbf{B}_1\mathbf{Y}_2\mathbf{Y}_2^{\mathrm{T}}\mathbf{B}_1^{\mathrm{T}}\right]$ and $\chi = Sp\left[\mathbf{B}_1\boldsymbol{\Psi}_{\mathrm{RI+II}}\mathbf{D}_1\left(\mathbf{B}_1\boldsymbol{\Psi}_{\mathrm{RI+II}}\right)^{\mathrm{T}}\right]$ of the second matrix terms in (4.10) and (4.12), respectively, be formed. Since the unknown vector parameter $\mathbf{Y}_2$ figures in $\widetilde{\chi}$, we can replace the trace $\widetilde{\chi}$ with its estimate $\widetilde{\chi}^{*} = Sp\left[\mathbf{B}_1\boldsymbol{\Psi}_{\mathrm{RI+II}}\boldsymbol{\gamma}_{\mathrm{RI+II}}^{*}\left(\boldsymbol{\Psi}_{\mathrm{RI+II}}\boldsymbol{\gamma}_{\mathrm{RI+II}}^{*}\right)^{\mathrm{T}}\mathbf{B}_1^{\mathrm{T}}\right]$. The next step is to verify the condition

$$\widetilde{\chi}^{*} > \chi. \tag{4.13}$$

If this condition fails, then the impact of the combined interference $\mathbf{Y}_2 = \mathbf{X}_2 + \mathbf{Z}$ on the estimate $\boldsymbol{\gamma}_1^*$ of the vector $\boldsymbol{\gamma}_1$ can be considered sufficiently small, and the vector of estimated parameters should not be expanded. Otherwise, it is necessary to apply ELS and, for HRM, ensure the invariance condition with respect to $\mathbf{Y}_2 = \mathbf{X}_2 + \mathbf{Z}$ considering the combined basis matrix $\boldsymbol{\Psi}_{\mathrm{RI+II}}$.

Formulas (4.1)–(4.13) can be used in practice to compare the three methods and settle many issues related to measurement experiment design.

According to a direct analysis of the expressions (3.5) and (3.6), HRM involves the inversion of matrices of dimensions $V_1 \times V_1$ and $V_2 \times V_2$ whereas ELS the inversion of a matrix of dimensions $(V_1 + V_2) \times (V_1 + V_2)$. Consequently, HRM-based estimation is preferable from the computational stability viewpoint: smaller matrices are inverted. Also, the computational process gets decomposed: the VLF is estimated for each component $x_i(t)$ independently (separately) in a dedicated measurement processing channel.

Formulas (4.1)–(4.13) can be used to tune the necessary HRM parameters minimizing the resulting VLF estimation error in each particular case. Necessary and sufficient conditions for the existence of a unique solution of the estimation problem with HRM require nonsingularity and some rank constraints for several matrices, by analogy with [15–17]. In practice, the given conditions are fulfilled by rationally choosing the functional bases and the number of degrees of freedom in the adopted models as well as by specifying appropriate observation conditions. Such issues are associated with computational experiment design and are not considered below: they require separate research in each particular case.

5. THE RECOGNITION ALGORITHM IN THE PRESENCE OF REGULAR AND IRREGULAR INTERFERENCES

Step 1. For each node $\mathbf{q}_{(l)}$, construct the basis matrices $\boldsymbol{\Psi}_{i(l)} = \boldsymbol{\Psi}_i(\mathbf{q}_{(l)})$ and $\boldsymbol{\Psi}_{j(l)} = \boldsymbol{\Psi}_j(\mathbf{q}_{(l)})$, where $i \neq j$.

Step 2. For each node $\mathbf{q}_{(l)}$, use (3.6) to find the matrix $\boldsymbol{\Lambda}_{j(l)} = \boldsymbol{\Lambda}_j\left(\mathbf{q}_{(l)}\right)$.

Step 3. For each node $\mathbf{q}_{(l)}$, use (3.8) to find the weight estimation matrix $\mathbf{P}^{\boldsymbol{\gamma}}_{i(l)} = \mathbf{P}^{\boldsymbol{\gamma}}_i\left(\mathbf{q}_{(l)}\right)$.

Step 4. For each node $\mathbf{q}_{(l)}$, use (3.11) to find the optimal estimate $\boldsymbol{\gamma}^*_{i(l)} = \boldsymbol{\gamma}^*_i\left(\mathbf{q}_{(l)}\right)$.

Step 5. For each node $\mathbf{q}_{(l)}$, use (3.13) to form the residual $\boldsymbol{\Delta}_{(l)} = \boldsymbol{\Delta}\left(\mathbf{q}_{(l)}\right)$.

