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NONLINEAR SYSTEMS

Robust Prescribed Performance Control under Given Constraints for Systems with Relative Degree One

K. A. Lastochkin

Trapeznikov Institute of Control Sciences, Russian Academy of Sciences, Moscow, Russia
e-mail: lastconst@yandex.ru

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Abstract—The problem of prescribed performance control under given constraints on the amplitude and rate of control signal is considered for dynamic systems with relative degree one in the presence of parametric uncertainty and external disturbances. Using the ideas of feedback design based on sat-function differentiable approximation and augmented perturbation estimation, a novel nonlinear robust control law is proposed, which, if the parameters are suitably chosen, provides a solution to the problem under consideration. The parameter calculation procedure is reduced to solving linear inequalities, which feasibility is proved analytically. Illustrative example corroborates theoretical results.

Keywords: robust control, transients, prescribed performance, control signal constraints, saturation function

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1. INTRODUCTION

The problem of prescribed performance control in the presence of input saturation for systems with unknown parameters and external perturbations remains open in modern control theory. Among the known attempts to address this problem, approaches should be mentioned that design a control law on the basis of *i*) barrier functions [1–8], which prevent the state/output from violating the prescribed bounds, and *ii*) smooth approximations of a saturation function [9–13], which ensure prescribed performance of the tracking error with respect to the reference signal using control signal with bounded amplitude.

The formal problem statement of the above-mentioned papers includes only control signal amplitude constraint, while its first derivative (rate) boundedness is not taken into account. However, actuators of real-world plants usually have limited rate of the control signal implementation. For example, considering aircraft, the maximum instantaneous deflection rate of control surfaces (ailerons, elevators, rudders, etc.) is always limited. If such constraints are not considered when designing the control law, then only a part of the algorithmically generated control signal will actually act the system. This can lead to violation of the prescribed transient performance specifications or even to loss of closed-loop stability. Therefore, in this paper, considering the prescribed performance control problem, it is proposed to take into account constraints on both the amplitude and the rate of the control signal.

The aim of this paper is to ensure prescribed transient performance under given constraints on both the control signal amplitude and rate for dynamic systems with relative degree one in the presence of parametric uncertainty and external disturbances. The class of systems with relative

degree one is considered for simplicity of presentation, and in the future the obtained results can be used to solve control problems for more complex systems.

To achieve this goal, a new control law is proposed, in which the proportional feedback, as in [9–13], is designed using a special differentiable approximation of the saturation function, but it is additionally filtered through a nonlinear filter of the type [14]. This technique allows one to *a priori* bound both the amplitude and the rate of the control signal within the prescribed limits without the need to require boundedness of the amplitude and first derivative of the system output. Furthermore, in contrast to [9–13], in addition to the classical proportional feedback, the control law is proposed to include a dynamic estimate of the augmented disturbance obtained on the basis of A.M. Tsykunov's auxiliary loop method [15, p. 201] that is combined with a nonlinear filter [14] to bound the rate of the augmented disturbance estimate within the given limits. It is shown in the paper that such a nonlinear robust feedback allows one to *a priori* bound the amplitude and rate of the control signal by given values and reduce the prescribed performance control problem with amplitude/rate constraints to the one of the control law parameters calculation via solution of linear inequalities.

The main distinctive features of the proposed control law compared to those in [1–13] are as follows:

1) in addition to ensure the prescribed transient performance, the amplitude and rate of the control signal are bounded within the given limits;
2) the choice of the control law parameters is reduced to solving linear inequalities and can be implemented numerically;
3) the feasibility of the obtained linear inequalities to calculate the control law parameters is shown analytically rather than numerically;
4) in contrast to [1–8], barrier functions are not used for feedback design, which eliminates discontinuities in the control law caused by division by zero and/or taking the natural logarithm of zero (see Fig. 4 and the comments thereto in Section 6);
5) in contrast to [9–13], the proposed control law *i*) contains not only a summand that is proportional to the tracking error, but also a dynamic estimate of the augmented disturbance designed by the auxiliary loop method [15, p. 201], and *ii*) is combined with nonlinear filters intended to bound the rate of the control signal.

The paper is organized as follows. Section 2 provides notation and definitions. Section 3 presents the problem statement and comments. Section 4 elucidates the main result of the paper and derives the control law that guarantees the prescribed transient performance under bounded amplitude and rate of the control signal. Section 5 provides an analytical comparison of the developed control law with the one proposed in [10]. Section 6 presents the results of numerical experiments. Section 7 formulates the conclusions and outlines future research scope.

2. PRELIMINARIES

The following notation is adopted: $t \geq t_0 \geq 0$ denotes time, $f : [t_0, \infty) \mapsto \mathbb{R}^n$ is a function of time, $f(t)$ stands for a value of the function f at the time instant t, the notation $F := \{f(t) \in \mathbb{R}^n : |f(t)| \leq f^{\mathrm{U}},\ |\dot{f}(t)| \leq v^{\mathrm{U}}\}$ denotes the space of functions of time whose amplitude and first derivative at time t are bounded by the numbers $f^{\mathrm{U}} > 0$, $v^{\mathrm{U}} > 0$, σ: $\mathbb{R} \mapsto \mathbb{R}$ is any differentiable approximation of the saturation function that meets the following conditions[1]:

a) $|\sigma'(s)| = \left|\frac{\partial \sigma(s)}{\partial s}\right| \leq \kappa \in (1, 2)$ for all $s \in \mathbb{R}$,

b) $s\sigma(s) > 0$ for all $s \neq 0$, $\sigma(0) = 0$,

[1] Hereafter $\operatorname{sat}\{s\} = \operatorname{sign}\{s\} \min\{|s|, 1\}, \operatorname{sign}\{s\} \in \begin{cases} 1, & s > 0 \\ [-1, 1], & s = 0 \\ -1, & s < 0. \end{cases}$

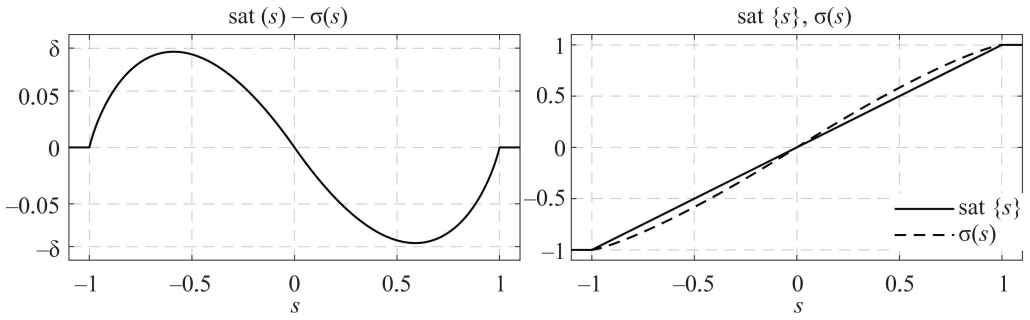


Fig. 1. Sat-function approximation error.

c) $\sigma(s) = \operatorname{sign}\{s\}$ for all $|s| \geq 1$,
d) $|s| < |\sigma(s)| < 1$ for $|s| < 1$,
e) there exists $\delta \in (0,\ 1)$ such that $|s - \sigma(s)| \leq \delta$ for all $|s| \leq 1$,
f) there exists $L > \kappa$ such that $|\sigma(a) - \sigma(b)| \leq L|a - b|$ for all $a,\ b \in \mathbb{R}$.

For instance, choosing $\kappa \simeq 1.26,\ \delta \simeq 0.088,\ L \simeq 1.3$, the following function meets the conditions *a*)–*f*):

$$\sigma(s) = \operatorname{sign}\{s\}\left(1 - (1 - \min\{|s|, 1\})^{1.26}\right).$$

Figure 1 demonstrates the comparison of sat-function with the above-given one.

3. PROBLEM STATEMENT

Consider a dynamic system

$$\dot{y}(t) = bu(t) + f(t, y, u),\ y(t_0) = y_0, \tag{3.1}$$

where $y(t) \in \mathbb{R}$ stands for a measurable output, $u(t) \in \mathbb{R}$ is a control signal, $f(t, y, u) \in \mathbb{R}$ denotes an unmeasurable augmented perturbation (a combination of parametric and external disturbances), $b > 0$ is a known estimate of the system gain, which is, without loss of generality, considered to be positive for simplicity.

The aim is to design a control law $u(t)$ for the system (3.1) in such a way that it holds for all $t \geq t_0$ that:

$$y(t) \in \mathcal{Y} := \left\{y(t) \in \mathbb{R}\colon \underline{g}(t) \leq y(t) \leq \overline{g}(t)\right\}, \tag{3.2}$$

$$u(t) \in \mathcal{U} := \left\{u(t) \in \mathbb{R}\colon\ |u(t)| \leq u^{\mathrm{U}},\ |\dot{u}(t)| \leq v^{\mathrm{U}}\right\}, \tag{3.3}$$

where $u^{\mathrm{U}} > 0,\ v^{\mathrm{U}} > 0$ are known scalars, $\underline{g}(t), \overline{g}(t)$ are measurable for all $t \geq t_0$.

Therefore, we consider the problem to ensure the prescribed transient performance for the system output using a control law from the class of signals with bounded amplitude and rate. To solve this problem, the following assumptions are adopted.

Assumption 1. The center of the set $\mathcal{Y}$ meets the following condition for all $t \geq t_0$:

$$g(t) \in \mathcal{G} := \left\{g(t) \in \mathbb{R}\colon |\dot{g}(t)| \leq g_1^{\Delta},\ |\ddot{g}(t)| \leq g_2^{\Delta}\right\},$$

where $g(t) = \frac{\overline{g}(t) + \underline{g}(t)}{2}$ is a measurable signal (the center of the set $\mathcal{Y}$), $g_1^{\Delta} > 0,\ g_2^{\Delta} > 0$ are known scalars.

Assumption 2. There exist known scalars $\gamma_u \in (0,\,1)$, $\gamma_v \in (0,\,1)$ such that for all $g(t) \in \mathcal{G}$, $u(t) \in \mathcal{U}$, $y(t) \in Y$, $\mathcal{Y} \subset \mathcal{Y}$ and all $t \geq t_0$ it holds that

$$F(t) \in \mathcal{F} := \left\{F(t) \in \mathbb{R}\colon |F(t)| \leq \gamma_u,\, \left|\dot{F}(t)\right| \leq \gamma_v \rho\right\},$$

where $F(t) := \frac{f(t,\,y,\,u)-\dot{g}(t)}{bu^{\mathrm{U}}}$, $\rho = \frac{v^{\mathrm{U}}}{u^{\mathrm{U}}}$.

Assumption 3. $y(t_0) \in \mathcal{Y}$, where $y(t)$ denotes a solution of (3.1).