Step 6. For the case where the function $\boldsymbol{\Delta}_{(l)}$ of the argument l has a minimum point, apply the extremum search algorithm (3.17) to find the optimal number l^* of the node $\mathbf{q}_{(l^*)}$ where this minimum is achieved and the resulting estimate $\boldsymbol{\gamma}^* = \boldsymbol{\gamma}^*_{(l^*)}$.

Step 7. This step is needed in the absence of an explicit minimum point. Then the resulting estimate $\boldsymbol{\gamma}^*$ is obtained by any other method, e.g., the averaging algorithm (3.18) or the majority algorithm (3.19), (3.20).

Step 8. Use (3.21) to compute the optimal resulting estimate $\boldsymbol{\Omega}^*_{i(l^*)}$ of the VLF $\boldsymbol{\Omega}_i$ for the component $x_i(t)$.

Step 9. Use (3.22) and (3.24) to find the correlation matrices $\mathbf{K}^{\boldsymbol{\Omega}}_{i(l^*)} = \mathbf{K}^{\boldsymbol{\Omega}}_i\left(\mathbf{q}_{(l^*)}\right)$ and $\mathbf{K}^{\boldsymbol{\gamma}}_{i(l^*)} = \mathbf{K}^{\boldsymbol{\gamma}}_i\left(\mathbf{q}_{(l^*)}\right)$.

6. SOME GENERALIZATIONS AND PRACTICAL RECOMMENDATIONS

It is possible to interpret HRM for broader conditions by setting $\mathfrak{I}_{\mathrm{opt}}$ as an operator that maximizes the corresponding conditional probability density (e.g., the posterior one or a likelihood function) depending on the parametrically defined correlation matrix $\mathbf{K}(\mathbf{q})$. For instance, for

Gaussian measurement noise, $\mathfrak{I}_{\text{opt}}$ can be taken as the operator maximizing the likelihood function

$$p\left(\mathbf{H}/\boldsymbol{\gamma}^*_{(l)}\right) = (2\pi)^{-N/2}\left(\det \mathbf{K}(\mathbf{q}_{(l)})\right)^{-1/2} \exp\left\{\left(\boldsymbol{\Delta}^*_{(l)}\right)^{\mathrm{T}}(\mathbf{q}_{(l)})\boldsymbol{\Delta}^*_{(l)}\right\}$$

with respect to the parameter l.

In contrast to algorithm (3.17), there is no need to construct the normalized weight matrix $\mathbf{W}_{(l)}$: the required normalization is performed naturally due to the properties of any probability density.

HRM for VLF estimation can be extended to observation models of a more general form. For example, it is possible to consider in (2.1) the basis functions $\psi_{jm}\left(t, \boldsymbol{\mu}_{jm}\right)$ that depend nonlinearly on a vector parameter $\boldsymbol{\mu}_{jm}$. In this case, one shall specify a common uncertainty region for all nonlinear parameters of the estimation problem and choose an appropriate cardinality L of the numerical grid to cover this region with a sufficient accuracy; for example, see [16]. The dimensions of all models (the information process and regular interference) shall be consistent with the sample size N in (2.3).

Obviously, such an approach to VLF estimation shall agree with the computing resources used, which directly depend on the number of alternative hypotheses under consideration. To reduce the requirements for these resources, it is reasonable to minimize the number of nonlinear parameters (i.e., the number of hypotheses) and increase the dimension of the vector of linear parameters. But even in this case, one should seek for a rational tradeoff to exclude possible paradoxes associated with practical VLF estimation. Such questions are the scope of the general theory of measurement experiment design; for example, see [3].

It is possible to combine HRM with the estimation method [17], oriented to form a family of reduced grids corresponding to all possible configurations of irregular interferences at the original grid nodes $\{t_1, \ldots, t_N\}$. Under a sufficiently large size of this grid, such a combination can significantly decrease the impact of regular and irregular interferences on the estimation results. For HRM, this will further increase the family of alternative hypotheses under consideration, and the resulting estimate will correspond to the optimal reduced grid for which the residual is minimal. A criterion and an algorithm for finding such a grid were presented in [17]; also, various options for forming a family of reduced grids, consistent with the available computing resources, were considered.

The normalized weight matrix can be specified as $\mathbf{W}(q)$, where the scalar argument $q \in \{0, 1, 2, \ldots\}$ corresponds to different observation options. Among them, one can provide options considering the properties of the measurement noise and, moreover, irregular interferences. For example, by appropriately selecting the diagonal elements of the matrix $\mathbf{W}(q)$, it is often possible to partially compensate for the irregular interference (via its rejection in the time domain) and smooth the noise.

Hopefully, with current progress in the field of parallel computing (especially based on new principles [34, 35]), HRM will not become an obstacle for advanced real-time systems. However, such an approach remains infeasible for many existing systems, where large samples and numerous alternative hypotheses may require a huge number of parallel data processing channels and significant computational costs. Some useful formulas to justify the computational costs for implementing HRM were provided in [15–17].

7. TEST EXAMPLES

Consider simple but application-relevant examples that are typical, for instance, for altimeters of certain aircraft and are related to smoothing trajectory measurements of small volume [3]. In model (2.2), $x_1(t) = \gamma_1 = \text{const}$ can be treated as the altitude of a loitering aircraft (the parameter γ_1 is a scalar linear functional), and $x_2(t) = \gamma_2 t^2$ with $\gamma_2 = \text{const}$ can be treated as a slowly varying power function representing a regular interference. The latter is often due to imperfections in the aircraft's control system or interferences of various origins on the side lobes of the altimeter antenna pattern.

For each fixed measurement sample, we introduce the relative energy coefficient $\varepsilon_E = \varepsilon_E(\mathbf{H}) = E_1\left(E_2 + E_{\mathbf{Z}} + E_{\mathbf{\Xi}}\right)^{-1} = \mathbf{X}_1^{\mathrm{T}}\mathbf{X}_1\left(\mathbf{X}_2^{\mathrm{T}}\mathbf{X}_2 + \mathbf{Z}^{\mathrm{T}}\mathbf{Z} + \mathbf{\Xi}^{\mathrm{T}}\mathbf{\Xi}\right)^{-1}$, which will be used in the comparative assessment of the developed method.

Example 1. This example is intended only to demonstrate the algorithmic features of HRM and assess its potential capabilities. The uncertainty regarding a nonstationary uncorrelated measurement noise is characterized by the matrix $\mathbf{K} = \mathbf{K}(q) = \operatorname{diag}\left[2\exp\{-q(n-1)\},\ n = \overline{1,N}\,\right]$, where $q > 0$, $\alpha \leqslant q \leqslant \beta$, $\alpha > 0$, and $\beta > 0$. The spectral analysis problem is solved: find the estimate γ_1^* of the coefficient γ_1 (for the information process) and the estimate γ_2^* of the coefficient γ_2 (for the regular interference). For the sake of simplicity, all problem parameters are appropriately scaled to dimensionless quantities.

To demonstrate HRM and perform its comparative analysis, we took a small sample ($N = 5$), with $[0,T] = [0,5]$, $t_n = n\cdot\Delta t$, $n = \overline{1,5}$, and $\Delta t = 1$.

In (2.3), let $V_1 = 1$, $\boldsymbol{\gamma}_1 = [\gamma_{11}]$, $\boldsymbol{\psi}_1(t) = [\psi_{11}(t)]$, $\psi_{11}(t) \equiv 1$, $\gamma_{11} = \gamma_1$, $V_2 = 1$, $\boldsymbol{\gamma}_2 = [\gamma_{21}]$, $\boldsymbol{\psi}_2(t) = [\psi_{21}(t)]$, $\psi_{21}(t) = t^2$, $\gamma_{21} = \gamma_2$, $\boldsymbol{\gamma} = [\gamma_{11}, \gamma_{21}]^{\mathrm{T}} = [\gamma_1, \gamma_2]^{\mathrm{T}}$, $M_1 = 1$, $\mathbf{\Omega}\{x_1(t)\} = [\omega_1\{x_1(t)\}] = [\varsigma_{11}] = [\gamma_1]$, $M_2 = 1$, $\mathbf{\Omega}\{x_2(t)\} = [\omega_1\{x_2(t)\}] = [\varsigma_{21}] = [\gamma_2]$, $Q = 1$, $\mathbf{q} = [q_1]^{\mathrm{T}}$, $q_1 = q$, $q > 0$, $\alpha_1 = \alpha$, and $\beta_1 = \beta$.

Let the true values of the parameters for the observation equation (2.3) be $\gamma_1 = 10$, $\gamma_2 = 10^{-1}$, $\alpha = 0$, $\beta = 9\times10^{-1}$, $L = 6$, $q_{(1)} = 0$, $q_{(2)} = 10^{-1}$, $q_{(3)} = 3\times10^{-1}$, $q_{(4)} = 5\times10^{-1}$, $q_{(5)} = 7\times10^{-1}$, $q_{(6)} = 9\times10^{-1}$. In other words, the parameter q belongs to the one-dimensional uncertainty region $[0,5]$ with six nodes uniformly located nodes. In this example, $\mathbf{\Psi}_1^{\mathrm{T}} = [1;1;1;1;1]$ and $\mathbf{\Psi}_2^{\mathrm{T}} = [1;4;9;16;25]$.