Discussing the problem under consideration, first of all, we would like to note that knowledge of the gain b is not a restrictive requirement. Indeed, using decomposition

$$\beta(t,\,y,\,u)\,u(t) = bu(t) + \Delta_b(t,\,y,\,u)\,u(t)$$

in case the following conditions are met

$$\begin{gathered} |\Delta_b(t,y,u)| < b, \\ \frac{1}{b}\left|\frac{d\Delta_b(t,\,y,\,u)}{dt}u(t) + \Delta_b(t,y,u)\,\dot{u}(t)\right| < v^{\mathrm{U}} \end{gathered} \tag{3.4}$$

for all $t \geq t_0$ and $u(t) \in \mathcal{U}$, $g(t) \in \mathcal{G}$, the summand $\Delta_b(t,\,y,\,u)\,u(t)$, caused by the uncertainty of the true gain $\beta(t,\,y,\,u)$, can be included into the external perturbation $f(t,\,y,\,u)$. If inequalities (3.4) cannot be satisfied, it may mean that the system gain changes its sign (loss of controllability) and definitely means that Assumption 2 is not met. In general, Assumption 2 implies that the augmented perturbation (the combination of external and parametric disturbances), divided by the available control effort bu^{U}, belongs to the same class of signals as the available control signal over the set $\mathcal{Y}$.

Assumptions 1 and 2 are interrelated, and therefore the scalars $g_1^{\Delta} > 0$ and $g_2^{\Delta} > 0$ from the first assumption cannot be chosen arbitrarily and must satisfy the constraints

$$\frac{g_1^{\Delta}}{bu^{\mathrm{U}}} < \gamma_u,\ \frac{g_2^{\Delta}}{bv^{\mathrm{U}}} < \gamma_v, \tag{3.5}$$

i.e., the available control effort amplitude (bu^{U}) and rate (bv^{U}) must exceed the velocity and acceleration of the center of the set $\mathcal{Y}$. For given $y(t_0)$, b, u^{U}, v^{U}, inequalities (3.5) and Assumption 3 can always be satisfied by a special choice of the set $\mathcal{Y}$.

3.1. Comments

At first glance, the stated control problem for the first-order system (3.1) seems to be non-substantive — in practice, there exist not so many systems described by first-order differential equations. However, it should be kept in mind that, by means of special transformations and changes of variables, substantive control problems for various complex high-order systems can be reduced to the considered control problem for a first-order system. Further, we consider some examples of such parametrizations.

Example 1. Consider a linear system

$$\begin{aligned} &\dot{x}(t) = Ax(t) + B\Big(u(t) + d(t)\Big),\ x(t_0) = x_0, \\ &y(t) = Cx(t), \end{aligned} \tag{3.6}$$

where $x(t) \in \mathbb{R}^n$ stands for a system state, $y(t) \in \mathbb{R}$ denotes a measurable output, $u(t) \in \mathbb{R}$ is a control signal, $d(t) \in \mathbb{R}$ denotes an unmeasurable perturbation, $A \in \mathbb{R}^{n\times n}$ is a state matrix, $B \in \mathbb{R}^{n\times 1}$

stands for an input vector, $C \in \mathbb{R}^{1\times n}$ denotes an output vector, the pairs (A, B) and (A, C) are completely observable and controllable, respectively. The matrices A, B, C are unknown, but we know the relative degree of the system, which is arbitrary but time-invariant, *i.e.*, $1 \le r \le n$, where the scalar r is defined by $CB = 0$, $CAB = 0, \ldots, CA^{r-1}B \neq 0$.

According to the analysis from [16, p. 21]; [17], under the adopted assumptions, there exists an unknown but non-singular change of variables:

$$\begin{bmatrix}\eta(t)\\ \xi(t)\end{bmatrix} = \begin{bmatrix}T_0\\ T_1\end{bmatrix} x(t) = \begin{bmatrix}T_0\\ C\\ CA\\ \vdots\\ CA^{r-1}\end{bmatrix} x(t),$$

which transforms (3.6) into:

$$\begin{aligned}
&\dot\eta(t) = A_{00}\eta(t) + A_{01}\xi(t),\\
&\dot\xi_1(t) = \xi_2(t),\\
&\dot\xi_j(t) = \xi_{j+1}(t),\ j = 2,\ldots,r-1,\\
&\dot\xi_r(t) = \beta\Big[u(t) + d(t) + w(t,\xi)\Big],\\
&y(t) = \xi_1(t),
\end{aligned} \tag{3.7}$$

where $\begin{bmatrix}M_0 & M_1\end{bmatrix}\begin{bmatrix}T_0\\ T_1\end{bmatrix} = I$, $T_0B = 0$ and

$$\begin{aligned}
&A_{00} = T_0AM_0,\ A_{01} = T_0AM_1,\\
&A_{10} = CA^rM_0,\ A_{11} = CA^rM_1,\ \beta = CA^{r-1}B,\\
&w(t,\xi) = \beta^{-1}\Big(A_{10}\eta(t) + A_{11}\xi(t)\Big).
\end{aligned}$$

If zeros of the transfer function from $u(t) + d(t)$ to $y(t)$ lie in the left half-plane, then the matrix A_{00} is Hurwitz [16, 17] and, consequently, for all trajectories $\xi(t)$, the signal $\beta^{-1}A_{10}\eta(t)$ can be treated as a bounded external disturbance.

If additionally $r = 1$, then $\xi(t) := y(t)$ and we immediately have from (3.7) that:

$$\dot y(t) = bu(t) + f(t, y, u),$$

where

$$f(t, y, u) := \beta d(t) + A_{10}\eta(t) + A_{11}y(t) + (\beta - b)u(t),$$

and the signal $\beta d(t) + A_{10}\eta(t)$ is identified with the component $f(t, \cdot, \cdot)$.

Therefore, for $r = 1$, the problem of keeping the output of system (3.6) within the given set (3.2) under the constraints (3.3) is equivalent to the stated one-dimensional problem.

If $n \ge r > 1$, then we shall assume that by means of differentiators (derivative observers) that are well studied in control theory [14, 18, 19], for all $t \ge t_0$ the estimates $\hat\xi(t)$ of the states $\xi(t)$ are available for measurement. In this case, we also assume that the following holds for the estimation error $\tilde\xi(t) = \hat\xi(t) - \xi(t)$ and its derivative:

$$\left|\tilde\xi_i(t)\right| \le \delta_1,\ \left|\dot{\tilde\xi}_i(t)\right| \le \delta_2,\ i = 1,\ldots,r,$$

where $\delta_1 > 0,\ \delta_2 > 0$.

These requirements are not restrictive, since the existing derivative observers guarantee their satisfaction [14].

Application of the linear operator to $\hat{\xi}_1(t)$ yields:

$$\lambda^{r-1}s(t) = (p+\lambda)^{r-1}\left[\hat{\xi}_1(t)\right] = (p+\lambda)^{r-1}\left[y(t)\right] + (p+\lambda)^{r-1}\left[\tilde{\xi}_1(t)\right],$$

where $p := \frac{d}{dt}[.]$, and $s(t)$ is measurable, as it can be calculated using r derivatives of the system output.

Differentiating $s(t)$ with respect to time, we obtain:

$$\begin{aligned}\dot{s}(t) &= \lambda^{1-r}\left[\dot{\xi}_r(t) + \epsilon(\xi_1,\ldots,\xi_{r-1}) + (p+\lambda)^{r-1}\left[\dot{\tilde{\xi}}_1(t)\right]\right]\\ &= \lambda^{1-r}\left[\beta\left[u(t)+d(t)+w(t,\xi)\right] + \epsilon(t) + (p+\lambda)^{r-1}\left[\dot{\tilde{\xi}}_1(t)\right]\right]\\ &= bu(t) + f(t,s,u),\end{aligned} \tag{3.8}$$

where

$$\begin{aligned}f(t,s,u) &:= \left(\lambda^{1-r}\beta - b\right)u(t)\\ &+ \lambda^{1-r}\left(\beta d(t) + A_{10}\eta(t) + A_{11}\xi(t) + \epsilon(t) + (p+\lambda)^{r-1}p\left[\tilde{\xi}_1(t)\right]\right)\end{aligned}$$

and the term $\epsilon(\xi_1,\ \ldots,\xi_{r-1}) := \epsilon(t)$ contains all other derivatives.

As

$$y(t) = \frac{\lambda^{r-1}}{(p+\lambda)^{r-1}}\left[s(t)\right] - \tilde{\xi}_1(t),$$

then for a sufficiently large value of the parameter $\lambda > 0$ and a sufficiently small upper bound δ_1, owing to the monotonic dependence of $y(t)$ on $s(t)$, we obtain that the problem of keeping the output of system (3.6) within the given set (3.2) under constraints (3.3) for the case $n \geq r > 1$ is reduced to the stated one-dimensional problem.

By complete analogy, the control problem for a class of nonlinear minimum-phase systems with arbitrary relative degree can be reduced to the control problem of a first-order system [16, p. 167].

Example 2. Consider a cascade system:

$$\begin{aligned}&\dot{x}_1(t) = b_1x_2(t) + f_1(t,x,u),\\ &\dot{x}_j(t) = b_jx_{j+1}(t) + f_j(t,x,u), \quad j = 2,\ldots,n-1,\\ &\dot{x}_n(t) = b_nu(t) + f_n(t,x,u),\\ &y(t) = x_1(t),\end{aligned} \tag{3.9}$$

where $x(t) \in \mathbb{R}^n$ is a measurable state, $f_i(t,x,u)$ stands for an unknown perturbations, $b_i > 0$ denotes known estimates of the system gains.

It is required to keep the output of system (3.9) within the given set (3.2) under the constraints (3.3) and, additionally, ensure boundedness of all elements of the state vector for all $t \geq t_0$:

$$y(t) \in \mathcal{Y} \subset \mathcal{X}_1,\ x_i(t) \in \mathcal{X}_i,\ i = 2,\ldots,n,$$

where $\mathcal{X}_i := \left\{x_i(t) \in \mathbb{R} : |x_i(t)| \leq x_i^{\mathrm{U}}\right\}$.

The following change of variables

$$\begin{aligned}&s_1(t) = x_1(t) - g(t),\\ &s_i(t) = x_i(t) - u_{i-1}(t),\ i = 2,\ldots,n,\end{aligned}$$

allows one to rewrite (3.9) as:

$$\begin{aligned}
\dot{s}_1(t) &= b_1\Big(u_1(t) + s_2(t)\Big) + f_1(t, x, u_n) - \dot{g}(t),\\
\dot{s}_j(t) &= b_j\Big(u_j(t) + s_{j+1}(t)\Big) - \dot{u}_{j-1}(t) + f_j(t, x, u_n),\ j = 2, \ldots, n-1,\\
\dot{s}_n(t) &= b_n u_n(t) - \dot{u}_{n-1}(t) + f_n(t, x, u_n),
\end{aligned} \tag{3.10}$$

where $u_i(t)$ are virtual control signals, and, for generality, we adopt $u_n(t) := u(t)$.