First, using (3.8), we found the coefficient vectors $\mathbf{P}_{1(l)}^{\boldsymbol{\gamma}}$ necessary for constructing the linear estimate $\gamma_{1(l)}^*$ of the coefficient γ_1, invariant with respect to the regular interference:

$$\mathbf{P}_{1(1)}^{\boldsymbol{\gamma}} = [0.4941; 0.4059; 0.2588; 0.0529; -0.2118]^{\mathrm{T}},$$

$$\mathbf{P}_{1(2)}^{\boldsymbol{\gamma}} = [0.4495; 0.4118; 0.2985; 0.0876; -0.2474]^{\mathrm{T}},$$

$$\mathbf{P}_{1(3)}^{\boldsymbol{\gamma}} = [0.3620; 0.4107; 0.3790; 0.1802; -0.3320]^{\mathrm{T}},$$

$$\mathbf{P}_{1(4)}^{\boldsymbol{\gamma}} = [0.2803; 0.3923; 0.4547; 0.3066; -0.4339]^{\mathrm{T}},$$

$$\mathbf{P}_{1(5)}^{\boldsymbol{\gamma}} = [0.2082; 0.3584; 0.5175; 0.4666; -0.5510]^{\mathrm{T}},$$

$$\mathbf{P}_{1(6)}^{\boldsymbol{\gamma}} = [0.1482; 0.3133; 0.5605; 0.6550; -0.6771]^{\mathrm{T}}.$$

We verified the unbiasedness condition $\mathbf{P}_{1(l)}^{\boldsymbol{\gamma}}\mathbf{X}_1 = \gamma_1$, where $\mathbf{X}_1 = [10;10;10;10;10]^{\mathrm{T}}$, and the invariance condition $\mathbf{P}_{1(l)}^{\boldsymbol{\gamma}}\mathbf{X}_2 = 0$, where $\mathbf{X}_2 = [0.1;0.4;0.9;1.6;2.5]^{\mathrm{T}}$, for all $l = \overline{1,6}$. The matrices $\mathbf{\Lambda}_{2(l)}$ for $\mathbf{\Lambda}_2$ (3.6) were constructed when finding the vectors. For instance, for $l = 1$,

$$\mathbf{\Lambda}_{2(1)} = \begin{bmatrix} 0.9990 & -0.0041 & -0.0092 & -0.0163 & -0.0255 \\ -0.0041 & 0.9837 & -0.0368 & -0.0654 & -0.1021 \\ -0.0092 & -0.0368 & 0.9173 & -0.1471 & -0.2298 \\ -0.0163 & -0.0654 & -0.1471 & 0.7385 & -0.4086 \\ -0.0255 & -0.1021 & -0.2298 & -0.4086 & 0.3616 \end{bmatrix}.$$

Next, using (3.12), we computed the estimation error variances for the coefficient γ_1 :

$$\sigma^2_{1(1)} = 1.0471, \quad \sigma^2_{1(2)} = 0.9503, \quad \sigma^2_{1(3)} = 0.7625,$$
$$\sigma^2_{1(4)} = 0.5888, \quad \sigma^2_{1(5)} = 0.4364, \quad \sigma^2_{1(6)} = 0.3103.$$

Clearly, for the uncertainty region in the parameter q, the spread of the estimation error variances changes by a factor of 3.37. Therefore, for HRM, the estimation accuracy of γ_1 significantly depends on the chosen hypothesis of the family $\left\{\Gamma_{(l)}\right\}_{l=1}^{6}$.

With the vectors $\mathbf{P}^{\boldsymbol{\gamma}}_{1(l)}$ replaced by the ones $\overline{\mathbf{P}}^{\boldsymbol{\gamma}}_{1(l)}$ neglecting the regular interference $\mathbf{X}_2$ (which is typical for OLS), we obtained the following biases of the estimate γ_1^* :

$$\left(\overline{\mathbf{P}}^{\gamma}_{1(1)}\right)^{\mathrm{T}} \mathbf{X}_2 = 1.1000, \quad \left(\overline{\mathbf{P}}^{\gamma}_{1(2)}\right)^{\mathrm{T}} \mathbf{X}_2 = 1.2207, \quad \left(\overline{\mathbf{P}}^{\gamma}_{1(3)}\right)^{\mathrm{T}} \mathbf{X}_2 = 1.4591,$$
$$\left(\overline{\mathbf{P}}^{\gamma}_{1(4)}\right)^{\mathrm{T}} \mathbf{X}_2 = 1.6733, \quad \left(\overline{\mathbf{P}}^{\gamma}_{1(5)}\right)^{\mathrm{T}} \mathbf{X}_2 = 1.8540, \quad \left(\overline{\mathbf{P}}^{\gamma}_{1(6)}\right)^{\mathrm{T}} \mathbf{X}_2 = 2.8015.$$

Thus, with the regular interference being neglected, an error of up to 28% arises when estimating the coefficient γ_1 by OLS.