Thus, the problem of keeping the output of system (3.9) within the given set (3.2) is reduced to solving one-dimensional and similar problems of keeping the states $s_i(t)$ within the given sets:

$$\begin{aligned}
\mathcal{S}_1 &:= \{s_1(t) \in \mathbb{R}\colon |s_1(t)| \le \varepsilon\},\\
\mathcal{S}_i &:= \Big\{s_i(t) \in \mathbb{R}\colon |s_i(t)| \le x_i^{\mathrm{U}} - u_{i-1}^{\mathrm{U}}\Big\},\ i = 2, \ldots, n
\end{aligned}$$

for all $t \ge t_0$ under conditions of boundedness of the amplitude and rate of a real control signal (3.3) and amplitude of the virtual ones:

$$U_i := \Big\{u_i(t) \in \mathbb{R} : |u_i(t)| \le u_i^{\mathrm{U}}\Big\},\ i = 1, \ldots, n-1,$$

where $u_i^{\mathrm{U}} < x_{i+1}^{\mathrm{U}}$ is a known scalar.

Moreover, if the amplitude constraint for the virtual control signals is augmented with the rate ones (where $v_i^{\mathrm{U}} > 0,\ i = 1, \ldots, n-1$, is a known scalar)

$$\mathcal{U}_i\colon = \Big\{u_i(t) \in \mathbb{R}\colon |u_i(t)| \le u_i^{\mathrm{U}},\ |\dot{u}_i(t)| \le v_i^{\mathrm{U}}\Big\},$$

then, for all $k = 2, \ldots, n$ in each kth equation of the system (3.10), the term $\dot{u}_{k-1}(t)$ is an external disturbance bounded by v_{k-1}^{U}. Consequently, solving the stabilization problems iteratively from the bottom to up (in the direction from $s_n(t)$ to $s_1(t)$) with the help of the control signals $u_i(t) \in \mathcal{U}_i$ for all $i = 1, \ldots, n-1$, we have that $s_{i+1}(t)$ in each ith equation is bounded by $\mathcal{S}_{i+1}$, and hence, at each step of the procedure, we have exactly the one-dimensional problem stated in this paper of keeping the output of system (3.1) within the given set (3.2) under the constraints on the amplitude and rate of the control signal.

Both considered examples demonstrate that sufficiently general and substantive control problems, which are often faced in practice, are reduced to solving the one-dimensional task considered in this paper. At the end of this subsection, it should be noted that the problems from Examples 1 and 2 are not solved in this study, but only given to illustrate the applicability of the main result of this paper for their solution. In other words, the results of this study are foundational on the way to solve the problems such as those considered in Examples 1 and 2. It is worth noting that, in fact, solving problem (3.2), (3.3) for system (3.1) means solving the same problem for the class of nonlinear systems with unknown parameters, relative degree one, and stable internal dynamics (such systems, according to [16, p. 167] and Example 1, are reduced to the form (3.1), and for them, under certain conditions, Assumption 2 will be satisfied). Defining the conditions on the right-hand side of the system equations, under which Assumption 2 holds, is also beyond the scope of this paper.

4. MAIN RESULT

In Section 4.1, the structure of the control law is derived, and the problem of choosing its parameters and achievement of the goal (3.2) under constraints (3.3) is reduced to finding a solution of a system of linear inequalities. In Section 4.2, the engineering interpretation of the obtained control law is discussed, and the MatLab code to solve the inequalities from Section 4.1 is provided.

4.1. Robust Prescribed Performance Control under Given Constraints

The problem of keeping the output $y(t)$ within the given set $\mathcal{Y}$ is weaker than the problem of tracking its center $g(t)$ with a dynamic error whose absolute value does not exceed the minimum width of the set, *i.e.*,

$$\text{IF } |y(t_0)-g(t_0)|\le\varepsilon \Rightarrow \forall t\ge t_0,\ |y(t)-g(t)|\le\varepsilon, \text{ THEN } y(t)\in\mathcal{Y},\ \forall t\ge t_0, \tag{4.1}$$

where $\varepsilon=\min\left\{\inf\limits_t|\overline{g}(t)-g(t)|,\ \inf\limits_t\left|\underline{g}(t)-g(t)\right|\right\}$ is a minimal width of $\mathcal{Y}$.

In other words, meeting the condition $|y(t)-g(t)|\le\varepsilon$ for all $t\ge t_0$ will immediately imply satisfaction of the prescribed transient performance (3.2), but not vice versa. Therefore, it is proposed to consider the prescribed performance control problem as a problem of tracking the set $\mathcal{Y}$ center with a given value of the dynamic error.

In order to solve such a problem under the constraints (3.3), the control law is portioned in the following way:

$$u(t)=(1-\gamma_u)\,u^{\mathrm{U}}u_s(t)+\gamma_u u^{\mathrm{U}}u_c(t), \tag{4.2}$$

where $u_s(t)$, $u_c(t)$ are stabilizing and compensation summands.

Then, owing to the following equalities

$$|u(t)|=\left|(1-\gamma_u)\,u^{\mathrm{U}}u_s(t)+\gamma_u u^{\mathrm{U}}u_c(t)\right|\le u^{\mathrm{U}},$$
$$|\dot u(t)|=\left|(1-\gamma_u)\,u^{\mathrm{U}}\dot u_s(t)+\gamma_u u^{\mathrm{U}}\dot u_c(t)\right|\le v^{\mathrm{U}},$$

the constraints (3.3) are weaker then

$$(1-\gamma_u)\,|u_s(t)|+\gamma_u\,|u_c(t)|\le 1, \tag{4.3a}$$
$$(1-\gamma_u)\,|\dot u_s(t)|+\gamma_u\,|\dot u_c(t)|\le \rho. \tag{4.3b}$$

Following decomposition (4.2), the problem to meet the control signal constraints (3.3) is reduced to satisfying inequalities (4.3a) and (4.3b) for all $t\ge t_0$.

To solve the tracking problem (4.1) under constraints (4.3a) and (4.3b), we define the control components as follows:

$$\begin{aligned}
\dot u_s(t)&=-\frac{(1-\mu\kappa\gamma_v)\,\rho}{1-\gamma_u}\sigma\left(\frac{1}{\lambda_1}\Big(u_s(t)+\sigma\big(\frac{K}{1-\gamma_u}z\big)\Big)\right),\quad |u_s(t_0)|\le 1,\\
\dot u_c(t)&=-\frac{\mu\kappa\gamma_v\rho}{\gamma_u}\sigma\left(\frac{1}{\lambda_2}\Big(u_c(t)+\sigma\big(\frac{1}{\gamma_u}\Phi\big)\Big)\right),\quad |u_c(t_0)|\le 1,
\end{aligned} \tag{4.4}$$

where $1<\mu<\frac{1}{\kappa\gamma_v}$, $\lambda_1>0$, $\lambda_2>0$, $K>0$, $z(t)=\frac{y(t)-g(t)}{bu^{\mathrm{U}}}$, and we will define $\Phi(t)$ further.

If $\Phi(t):=F(t)$, then $-\gamma_u u_c(t)$ is an augmented perturbation with the amplitude and rate bounded by given values. As the augmented perturbation $F(t)$ is unmeasurable, then $\Phi(t)$ is proposed to be obtained with the help of A.M. Tsykunov's auxiliary loop method [15, p. 201]:

$$\Phi(t):=\chi(t)-u_f(t), \tag{4.5}$$

where $l>0$ and

$$\chi(t)=l\Big(z(t)-z_f(t)\Big),$$
$$\dot z_f(t)=\chi(t),\quad z_f(t_0)=z(t_0),$$
$$\dot u_f(t)=l\left(\frac{1}{u^{\mathrm{U}}}u(t)-u_f(t)\right),\quad u_f(t_0)=u_{0f},$$

and then $-\gamma_u u_c(t)$ is an estimate of the augmented perturbation with the amplitude and rate bounded by given values.

It should be noted that, owing to (4.4), the absolute value of the rates of $u_s(t)$, $u_c(t)$ are bounded for all $t \geq t_0$ by given positive scalars $\frac{(1-\mu\kappa\gamma_v)\rho}{1-\gamma_u}$ and $\frac{\mu\kappa\gamma_v\rho}{\gamma_u}$, and therefore, inequality (4.3b) is an identity:

$$(1-\mu\kappa\gamma_v)\rho + \gamma_u \frac{1}{\gamma_u}\kappa\mu\gamma_v\rho \leq \rho. \tag{4.6a}$$

On the other hand, if for all $t \geq t_0$ we have that $|u_s(t)| \leq 1$, $|u_c(t)| \leq 1$, then inequality (4.3a) is also an identity:

$$1 - \gamma_u + \gamma_u \leq 1. \tag{4.6b}$$

Thus, the problem to meet the constraints (3.3) for the amplitude and rate of the control signal is reduced to satisfaction of the conditions $|u_s(t)| \leq 1$, $|u_c(t)| \leq 1$ for all $t \geq t_0$. The conditions, under which such requirements hold and constraints (3.3) are satisfied, are given in the following theorem.

Theorem 1. *Let*

$$\gamma_u \in (0,\ 1),\ \gamma_v \in (0,\ 1),\ \kappa \in (1,\ 2),\ \rho > 0,\ 1 < \mu < \tfrac{1}{\kappa\gamma_v},$$
$$\lambda_1 > 0,\ \lambda_2 > 0,\ K > 0,\ l > 0,$$

then the control law (4.4), (4.5) *with the initial conditions*

$$|u_s(t_0)| \leq 1,\ |u_c(t_0)| \leq 1 \tag{4.7}$$

ensures $u(t) \in \mathcal{U}$ *for all* $t \geq t_0$.

Proof of Theorem 1 is given in Appendix.

Since $u_s(t)$, $u_c(t)$ are internal states of the feedback law, it is always possible to set $u_s(t_0)$, $u_c(t_0)$ such that the constraint $u(t) \in \mathcal{U}$ is satisfied for all $t \geq t_0$. Despite the parameters γ_u, γ_v are used in the control law, the existence of bounds for the disturbance from Assumption 2 is not required for the proof of Theorem 1, and therefore boundedness of the amplitude and rate of the control signal is ensured independently of the satisfaction of Assumption 2 and, moreover, leads to the fulfillment of all its prerequisites. This result allows one to obtain conditions, under which the chosen control law provides a solution to the prescribed performance control problem (3.2) under the given constraints for the amplitude and rate of the control signal (3.3).