The estimates $\gamma^*_{2(l)}$ for γ_2 are also needed to implement the optimal algorithm (3.17). The corresponding weight vectors were computed as

$$\mathbf{P}^{\boldsymbol{\gamma}}_{2(1)} = [-0.0267; -0.0187; -0.0053; 0.0134; 0.0374]^{\mathrm{T}},$$
$$\mathbf{P}^{\boldsymbol{\gamma}}_{2(2)} = [-0.0235; -0.0191; -0.0082; 0.0107; 0.0401]^{\mathrm{T}},$$
$$\mathbf{P}^{\boldsymbol{\gamma}}_{2(3)} = [-0.0179; -0.0189; -0.0134; 0.0046; 0.0456]^{\mathrm{T}},$$
$$\mathbf{P}^{\boldsymbol{\gamma}}_{2(4)} = [-0.0133; -0.0177; -0.0177; -0.0028; 0.0515]^{\mathrm{T}},$$
$$\mathbf{P}^{\boldsymbol{\gamma}}_{2(5)} = [-0.0095; -0.0159; -0.0210; -0.0113; 0.0577]^{\mathrm{T}},$$
$$\mathbf{P}^{\boldsymbol{\gamma}}_{2(6)} = [-0.0066; -0.0137; -0.0231; -0.0206; 0.0639]^{\mathrm{T}}.$$

When finding these vectors, we constructed the matrices $\mathbf{\Lambda}_{1(l)}$ for $\mathbf{\Lambda}_1$ (3.6). For instance, for $l = 3$,

$$\mathbf{\Lambda}_{1(3)} = \begin{bmatrix} 0.8995 & -0.1005 & -0.1005 & -0.1005 & -0.1005 \\ -0.1356 & 0.8644 & -0.1356 & -0.1356 & -0.1356 \\ -0.1831 & -0.1831 & 0.8169 & -0.1831 & -0.1831 \\ -0.2472 & -0.2472 & -0.2472 & 0.7528 & -0.2472 \\ -0.3336 & -0.3336 & -0.3336 & -0.3336 & -0.6664 \end{bmatrix}.$$

Next, using (3.12), we computed the estimation error variances for the parameter $\gamma_2 \times 10^{-3}$:

$$\sigma^2_{2(1)} = 5.3476, \quad \sigma^2_{2(2)} = 4.2002, \quad \sigma^2_{2(3)} = 2.6396,$$
$$\sigma^2_{2(4)} = 1.6881, \quad \sigma^2_{2(5)} = 1.0861, \quad \sigma^2_{2(6)} = 0.6955.$$

In this case, the spread of the estimation error variances changes by a factor of 7.69.

According to the above data, the implementation of both HRM and well-known methods (OLS, ELS, and GIUE) with only one pre-fixed (nominal) value $q_0 \in [\alpha, \beta]$ can lead to a considerable spread of the estimates. Therefore, it is possible to significantly improve the estimation quality by HRM, which chooses an optimal node in the uncertainty region $[\alpha, \beta]$.

With no need for matrix inversion in this example, in the case of ELS, the combined vector estimate $\gamma^*_{\mathrm{ELS}} = [\gamma^*_{\mathrm{ELS1}}, \gamma^*_{\mathrm{ELS2}}]^{\mathrm{T}}$ was found for all hypotheses $\Gamma_{(l)}$; this required inverting the matrices $\mathbf{C}_{(l)} = \mathbf{\Psi}^{\mathrm{T}}\mathbf{W}_{(l)}\mathbf{\Psi}$, where

$$\mathbf{\Psi}^{\mathrm{T}} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 4 & 9 & 16 & 25 \end{bmatrix}.$$

For the family of all these matrices, we found their condition numbers $\nu_{(l)} = \nu(\mathbf{C}_{(l)})$ (in the Euclidean norm): $\nu_{(1)} = 515.7839$, $\nu_{(2)} = 666.9735$, $\nu_{(3)} = 1101.7277$, $\nu_{(4)} = 1778.5476$, $\nu_{(5)} = 2789.8688$, $\nu_{(6)} = 4239.6586$. Obviously, even for the simplest low-dimensional test example and a small sample, we are dealing with ill-conditioned matrices and, hence, with an additional computational error to find the desired estimates. This fact is characteristic of most estimation problems with a power basis [27–29].

Example 2. Let the matrix $\mathbf{K} = \mathbf{K}(q) = \mathrm{diag}\big[[1 + q(n-1)]^{-2},\ n = \overline{1,5}\,\big]$ correspond to the case of uncorrelated nonstationary measurements of a small volume. In this case, $\mathbf{W}_{(l)} = \mathrm{diag}\big[w_{nn}(q_{(l)}),\ n = \overline{1,5}\,\big]$, where $w_{nn}(q_{(l)}) = [1 + q(n-1)]^2 \left(\sum\limits_{n=1}^{5} [(1 + q(n-1))]^2\right)^{-1}$, which holds for the norm $\|\mathbf{W}_{(l)}\| = \sum\limits_{n=1}^{n} w_{nn(l)}$ with $w_{nn(l)} > 0$. We set $[0, T] = [0, 4]$, $t_n = (n-1)\Delta t$, $n = \overline{1,5}$, $\Delta t = 1$, $\alpha = -0.25$, $\beta = 1$, $L = 6$, $q_{(1)} = -0.2$, $q_{(2)} = -0.125$, $q_{(3)} = 0$, $q_{(4)} = 0.3$, $q_{(5)} = 0.7$, and $q_{(6)} = 1$. For each node in the uncertainty region, Table 1 presents the estimation results of the coefficient γ_1 by HRM for the family of characteristic samples $\{\mathbf{H}_k\}_{k=1}^{5}$ corresponding to different variants of irregular interference occurrence (for particular values of the energy parameter $\{\varepsilon_{Ek}\}_{k=1}^{5}$ and a threshold $\delta_{F0} = 15\%$).

Table 1

$\varepsilon_{E1} = 5.44$, $\mathbf{H}_1 = [9.5; 11.5; 12.3; 14.1; 18.2]^{\mathrm{T}}$					
$\gamma^*_{1(1)} = 11.92$	$\gamma^*_{1(2)} = 10.11$	$\gamma^*_{1(3)} = 10.12$	$\gamma^*_{1(4)} = 10.11$	$\gamma^*_{1(5)} = 10.09$	$\gamma^*_{1(6)} = 10.09$
$F_{(1)} = 3.74$	$F_{(2)} = 0.37$	$F_{(3)} = 0.29$	$F_{(4)} = 0.21$	$F_{(5)} = 0.16$	$F_{(6)} = 0.15$
$\delta_{F(1)} = 100$	$\delta_{F(2)} = 6.13$	$\delta_{F(3)} = 3.90$	$\delta_{F(4)} = 1.67$	$\delta_{F(5)} = 0.28$	$\delta_{F(6)} = 0$
$\varepsilon_{E2} = 7.30$, $\mathbf{H}_2 = [9; 11.5; 10; 17.5; 13]^{\mathrm{T}}$					
$\gamma^*_{1(1)} = 11.92$	$\gamma^*_{1(2)} = 10.11$	$\gamma^*_{1(3)} = 10.12$	$\gamma^*_{1(4)} = 10.11$	$\gamma^*_{1(5)} = 10.09$	$\gamma^*_{1(6)} = 10.09$
$F_{(1)} = 2.37$	$F_{(2)} = 4.30$	$F_{(3)} = 7.36$	$F_{(4)} = 10.33$	$F_{(5)} = 11.59$	$F_{(6)} = 13.77$
$\delta_{F(1)} = 0$	$\delta_{F(2)} = 17$	$\delta_{F(3)} = 43.88$	$\delta_{F(4)} = 69.82$	$\delta_{F(5)} = 80.88$	$\delta_{F(6)} = 100$
$\varepsilon_{E3} = 3.28$, $\mathbf{H}_3 = [17; 16.5; 13; 14; 16]^{\mathrm{T}}$					
$\gamma^*_{1(1)} = 14.88$	$\gamma^*_{1(2)} = 13.91$	$\gamma^*_{1(3)} = 12.30$	$\gamma^*_{1(4)} = 10.47$	$\gamma^*_{1(5)} = 9.87$	$\gamma^*_{1(6)} = 9.67$
$F_{(1)} = 7.77$	$F_{(2)} = 11.31$	$F_{(3)} = 12.76$	$F_{(4)} = 9.09$	$F_{(5)} = 6.46$	$F_{(6)} = 5.46$
$\delta_{F(1)} = 31.64$	$\delta_{F(2)} = 80.19$	$\delta_{F(3)} = 100$	$\delta_{F(4)} = 49.72$	$\delta_{F(5)} = 13.66$	$\delta_{F(6)} = 0$
$\varepsilon_{E4} = 2.83$, $\mathbf{H}_4 = [9; 18.5; 7; 15.5; 18]^{\mathrm{T}}$					
$\gamma^*_{1(1)} = 11.04$	$\gamma^*_{1(2)} = 10.92$	$\gamma^*_{1(3)} = 10.60$	$\gamma^*_{1(4)} = 10.10$	$\gamma^*_{1(5)} = 10.06$	$\gamma^*_{1(6)} = 10.04$
$F_{(1)} = 21.78$	$F_{(2)} = 21.01$	$F_{(3)} = 17.84$	$F_{(4)} = 12.73$	$F_{(5)} = 9.99$	$F_{(6)} = 9.08$
$\delta_{F(1)} = 100$	$\delta_{F(2)} = 93.97$	$\delta_{F(3)} = 69.98$	$\delta_{F(4)} = 28.74$	$\delta_{F(5)} = 7.18$	$\delta_{F(6)} = 0$
$\varepsilon_{E5} = 6.89$, $\mathbf{H}_5 = [17; 11.5; 11; 14.8; 10]^{\mathrm{T}}$					
$\gamma^*_{1(1)} = 13.02$	$\gamma^*_{1(2)} = 11.81$	$\gamma^*_{1(3)} = 9.8$	$\gamma^*_{1(4)} = 7.52$	$\gamma^*_{1(5)} = 6.68$	$\gamma^*_{1(6)} = 6.39$
$F_{(1)} = 13.97$	$F_{(2)} = 18.95$	$F_{(3)} = 22.96$	$F_{(4)} = 20.24$	$F_{(5)} = 18.18$	$F_{(6)} = 17.24$
$\delta_{F(1)} = 0$	$\delta_{F(2)} = 55.39$	$\delta_{F(3)} = 100$	$\delta_{F(4)} = 71.74$	$\delta_{F(5)} = 46.82$	$\delta_{F(6)} = 36.37$