Theorem 2. *Let Assumptions 1–3 be met, under given parameters*

$$\gamma_u \in (0,\ 1),\ \gamma_v \in (0,\ 1),\ bu^{\mathrm{U}} > 0,\ \delta \in (0,\ 1),\ \kappa \in (1,\ 2),\ L > \kappa,\ \rho > 0,\ \varepsilon > 0 \tag{4.8a}$$

and some scalar

$$1 < \mu^* < \tfrac{1}{\kappa\gamma_v} \tag{4.8b}$$

the following linear inequalities with respect to $\Psi \in \mathbb{R}^5$ *be feasible:*

$$\begin{aligned} &e_i^\top \Gamma \Psi < e_i^\top \Theta,\ i = 1, \ldots, 4, \\ &e_j^\top \Psi > 0,\ j = 1, \ldots, 5, \end{aligned} \tag{4.9}$$

where $e_i \in \mathbb{R}^4$, $e_j \in \mathbb{R}^5$ *are coordinate vectors of the respective spaces,*

$$\Gamma = \begin{bmatrix} 0 & 0 & 0 & 1-\mu^* & 1 \\ 0 & 1-\gamma_u & \gamma_u & L\gamma_v\rho & L\gamma_v\rho \\ -\frac{(1-\mu^*\kappa\gamma_v)\rho}{\kappa} & 1-\gamma_u & \gamma_u & L\gamma_v\rho & L\gamma_v\rho \\ (1-\gamma_u)\, bu^{\mathrm{U}} & 0 & 0 & 0 & 0 \end{bmatrix}, \Theta = \begin{bmatrix} 0 \\ 1-(1+\delta)\gamma_u \\ -1+(1-\delta)\gamma_u \\ \varepsilon \end{bmatrix}.$$

Then the control law (4.4), (4.5) *with the parameters*

$$K = \hat{\Psi}_1^{-1},\ \lambda_1 = \hat{\Psi}_2,\ \lambda_2 = \hat{\Psi}_3,\ l = \hat{\Psi}_4^{-1},\ \tau = \hat{\Psi}_5,\ \mu = \mu^*, \tag{4.10}$$

initial conditions (4.7) *and*

$$|z(t_0)| \le \frac{1-\gamma_u}{K}, \quad \left|u_s(t_0) + \sigma\left(\frac{K}{1-\gamma_u} z(t_0)\right)\right| \le \lambda_1, \tag{4.11a}$$

$$|-u_f(t_0) - F(t_0)| \le \tau\gamma_v\rho, \quad \left|u_c(t_0) - \sigma\left(\frac{1}{\gamma_u} u_f(t_0)\right)\right| \le \lambda_2 \tag{4.11b}$$

ensures $u(t) \in \mathcal{U}$ *and* $y(t) \in \mathcal{Y}$ *for all* $t \ge t_0$.

Proof of Theorem 2 is postponed to Appendix.

Theorem 2 shows that the control law parameters calculation with the help of (4.10) under conditions (4.8a), (4.8b) and (4.9) guarantees achievement of the goal (3.2) under the given constraints (3.3) for the amplitude and rate of the control signal. However, Theorem 2 does not address feasibility of the linear inequalities (4.9) for all values of parameters (4.8a) and scalar (4.8b). The following lemma states the conditions, under which inequalities (4.9) are feasible.

Lemma 1. *Let*

$$\gamma_u \in \left(0,\ \frac{1}{1+(1+\alpha)\,\delta}\right), \quad \alpha \in (0,\ 1), \quad \gamma_v \in (0\ \ 1),$$
$$bu^{\mathrm{U}} > 0, \quad \delta \in (0,\ 1), \quad \kappa \in (1,\ 2), \quad L > \kappa, \quad \rho > 0$$

and

$$1 < \mu^* < \frac{1}{\kappa\gamma_v}.$$

Then inequalities (4.9) *are feasible wih respect to* $\Psi \in \mathbb{R}^5$ *for any* $\varepsilon > \underline{\varepsilon} > 0$.

Proof of Lemma 1 and equation for $\underline{\varepsilon}$ are presented in Appendix.

Thus, in accordance with Lemma 1, there exists ε, for which inequalities (4.9) are feasible. However, according to Lemma 1, in order to ensure existence of inequalities (4.9) solution, it is required that $\gamma_u \in \left(0,\ \frac{1}{1+(1+\alpha)\delta}\right)$, $\alpha \in (0,\ 1)$. This reduced range of admissible values of γ_u does not coincide with the one given in Assumption 2. However, this circumstance is not a serious obstacle to achieve the goal (3.2) under constraints (3.3), since the parameter δ is a parameter of the function $\sigma(.)$ from the condition e). For the following differentiable function satisfying conditions a)–f)

$$\sigma(s) = \operatorname{sign}\{s\}\left(1 - (1 - \min\{|s|, 1\})^{1.0178}\right) \tag{4.12}$$

it holds that $\delta = 6.5 \cdot 10^{-3}$, and consequently, $\frac{1}{1+(1+\alpha)\delta} \le 0.9872$, and the range from Assumption 2 coincides with sufficient accuracy with the interval, for which inequalities (4.9) are feasible.

The conditions of Lemma 1 are only sufficient, and their violation does not imply that (4.9) is infeasible, *i.e.*, Lemma 1 conditions violation does not mean that the proposed control law cannot ensure achievement of the goal (3.2) under the given constraints (3.3) and the required $\varepsilon < \underline{\varepsilon}$. However, the result of Lemma 2 is important from a theoretical point of view, since it allows one to conclude that it is fundamentally possible to solve (4.9) for a set of given width.

Inequalities (4.9) can be solved using various algorithms and software. For example, one can apply algorithms to solve linear (linprog function in MatLab) or semidefinite programming problems (cvx, SEDUMI, YALMIP in MatLab).

According to Theorem 2, any solution of the inequalities allows one to obtain a control law that ensures the prescribed transient performance for the output (3.2) under the given control constraints (3.3). Therefore, the question of how to choose the most suitable solver or obtain the optimal solution of the considered inequalities is beyond the scope of the theoretical problem statement of this paper and requires its refinement or extension by some optimality criterion. From practical point of view, the solutions obtained via different numerical algorithms are identical with respect to the performance requirements (3.2) and constraints (3.3), but may differ, for example, in terms of the oscillations of the system output within the set $\mathcal{Y}$ or the boundedness of the control signal acceleration. In the author's opinion, the most successful solutions of inequalities (4.9) with respect to the mentioned criteria can be obtained using the interior-point method (feasp function in MatLab).

The program code below, written in the MatLab language using the feasp function, implements the solution of inequalities (4.9) for given parameters (4.8a) and a chosen scalar (4.8b). The result of the code execution is the control law parameters.

```
% SYSTEM PARAMETERS
rho=1.5; g_u=0.98; g_v=0.45; bU=0.8*(1.4522e+04)*1/3;
kappa=1.04; delta=6.5*10^-3; L=1.1; mu=1.01; eps=5;
% INEQUALITIES INITIALIZATION AND SOLUTION
Gamma=[0 0 0 1-mu 1; 0 1-g_u g_u L*g_v*rho L*g_v*rho;
-(1-mu*kappa*g_v)*rho/kappa 1-g_u g_u L*g_v*rho L*g_v*rho;
(1-g_u)*bU 0 0 0 0];
Theta=[0; 1-(1+delta)*g_u;-1+(1-delta)*g_u; eps];
e1=[1 0 0 0]; e2=[0 1 0 0]; e3=[0 0 1 0]; e4=[0 0 0 1];
setlmis([]); Psi=lmivar(2,[5 1]);
lmiterm([-1 1 1 Psi],.5*[1,0,0,0,0],1,'s');
lmiterm([-2 1 1 Psi],.5*[0,1,0,0,0],1,'s');
lmiterm([-3 1 1 Psi],.5*[0,0,1,0,0],1,'s');
lmiterm([-4 1 1 Psi],.5*[0,0,0,1,0],1,'s');
lmiterm([-5 1 1 Psi],.5*[0,0,0,0,1],1,'s');
lmiterm([6 1 1 Psi],.5*e1*Gamma,1,'s');
lmiterm([7 1 1 Psi],.5*e2*Gamma,1,'s');
lmiterm([8 1 1 Psi],.5*e3*Gamma,1,'s');
lmiterm([9 1 1 Psi],.5*e4*Gamma,1,'s');
lmiterm([-6 1 1 0],e1*Theta); lmiterm([-7 1 1 0],e2*Theta);
lmiterm([-8 1 1 0],e3*Theta); lmiterm([-9 1 1 0],e4*Theta);
LM1=getlmis;[tmin,xfeas] = feasp(LM1, [0 1000 1e22 0 0]);
% if tmin ¡ 0, then the inequalities (4.9) is feasible
PS1 = [1 0 0 0 0]*dec2mat(LM1,xfeas,Psi);
PS2 = [0 1 0 0 0]*dec2mat(LM1,xfeas,Psi);
PS3 = [0 0 1 0 0]*dec2mat(LM1,xfeas,Psi);
PS4 = [0 0 0 1 0]*dec2mat(LM1,xfeas,Psi);
PS5 = [0 0 0 0 1]*dec2mat(LM1,xfeas,Psi);
K=PS1^-1; lambda1=PS2; lambda2=PS3; l=PS4^-1; mu=mu; tau=PS5;
```

Remark 1. By choosing the initial condition $y(t_0)$ inside a sufficiently small neighborhood of the center of the set $\mathcal{Y}$, the inequality $|z(t_0)| \le \frac{1-\gamma_u}{K}$ can be satisfied. Conditions (4.11b) are not restrictive for practical scenarios, since usually, at the initial time instant t_0, we have $y(t_0) = 0$, $u(t_0) = 0$, $\dot{g}(t_0) = 0$ and the disturbance $f(t, y, u)$ satisfies $f(t_0, 0, 0) = 0$. Therefore, to satisfy conditions (4.11b), it is usually sufficient to choose $u_f(t_0) = u_c(t_0) = 0$. If this is not the case,

then the following simple technique is proposed. Since $\Phi(t)$ is an estimate of $F(t)$, one can obtain the estimate $\Phi(t)$ over $t < t_0$ and, at the moment of the control system initialization t_0, set $u_f(t_0) = -\Phi(t_0)$. If, for $t < t_0$, a sufficiently large parameter $l > 0$ is used in (4.5), then the equality $u_f(t_0) = -\Phi(t_0)$ will be satisfied with any prescribed accuracy [15]. In other words, if nothing is known *a priori* about whether conditions (4.11b) are satisfied, then it makes sense to first run the disturbance observer (4.5), and only then initialize the control system (4.4) taking into account the disturbance estimate, which has already been obtained by the time t_0.