Next, Table 2 provides the estimation results of the coefficient by HRM with algorithms (3.17), (3.18), and (3.20), where Δ_1, $\overline{\Delta}_1$, and $\overline{\overline{\Delta}}_1$ are the absolute estimation errors.

Table 2

$\mathbf{H}_1$	$\mathbf{H}_2$	$\mathbf{H}_3$	$\mathbf{H}_4$	$\mathbf{H}_5$
$\gamma^*_{1(1)} = 10.09$	$\gamma^*_{1(1)} = 9.64$	$\gamma^*_{1(1)} = 9.67$	$\gamma^*_{1(1)} = 10.04$	$\gamma^*_{1(1)} = 13.02$
$\Delta_1 = 0.09$	$\Delta_1 = 0.36$	$\Delta_1 = 0.33$	$\Delta_1 = 0.04$	$\Delta_1 = 3.02$
$\overline{\gamma}^*_{1(1)} = 10.1$	$\overline{\gamma}^*_{1(1)} = 9.64$	$\overline{\gamma}^*_{1(1)} = 9.77$	$\overline{\gamma}^*_{1(1)} = 10.05$	$\overline{\gamma}^*_{1(1)} = 13.02$
$\overline{\Delta}_1 = 0.1$	$\overline{\Delta}_1 = 0.36$	$\overline{\Delta}_1 = 0.23$	$\overline{\Delta}_1 = 0.05$	$\overline{\Delta}_1 = 3.02$
$\overline{\overline{\gamma}}^*_{1(1)} = 10.11$	$\overline{\overline{\gamma}}^*_{1(1)} = 9.64$	$\overline{\overline{\gamma}}^*_{1(1)} = 9.77$	$\overline{\overline{\gamma}}^*_{1(1)} = 10.05$	$\overline{\overline{\gamma}}^*_{1(1)} = 13.02$
$\overline{\overline{\Delta}}_1 = 0.11$	$\overline{\overline{\Delta}}_1 = 0.36$	$\overline{\overline{\Delta}}_1 = 0.23$	$\overline{\overline{\Delta}}_1 = 0.05$	$\overline{\overline{\Delta}}_1 = 3.02$

The data in this table clearly show the possibility of reliable information process estimation in the presence of regular and irregular interferences in the input observation.

Finally, Table 3 presents the estimation results by OLS and ELS with the nominal value $q_0 = q_{(3)} = 0$ used in the processing algorithms, where δ_1 is the relative error expressed as a percentage.