Remark 2. In accordance with the proof of Theorem 2, if the constraints for the initial conditions (4.11a), (4.11b) are not satisfied, but inequalities (4.7), (4.9) hold, then $u(t) \in \mathcal{U}$ for all $t \geq t_0$, and for all $y(t_0) \in Y$ there exists a time instant $T > t_0$ such that $y(t) \in \mathcal{Y}$ for all $t \geq T$.

Remark 3. To reduce the values of the the control law (4.4) parameters, inequalities (4.9) can be augmented with constraints of the form $\Psi_i \geq \underline{\Psi}_i > 0$, where $\underline{\Psi}_i > 0$ are given numbers.

4.1.1. Estimation of augmented perturbation initial condition. In accordance with Theorem 2, the goal (3.2) achievement essentially depends on the satisfaction of inequalities (4.11a) and (4.11b). According to the arguments presented in Remark 1, inequalities (4.11a) can always be satisfied by choosing $\overline{g}(t_0), \underline{g}(t_0), u_s(t_0)$ for a given $y(t_0)$. In this subsection, we develop the idea proposed in the same remark on how to satisfy inequalities (4.11b) via estimation of the initial condition $F(t_0)$ of the augmented disturbance. To this end, let $t_0 > 0$ be the time instant of initialization of the control law (4.4), (4.5) for system (3.1), and let $t = 0$ be the time instant when the states of system (3.1) start being measured. Then, to satisfy inequalities (4.11b), the following initialization of the control law initial conditions is proposed:

$$\begin{aligned} u_f(t_0) &= -\hat{F}(t_0), \\ u_c(t_0) &= \sigma\left(\frac{1}{\gamma_u} u_f(t_0)\right), \end{aligned} \tag{4.13}$$

where $\hat{F}(t)$ is an estimate of $F(t_0)$ obtained over $[0,\ t_0]$.

Therefore, satisfaction of (4.11b) is reduced to estimation of the augmented perturbation over $[0,\ t_0]$ in such a way that

$$\left|\hat{F}(t_0) - F(t_0)\right| \leq \tau\gamma_v\rho,\ \tau > 0. \tag{4.14}$$

To achieve this goal, we introduce a disturbance observer that is structurally identical to (4.5) but starts its operation at $t = 0$:

$$\hat{F}(t) = \hat{\chi}(t) - \hat{u}(t) - \frac{\dot{g}(t)}{bu^{\mathrm{U}}}, \tag{4.15}$$

where

$$\begin{aligned} \hat{\chi}(t) &= k\Big(z(t) - \hat{z}(t)\Big), \\ \dot{\hat{z}}(t) &= \hat{\chi}(t), \quad \hat{z}(0) = z(0), \\ \dot{\hat{u}}(t) &= k\left(\frac{1}{u^{\mathrm{U}}} u(t) - \hat{u}(t) - \frac{\dot{g}(t)}{bu^{\mathrm{U}}}\right), \quad \hat{u}(0) = 0 \end{aligned}$$

and $k > 0$, $g(t) = 0$ and $\dot{g}(t) = 0$ for all $t_0 > t \geq 0$.

Then the estimation error $\tilde{F}(t) = \hat{F}(t) - F(t)$ satisfies the following equation for all $t_0 \geq t \geq 0$:

$$\dot{\tilde{F}}(t) = k\left(\dot{z}(t) - \dot{\hat{z}}(t)\right) - \dot{\hat{u}}(t) - \frac{\ddot{g}(t)}{bu^{\mathrm{U}}} - \dot{F}(t) = -k\tilde{F}(t) - \frac{\dot{f}(t,x,u)}{bu^{\mathrm{U}}}, \tag{4.16}$$

where $\tilde{F}(0) = -F(0)$.

Owing to equation (4.16), it is easy to show the following.

Proposition 1. *Let* $\left|\frac{\dot{f}(t,\,x,\,u)}{bu^{\mathrm{U}}}\right| \leq \gamma_v\rho$ *for all* $0 \leq t \leq t_0$ *and the initial conditions for a compensation component of the control law* (4.4), (4.5) *be defined in accordance with* (4.13), (4.15). *Then, for all* $F(0)$ *and* $t_0 > 0$, *there exists* $k^* > 0$ *such that the choice*

$$k > k^* \tag{4.17}$$

ensures that inequality (4.11b) *holds.*

Proof of proposition 1 is presented in Appendix.

Thus, using the proposed initialization algorithm for the compensation component of the control law, it is always possible to satisfy inequalities (4.11b). This allows one to conclude that all requirements on the initial conditions (4.7), (4.11a), (4.11b) from Theorem 2 are not restrictive and can be satisfied by: *a*) choosing $\overline{g}(t_0), \underline{g}(t_0), u_s(t_0)$ for a given $y(t_0)$, *b*) initializing the initial conditions of the compensation component of the control law (4.4), (4.5) using the algorithm (4.13), (4.15), (4.17). It should also be noted that inequality $\left|\frac{f(t,\,x,\,u)}{bu^{\mathrm{U}}}\right| \leqslant \gamma_v\rho$ can be satisfied even for unstable systems, and hence, proposition 1 does not require the system to be stable and/or be at the equilibrium over $[0, t_0]$.

4.2. Engineering Interpretation

The control law (4.4), (4.5) is contains a proportional feedback, whose amplitude and rate are bounded at all time instants by $(1-\gamma_u)u^{\mathrm{U}}$ and $(1-\mu\kappa\gamma_v)v^{\mathrm{U}}$ and the estimate of the augmented disturbance, whose amplitude and rate are bounded at all time instants by $\gamma_u u^{\mathrm{U}}$ and $\mu\kappa\gamma_v v^{\mathrm{U}}$, respectively.

To calculate the parameters of such a control law, Theorem 2 requires to solve linear inequalities (4.9) and know the following parameters:

— the maximum value of the control effort bu^{U}, where b is a known estimate of the unknown system gain (see Section 3.1.2);
— the constraints u^{U} and v^{U} for the amplitude and rate of the control signal;
— the scalars γ_u and γ_v, which indicate how many times the amplitude and rate of the disturbance, divided by maximum control effort, are less than unity;
— the minimum width ε of the set $\mathcal{Y}$.

Considering practical scenarios, an assumption that all these parameters are known is not restrictive, since some of the parameters are chosen at the stage of the system actuator design (u^{U}, v^{U}, b) or at the specification stage of the control system development (ε). The other part, namely the scalars $\gamma_u \in (0,\,1)$, $\gamma_v \in (0,\,1)$, from theoretical point of view, can be chosen arbitrarily close to the right boundaries of their admissible intervals. However, from practical point of view and due to the conservativeness imposed by derivation of inequalities (4.9), there is no need in that, and the choice of γ_u, γ_v should be aimed at inequalities (3.5) satisfaction.

To solve inequalities (4.9), it is also necessary to know the parameters κ, L, δ from properties *a*)–*f*) of the saturation function approximation. This requirement is not restrictive, since for each specific approximation these parameters can always be calculated numerically or found analytically. Moreover, in this paper, for two functions (see Section 2 and equation (4.12)), the values of these parameters have already been given.

If inequalities (4.9) are feasible, then, according to Theorem 2, the control law (4.4), (4.5) with parameters (4.10) and initial conditions (4.7), (4.11a), (4.11b) is guaranteed to ensure the following property:

If Assumptions 1–3 are met, then for any augmented disturbances $F(t)$, *whose amplitude does not exceed* $100\gamma_u$ % *of* u^{U}, *and rate is not higher than* $100\gamma_v$ % *of* v^{U}, *it is ensured that* $y(t) \in \mathcal{Y}$, $u(t) \in \mathcal{U}$ *for all* $t \geq t_0$.

5. ANALYTIC COMPARISON

In terms of structure and properties, the control law (4.2) is most closely related to the one from [10]. In accordance with the results of [10], the following control law is introduced to ensure a given bound for the tracking error with respect to the center of the set $\mathcal{Y}$:

$$u(t) = -u^{\mathrm{U}} m \sigma(kz(t)), \tag{5.1}$$

where $m > 0,\ k > 0$ and

$$\sigma(s) := \frac{2}{1+\exp(-s)} - 1$$

is a sigmoid function (in contrast to the saturation function approximation used in this paper, for the sigmoid one, condition c) from Section 2 is not satisfied).

According to lemma and Remark 1 from [10], if there exists $\gamma_u \in (0,\ 1)$ such that for all $t \geq t_0$ it holds that $|F(t)| \leq \gamma_u$, and the following inequalities are satisfied:

$$\frac{\gamma_u}{\sigma(2.2)} < m \leq 1, \tag{5.2a}$$

$$\frac{2.2}{\Delta} \leq k < \frac{\rho}{m^2}, \quad \Delta > 0, \tag{5.2b}$$

$$bu^{\mathrm{U}}\Delta \leq \varepsilon \tag{5.2c}$$

then the control law (5.1) ensures that $y(t) \in \mathcal{Y}$ and $u(t) \in \mathcal{U}$ for all $t \geq t_0$.

First of all, it should be noted that inequalities (5.2a) and (5.2b) are feasible only if it holds that

$$\begin{aligned} v^{\mathrm{U}} &> \frac{2.2}{\Delta} m^2 u^{\mathrm{U}}, \\ \varepsilon &> 2.2 b u^{\mathrm{U}} \frac{\gamma_u^2}{\sigma^2(2.2)\rho} = \underline{\varepsilon}, \end{aligned} \tag{5.3}$$

which are rather restrictive requirements: i) not for all $v^{\mathrm{U}} > 0$ and $u^{\mathrm{U}} > 0$ it is possible to bound the control signal rate, ii) the comparison of the lower bounds (5.3) and (A.14) allows one to conclude that the proposed control law, for the same $u^{\mathrm{U}} > 0,\ v^{\mathrm{U}} > 0$ and $\gamma_u \to 1$, ensures $y(t) \in \mathcal{Y}$ for all $t \geq t_0$ for sets of substantially smaller width ε.

Secondly, the following upper bound was used to obtain (5.2b) in [10]:

$$\begin{aligned} |\dot{u}(t)| &= \left| -\frac{u^{\mathrm{U}} m k}{2} \left[1 - \sigma^2(kz(t))\right] \dot{z}(t) \right| \\ &\leq \left| -\frac{u^{\mathrm{U}} m k}{2} \left[1 - \sigma^2(kz(t))\right] \right| \left| -m\sigma^2(kz(t)) + F \right| \\ &< \left| -u^{\mathrm{U}} m^2 k \left[1 - \sigma^2(kz(t))\right] \right| < u^{\mathrm{U}} m^2 k. \end{aligned} \tag{5.4}$$

Such an estimate can only be obtained when the output signal $y(t)$ is not affected by the measurement noise, *i.e.*, when inequalities (5.2a), (5.2b), (5.2c) are met, but the output is corrupted by the measurement noise, the control law (5.1) does not guarantee boundedness of the control signal rate by a given value. As follows from (4.4), the control law proposed in this paper, owing to the nonlinear filters of the type [14], guarantees boundedness of the control signal rate regardless of the presence of measurement noise, which is a significant advantage for practical scenarios.

Therefore, the main advantages of the control law (4.4), (4.5) in comparison with (5.1) are as follows:

1) the given constraints for the amplitude and rate of the control signal are satisfied in addition to the one for the tracking error;
2) the proposed control law contains not only a component proportional to the tracking error, but also a dynamic estimate of the external augmented disturbance designed with the help of the auxiliary loop method [15, p. 201];
3) the procedure to choose the control law parameters is reduced to solving linear inequalities, which can be done numerically using software packages (the inequalities from [10] are solved manually).

6. NUMERICAL EXPERIMENTS

Simulations have been conducted in MatLab Simulink using forward Euler integration method with a fixed step of 10^{-4} seconds.

To demonstrate the theoretical results, we considered a linearized model of the longitudinal motion of an aircraft at an altitude of 1200 meters with an airspeed of Mach 2 [20, p. 565]:

$$\begin{aligned} &\dot{x}(t) = Ax(t) + Bu(t) + Dd(t),\ x(t_0) = x_0, \\ &y(t) = Cx(t), \end{aligned} \tag{6.1}$$

where

$$A = \begin{bmatrix} -1.364 & -92.82 & -128.46 \\ 1 & -4.68 & -0.087 \\ 0 & 0 & -190 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ 0 \\ 190 \end{bmatrix},\ D = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 1.36 & -184.26 & 76.43 \end{bmatrix},\ x(t_0) = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix},\ d(t) = 0,$$

$u(t)$ is the elevator control signal, rad, $y(t)$ stands for the vertical acceleration, m/s^2, $x_1(t)$ denotes the pitch rate, rad/s, $x_2(t)$ is the angle of attack, rad, $x_3(t)$ denotes the elevator deflection angle, rad, $d(t)$ is an external disturbance. The maximum elevator deflection angle is $\frac{1}{3}$ rad, and its rate does not exceed $\frac{1}{2}$ rad/s, *i.e.* the control constraints are defined as $u^{\mathrm{U}} = \frac{1}{3}$, $v^{\mathrm{U}} = \frac{1}{2}$. The matrices A, B, C and D are assumed to be unknown, but we know the estimate $b = 1.16 \times 10^4$ of the gain $CB = 1.45 \times 10^4$. The relative degree of system (6.1) is equal to one, and therefore, according to Example 1, it is reducible to the form (3.1).

The prescribed transient performance for the system (6.1) is defined by the set $\mathcal{Y}$ with the following bounds:

$$\overline{g}(t) = 75\sin(0.1\pi t) + 5,\ \underline{g}(t) = 75\sin(0.1\pi t) - 5, \tag{6.2}$$

and then $\varepsilon = 5$ is the minimal width of the set $\mathcal{Y}$.

Assume that the amplitude of the augmented disturbance $F(t)$ does not exceed 98% of u^{U}, and its rate is not higher than 45% of v^{U} over the trajectories of the closed-loop system, *i.e.*,

$$\gamma_u = 0.98,\ \gamma_v = 0.45. \tag{6.3}$$

This hypothesis corresponds to Assumption 2 and cannot be verified *a priori* for the system (6.1) with unknown parameters and affected by external disturbances. The verification of this hypothesis can only done *a posteriori* by application of the control law (4.4), (4.5) and analysis of the system output behavior. From theoretical point of view, one can choose $\gamma_u \to 1$ and $\gamma_v \to 1$, then the class of admissible augmented disturbances is maximally extended. From practical point of view, there is no need for such a choice, since in many practical problems the augmented disturbance rarely exceeds $50 - 60\%$ of the maximal amplitude and rate of the control signal. Moreover, the solution of inequalities (4.9) may result in relatively large values of the control law parameters for $\gamma_u \to 1$ and $\gamma_v \to 1$.

In fact, a particular choice of γ_u and γ_v defines a class of systems with unknown parameters and external disturbances, for which the proposed control law ensures achievement of the goal (3.2) under constraints (3.3). Unfortunately, the fact that a particular system with unknown parameters and external disturbances is a member of such a class can only be verified *a posteriori* after testing the control law. For example, for system (6.1), the values from (6.3) were chosen via iterative process consisted of *i*) solution of inequalities (4.9) for various values of γ_u and γ_v, and *ii*) application of the control law (4.4), (4.5) with the calculated parameters. Such iterations were terminated, when we obtained such values of parameters (4.10) that the application of the control law ensured achievement of the stated goal. Note that the discussed *a posteriori* validation is conventional for many robust control laws. The only difference is that, usually, the ranges for the system parameters and the amplitude of external disturbances are chosen on the basis of some hypothesis, whereas in this paper, we choose the maximum values of the amplitude and rate of the augmented disturbance, which is an unknown function of the system output, its parameters and disturbances (see Example 1).

Thus, the given sets $\mathcal{Y}$ and $\mathcal{U}$, together with the chosen parameters (6.3), correspond to the following specification for the automatic control system design: *it is required to ensure tracking of the reference signal for the vertical acceleration with a dynamic error not exceeding 5 m/s^2 in the presence of matched augmented disturbances that are not higher than* 98% *of the maximum amplitude and* 45% *of the maximum rate of the control signal.*

To solve such a problem using the control law (4.4), (4.5), in accordance with Theorem 2, it is necessary to solve inequalities (4.9). To this end, we have chosen the function (4.12) and pick $\mu^* = 1.01$.

As a result, the following solution of (4.9) was obtained:

$$K = 17.7436,\ \lambda_1 = 0.1434,\ \lambda_2 = 0.0029,\ l = 194.912,\ \tau = 3.5 \times 10^{-5}. \tag{6.4}$$

The inequalities were solved in MatLab using code presented in Section 4.2. To achieve the goal (3.2) under constraints (3.3), in accordance with Theorems 1 and 2, the initial conditions of the control law were picked as follows:

$$z_f(t_0) = z(t_0) = 0,\ u_s(t_0) = 0,\ u_f(t_0) = -F(t_0) = 0,\ u_c(t_0) = 0, \tag{6.5}$$

which ensured satisfaction of the conditions (4.7) and (4.11a), (4.11b).

Figures 2 and 3 depicts the behavior of the control signal $u(t)$ and its rate $\dot{u}(t)$, the output $y(t)$, as well as the internal variables $\eta_1(t)$, $\eta_2(t)$, $\eta_3(t)$ of the control law (see the proof of Theorem 2).

Figures 2 and 3 validate theoretical results obtained in Theorem 2. The satisfaction of inequalities (4.9) and other prerequisites of Theorem 2 leads to the achievement of the stated goal (3.2) under constraints (3.3). Moreover, the internal variables of the control law also do not leave the range specified by inequalities (4.9) (see the proof of Theorem 2).

Next the proposed control law was tested for a system affected by measurement noise and discontinuous external disturbances. For such an experiment, the output of the system (6.1) and the perturbation were defined as follows:

$$\begin{gathered} y(t) = Cx(t) + w_0(t) + w_1(t), \\ d(t) = 0.1 + \operatorname{sign}\{\sin(0.4\pi t)\}, \\ w_1(t) = \begin{cases} -Cx(t) - w_0(t) + \overline{g}(t), & \text{if} \quad t \in [7,\ 7.1), \\ 0 \quad \text{otherwise}, \end{cases} \end{gathered}$$

where the signal $w_0(t)$ was implemented in the MatLab Simulink with the help of the block "Random Number" with zero mean, variance of 10^{-1} and sampling time equaled to 10^{-4}. All other parameters of the system and control law were left unchanged.

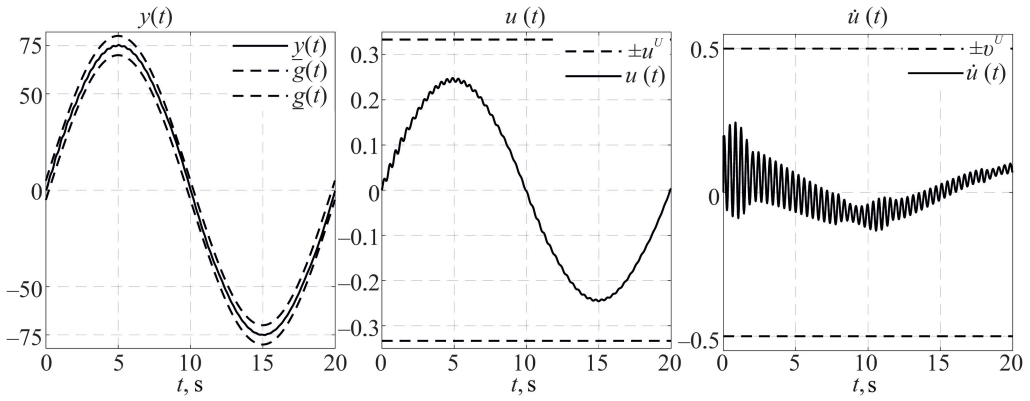


Fig. 2. Behavior of $u\,(t)$, $\dot{u}\,(t)$ and $y\,(t)$.

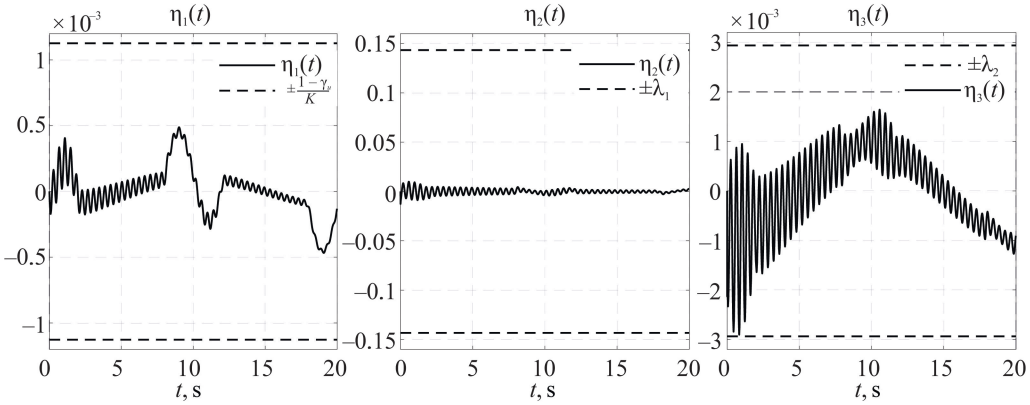


Fig. 3. Behavior of $\eta_1\,(t)$, $\eta_2\,(t)$, $\eta_3\,(t)$.