Table 3

$\varepsilon_{E1} = 5.44$, $\mathbf{H}_1 = [9.5; 11.5; 12.3; 14.1; 18.2]^{\mathrm{T}}$				
OLS	$\gamma^*_{1(1)} = 13.12$	$F = 8.63$	$\Delta_1 = 3.12$	$\delta_1 = 31.2$
ELS	$\gamma^*_{1(1)} = 10.12$	$F = 0.29$	$\Delta_1 = 0.12$	$\delta_1 = 1.2$
$\varepsilon_{E2} = 7.30$, $\mathbf{H}_2 = [9; 11.5; 10; 17.5; 13]^{\mathrm{T}}$				
OLS	$\gamma^*_{1(1)} = 12.2$	$F = 8.86$	$\Delta_1 = 2.2$	$\delta_1 = 22$
ELS	$\gamma^*_{1(1)} = 9.2$	$F = 7.36$	$\Delta_1 = 0.8$	$\delta_1 = 8$
$\varepsilon_{E3} = 3.28$, $\mathbf{H}_3 = [17; 16.5; 13; 14; 16]^{\mathrm{T}}$				
OLS	$\gamma^*_{1(1)} = 15.31$	$F = 2.36$	$\Delta_1 = 5.3$	$\delta_1 = 53$
ELS	$\gamma^*_{1(1)} = 12.3$	$F = 12.76$	$\Delta_1 = 2.3$	$\delta_1 = 18.7$
$\varepsilon_{E4} = 2.83$, $\mathbf{H}_4 = [9; 18.5; 7; 15.5; 18]^{\mathrm{T}}$				
OLS	$\gamma^*_{1(1)} = 13.6$	$F = 22.34$	$\Delta_1 = 3.6$	$\delta_1 = 36$
ELS	$\gamma^*_{1(1)} = 10.6$	$F = 17.84$	$\Delta_1 = 0.6$	$\delta_1 = 6$
$\varepsilon_{E5} = 6.89$, $\mathbf{H}_5 = [17; 11.5; 11; 14.8; 10]^{\mathrm{T}}$				
OLS	$\gamma^*_{1(1)} = 12.8$	$F = 6.66$	$\Delta_1 = 2.8$	$\delta_1 = 28$
ELS	$\gamma^*_{1(1)} = 9.8$	$F = 22.96$	$\Delta_1 = 0.2$	$\delta_1 = 2$

We also considered the case of correlated nonstationary measurements of large volume ($N = 41$) with the matrix $\mathbf{K} = \mathbf{K}(q)$ composed of the diagonal elements $k_{nn}(q) = [1 + q\Delta t(n-1)]^{-2}$, $n = \overline{1, 41}$, where $\Delta t = 0.1$, and the off-diagonal elements $k_{nm} = \exp\{-|n - m|\Delta t\}$, $n, m = \overline{1, 41}$, $n \neq m$. Using a random number generator, one thousand input observation samples were generated, with each sample containing regular and irregular interferences. The parameters q and ε_E were specified as random variables with the uniform distribution on the intervals from -0.2 to 1 and from 1.7 to 2.5, respectively. The number of anomalous outliers in each realization corresponded to $\rho_{\mathbf{Z}} = 30\%$. The entire observation interval was divided into three equal-size segments (initial, middle, and final). In each experiment, the irregular interference was concentrated on one of the segments (with a uniform distribution law). Based on the simulation results, the coefficient γ_1 was estimated with the following averaged (over all samples) absolute errors: for OLS, $\Delta_1 = 4.78$; for ELS and GIUE, $\Delta_1 = 2.11$; for HRM, $\Delta_1 = 0.32$.

ELS and GIUE coincide in accuracy since a small-dimensional case is considered. The gain from GIUE, due to the decomposition of the estimation algorithm, manifests itself only for large-dimensional problems.

Thus, even for one degree of freedom (the parameter q is scalar) in uncertainty description, it is possible to significantly improve the estimation accuracy of information processes with regular and irregular interferences. Obviously, when solving more complex applied problems, one should choose a multidimensional uncertainty region and an appropriate number of nodes $\mathbf{q}_{(l)}$, $l = \overline{1, L}$.

8. CONCLUSIONS

Under essential uncertainty, HRM allows studying in detail any component of the observation equation using VLF while ensuring the invariance condition of the estimation result with respect to a regular interference and minimizing (within the adopted criterion) the impact of an irregular interference on the estimation results. It is possible to organize several parallel computation channels for VLF in the entire uncertainty region of the correlation function parameters of the semi-defined measurement noise. Within a single channel, all advantages of linear estimation are preserved without expanding the vector of estimated parameters. Analytical formulas have been derived to assess the potential capabilities of HRM and the conditions for its most effective application.

HRM can be modified for other known problems, e.g., estimation from a growing-volume sample, signal detection/recognition/resolution, and the linear or nonlinear filtering of signal parameters. This method is well suited for multichannel information-measuring systems designed to operate in real time. All computational procedures in each channel reduce to simple mathematical operations on vectors and matrices; and HRM can be combined with conventional approaches to applications-relevant problems of optimal and quasi-optimal measurement processing.

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