For comparison purposes, we chose the control law (5.1) with parameters $m = 1,\ k = 1692$ calculated via (5.2) with $\varepsilon = 5$, $b = 1.16 \times 10^4$ and $u^{\mathrm{U}} = \frac{1}{3}$, and the law from Section 4 of [1]:

$$u(t) = -(CB)^{-1}\Big[CAx(t) + K\varepsilon(t)\Big], \quad \varepsilon(t) := ln\left(\frac{\underline{g}(t) - y(t)}{y(t) - \overline{g}(t)}\right), \tag{6.6}$$

where $K > 0$.

The control law (6.6) requires knowledge of the system (6.1) parameters and does not take into account the constraints (3.3) for the control signal and its rate. However, despite these circumstances, its comparison with the proposed control law is admissible and correct, since the purpose of the comparison is primarily to verify the sensitivity to measurement noise and discontinuous perturbations. The conclusions drawn will remain valid both for adaptive/robust versions of the control law (6.6) and for versions that take into account the control input constraint [1, 2].

Figure 4 depicts that the control law (6.6) ensures that the system output remains in the set $\mathcal{Y}$ in the presence of measurement noise for all $t < 7$. However, when $y\,(t)$ approaches the boundaries of the set $\mathcal{Y}$, the rate of the control signal increases. As a result, a control signal that cannot be implemented by the actuator may be generated. Moreover, at the time instant $t = 7$, due to the influence of the noise component $w_1\,(t)$, a feedback discontinuity occurs (see (6.6)), and therefore, the simulation cannot be continued for all $t > 7$. Considering practical scenarios, even a potential occurrence of such discontinuities is unacceptable, since feedback discontinuity can lead to equally unpredictable and catastrophic consequences. The control law proposed in this paper is free from this drawback.

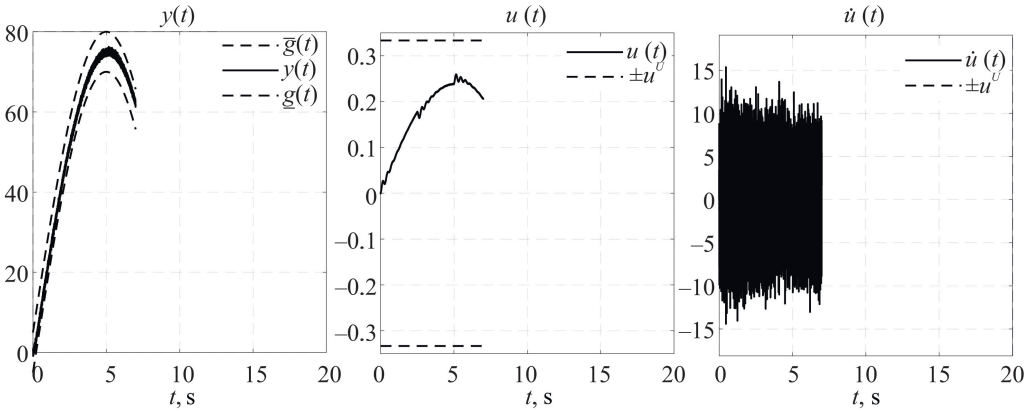


Fig. 4. Behavior of $u(t)$, $\dot{u}(t)$ and $y(t)$ for (6.6) when $K = 2.5$.

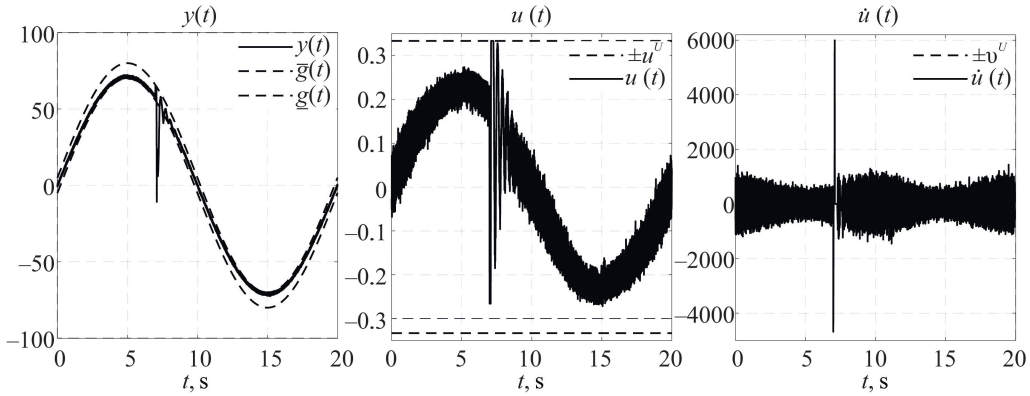


Fig. 5. Behavior of $u(t)$, $\dot{u}(t)$ and $y(t)$ for (5.1) when $m = 1,\ k = 1692$.

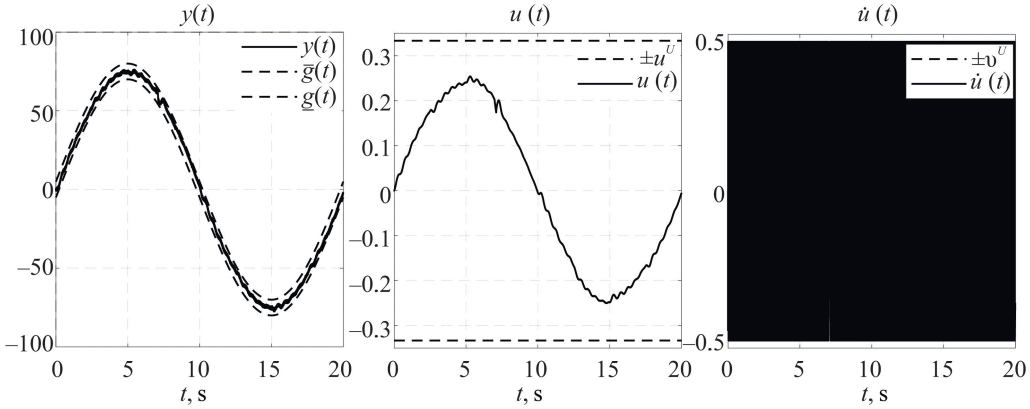


Fig. 6. Behavior of $u(t)$, $\dot{u}(t)$ and $y(t)$ for (4.4), (4.5).

In Figure 5, the behavior of $u(t)$, $\dot{u}(t)$ and $y(t)$ for (5.1) is presented.

The control law (5.1) solves the tracking problem with prescribed accuracy, which leads to keeping the system output within the given set $\mathcal{Y}$. However, the rate of the control signal is not bounded by a given value, and the control signal is significantly affected by noise, which illustrates the conclusions of the analysis from Section 5.

Figure 6 presents the behavior of $u(t)$, $\dot{u}(t)$ and $y(t)$ for the proposed control law for the system affected by discontinuous disturbances and measurement noise.

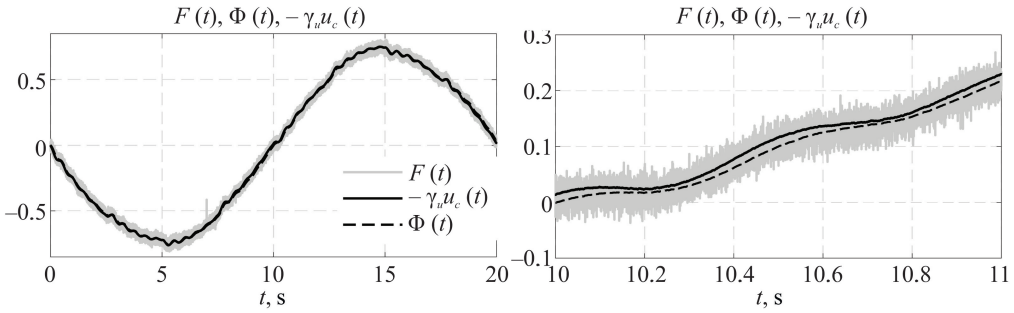


Fig. 7. Behavior of $F(t)$ and $\Phi(t)$, $-\gamma_u u_c(t)$.

According to Assumption 2, the proposed control law requires all external disturbances to be differentiable. In this simulation, this requirement is violated, but the prescribed performance (3.2) under the given constraints (3.3) is still ensured. This allows us to conclude that the conditions adopted for the proof of Theorem 2 are conservative and stimulates further theoretical research aimed at their relaxation. Moreover, in comparison with the control law (6.6), the proposed one is free from the problem of feedback discontinuities and ensures that the given constraints for the amplitude and rate of the control signal are met even in the presence of measurement noise. Comparing to the control law (5.1), the proposed one, even under such conditions, ensures boundedness of the control rate by a given value. It should be noted that the properties of the proposed control law (4.4), (4.5) are strictly justified only for the situation when we have no discontinuous external disturbances and measurement noise. Therefore, the conclusions drawn from Figures 3-5 are valid only for the considered system (6.1) under the given specific initial conditions and the integration algorithm in use.

Figure 7 depicts behavior of $F(t)$ and estimates $\Phi(t)$, $-\gamma_u u_c(t)$.

It follows from Fig. 7 that the disturbance estimate obtained by the auxiliary loop method [15, p. 201] is noisy due to the presence of noise in the measured output. In practice, using such disturbance estimates for compensation purposes can result in system instability because of *i*) the bounded control signal rate and *ii*) the excitation of system unmodeled dynamics. To reduce the influence of noise on the estimate $\Phi(t)$, the auxiliary loop method requires to decrease the parameter l. However, the developed control law employs a nonlinear filter of the type [14], which forms the estimate $\Phi(t)$ with a rate not exceeding the control signal rate constraint. This makes it possible, regardless of the values of the parameter l, to attenuate the noise influence and, as a result, satisfy the control signal rate constraint without causing excitation of the unmodeled dynamics. An analytical study of the filtering properties of the nonlinear filter used in the proposed control law is one of the promising directions of future research.

7. CONCLUSION

A method is proposed to solve a robust prescribed performance control problem under given constraints for the amplitude and rate of the control signal for a class of nonlinear minimum-phase systems with unknown parameters and relative degree equal to one (such systems, according to [16, p. 167], can be reduced to the form (3.1), and Assumption 2 can be satisfied for them). It follows from Example 1 that one of the representatives of such a class of systems is linear systems with unknown parameters and relative degree equal to one.

The scope of further theoretical research is, first, to extend the obtained results to the class of nonlinear systems with stable internal dynamics and relative degree greater than one (see Example 1), and second, to use the results of the paper to solve prescribed performance control problem for cascade systems under given constraints on the states and the amplitude/rate of the control signal (see Example 2).

APPENDIX

Proof of Theorem 1. Consider a function:

$$V = \frac{1}{2}u_s^2. \tag{A.1}$$

The derivative of (A.1) is written as

$$\dot{V} = -\frac{(1-\mu\kappa\gamma_v)\,\rho}{1-\gamma_u}u_s\sigma\left(\frac{1}{\lambda_1}\left(u_s+\sigma\left(\frac{K}{1-\gamma_u}z\right)\right)\right)$$
$$= -\frac{(1-\mu\kappa\gamma_v)\,\rho}{1-\gamma_u}u_s\operatorname{sign}\left\{u_s+\sigma\left(\frac{K}{1-\gamma_u}z\right)\right\}\left|\sigma\left(\frac{1}{\lambda_1}\left(u_s+\sigma\left(\frac{K}{1-\gamma_u}z\right)\right)\right)\right|.$$

For all $u_s \notin \Lambda = \{u_s \in \mathbb{R}\colon |u_s| \le 1\}$, it holds that

$$\operatorname{sign}\left\{u_s+\sigma\left(\frac{K}{1-\gamma_u}z\right)\right\} = \operatorname{sign}\{u_s\},$$

therefore, we have

$$\dot{V} = -\frac{(1-\mu\kappa\gamma_v)\,\rho}{1-\gamma_u}\left|u_s\sigma\left(\frac{1}{\lambda_1}\left(u_s+\sigma\left(\frac{K}{1-\gamma_u}z\right)\right)\right)\right| \le 0$$

for all $u_s \notin \Lambda = \{u_s \in \mathbb{R}\colon |u_s| \le 1\}$.

On the other hand, it holds for all $u_s \in \Lambda$ that

$$\dot{V} = -\frac{(1-\mu\kappa\gamma_v)\,\rho}{1-\gamma_u}u_s\sigma\left(\frac{1}{\lambda_1}\left(u_s+\sigma\left(\frac{K}{1-\gamma_u}z\right)\right)\right)$$
$$= -\frac{(1-\mu\kappa\gamma_v)\,\rho}{1-\gamma_u}u_s\operatorname{sign}\left\{u_s+\sigma\left(\frac{K}{1-\gamma_u}z\right)\right\}\left|\sigma\left(\frac{1}{\lambda_1}\left(u_s+\sigma\left(\frac{K}{1-\gamma_u}z\right)\right)\right)\right|$$
$$\le \frac{(1-\mu\kappa\gamma_v)\,\rho}{1-\gamma_u}\left|u_s\sigma\left(\frac{1}{\lambda_1}\left(u_s+\sigma\left(\frac{K}{1-\gamma_u}z\right)\right)\right)\right|,$$

thus, when $|u_s(t_0)| \le 1$, we have that $|u_s(t)| \le 1$ for all $t \ge t_0$.

Having considered the function $V = \frac{1}{2}u_c^2$ and repeated the above-given derivations, we obtain that, when $|u_c(t_0)| \le 1$, it holds that $|u_c(t)| \le 1$ for all $t \ge t_0$, and therefore, in accordance with the derivations (4.6a) and (4.6b), we have $u(t) \in \mathcal{U}$ for all $t \ge t_0$.

Proof of Theorem 2. Consider the following change of variables:

$$\eta_1 = z,$$
$$\eta_2 = u_s + \sigma\left(\frac{K}{1-\gamma_u}\eta_1\right),$$
$$\eta_3 = u_c + \sigma\left(\frac{1}{\gamma_u}\Phi\right)$$

and obtain the following differential equations with respect to η_1, η_2, η_3:

$$\begin{aligned}
\dot{\eta}_1 &= (1-\gamma_u)\,u_s + \gamma_u u_c + F(t)\\
&= (1-\gamma_u)\left(u_s \pm \sigma\left(\frac{K}{1-\gamma_u}\eta_1\right)\right) + \gamma_u u_c + F(t)\\
&= (1-\gamma_u)\left(-\sigma\left(\frac{K}{1-\gamma_u}\eta_1\right)+\eta_2\right) + \gamma_u u_c + F(t),\\
\dot{\eta}_2 &= -\frac{(1-\mu\kappa\gamma_v)\,\rho}{1-\gamma_u}\sigma\left(\frac{1}{\lambda_1}\eta_2\right) + \frac{K}{1-\gamma_u}\sigma'\left(\frac{K}{1-\gamma_u}\eta_1\right)\dot{\eta}_1,\\
\dot{\eta}_3 &= -\frac{\mu\kappa\gamma_v\rho}{\gamma_u}\sigma\left(\frac{1}{\lambda_2}\eta_3\right) + \sigma'\left(\frac{1}{\gamma_u}\Phi\right)\frac{1}{\gamma_u}\dot{\Phi}.
\end{aligned} \tag{A.2}$$

Consider the sets ($\eta = \begin{bmatrix}\eta_1 & \eta_2\end{bmatrix}^\top$):

$$\Omega_1 = \left\{\eta \in \mathbb{R}^2\colon\ |\eta_1| \le \frac{1-\gamma_u}{K},\quad |\eta_2| \le \lambda_1\right\},$$
$$\Omega_2 = \left\{\eta \in \mathbb{R}^2\colon\ |\eta_2| \le \lambda_1\right\},$$

for which, when it holds that

$$bu^{\mathrm{U}}(1-\gamma_u) < K\varepsilon, \tag{A.3}$$

then $y \in \mathcal{Y}$ follows from $\eta \in \Omega_1$.

First of all, we study boundedness of η_3 in case $\eta \in \Omega_1$. To that end, the estimate of $\dot{\Phi}(t)$ is obtained as follows. Having differentiated $\Phi(t)$, it is written:

$$\dot{\Phi}(t) = l\Big(\dot{z}(t) - \dot{z}_f(t)\Big) - \dot{u}_f(t) = -l\Big(\Phi(t) - F(t)\Big) = -l\tilde{\Phi}(t), \tag{A.4}$$

where $\Phi(t_0) = -u_f(t_0)$ and $\tilde{\Phi}(t) = \Phi(t) - F(t)$.

Then $\dot{\Phi}(t)$ is bounded if the error $\tilde{\Phi}(t)$ is bounded. A differential equation for $\tilde{\Phi}(t)$ is written as:

$$\dot{\tilde{\Phi}}(t) = -l\tilde{\Phi}(t) - \dot{F}(t),\ \tilde{\Phi}(t_0) = -u_f(t_0) - F(t_0). \tag{A.5}$$

When $g(t) \in \mathcal{G}$, $u(t) \in \mathcal{U}$, ($\eta(t) \in \Omega_1 \Rightarrow y(t) \in \mathcal{Y}$) for all $t \ge t_0$, Assumptions 1, 2 and condition $|-u_f(t_0) - F(t_0)| \le \tau\gamma_v\rho$, $\tau > 0$ are met, then there exists the following upper bound for the solution of equation (A.5):

$$\begin{aligned}\left|\tilde{\Phi}(t)\right| &= \left|e^{-l(t-t_0)}\tilde{\Phi}(t_0) - \int_{t_0}^{t} e^{-l(t-\tau)}\dot{F}(\tau)\,d\tau\right| \\ &\le \tau\gamma_v\rho + \gamma_v\rho\int_{t_0}^{t} e^{-l(t-\tau)}d\tau \le \left(\tau + l^{-1}\right)\gamma_v\rho\end{aligned} \tag{A.6}$$

for all $t \ge t_0$.

Then the derivative of the quadratic function $V = \frac{1}{2}\eta_3^2$ for all $\eta \in \Omega_1$ is obtained as:

$$\begin{aligned}\dot{V} &= \eta_3\left[-\frac{\mu\kappa\gamma_v\rho}{\gamma_u}\sigma\left(\frac{1}{\lambda_2}\eta_3\right) - \sigma'\left(\frac{1}{\gamma_u}\Phi(t)\right)\frac{1}{\gamma_u}l\tilde{\Phi}(t)\right] \\ &\le |\eta_3|\left[-\frac{\mu\kappa\gamma_v\rho}{\gamma_u}\left|\sigma\left(\frac{1}{\lambda_2}\eta_3\right)\right| + \kappa\frac{1}{\gamma_u}l\left|\tilde{\Phi}\right|\right] \\ &\le |\eta_3|\left[-\frac{\mu\kappa\gamma_v\rho}{\gamma_u}\left|\sigma\left(\frac{1}{\lambda_2}\eta_3\right)\right| + \frac{\kappa(l\tau+1)\gamma_v\rho}{\gamma_u}\right].\end{aligned} \tag{A.7}$$

On the other hand, for all $\eta_3 \notin \{\eta_3 \in \mathbb{R}\colon\ |\eta_3| < \lambda_2\}$, owing to property c) of the function $\sigma(.)$ and conditions $\mu > 1$, $\tau > 0$, $\tau + \frac{1-\mu}{l} < 0$, it holds that:

$$\dot{V} \le -\underbrace{\Big(\mu - (l\tau+1)\Big)}_{>0}\frac{\kappa\gamma_v\rho}{\gamma_u}|\eta_3|.$$

Therefore, when $|\eta_3(t_0)| \le \lambda_2$, we have $|\eta_3(t)| \le \lambda_2$ for $\eta \in \Omega_1$ and all $t \ge t_0$ independently from the sign of $\dot{V}$ over a set $\{\eta_3 \in \mathbb{R}\colon |\eta_3| < \lambda_2\}$.

Next aim is to prove that $\eta(t) \in \Omega_1$ for all $t \geq t_0$ follows from $\eta(t_0) \in \Omega_1$.

The derivative of the quadratic function $V_1 = \frac{1}{2}\eta_1^2$ for all $\eta \in \Omega_1$ is written as:

$$\begin{aligned}\dot{V}_1 &= \eta_1\left[(1-\gamma_u)\left(-\sigma\left(\frac{K}{1-\gamma_u}\eta_1\right)+\eta_2\right)+F+\gamma_u u_c \pm \gamma_u\sigma\left(\frac{F}{\gamma_u}\right)\right]\\ &= \eta_1(1-\gamma_u)\left(-\sigma\left(\frac{K}{1-\gamma_u}\eta_1\right)+\eta_2\right)\\ &+\eta_1\left[F-\gamma_u\sigma\left(\frac{F}{\gamma_u}\right)+\gamma_u\left(u_c \pm \sigma\left(\frac{\Phi}{\gamma_u}\right)+\sigma\left(\frac{F}{\gamma_u}\right)\right)\right].\end{aligned} \tag{A.8}$$

On the other hand, when we satisfy (4.7), Assumptions 1 and 2 for some

$$\delta := \sup_{|F|\leq\gamma_u}\left|\frac{F}{\gamma_u}-\sigma\left(\frac{F}{\gamma_u}\right)\right|$$

and all $\eta \in \Omega_1$, the following inequality holds:

$$\left|F-\gamma_u\sigma\left(\frac{1}{\gamma_u}F\right)\right| = \left|\frac{F}{\gamma_u}-\sigma\left(\frac{F}{\gamma_u}\right)\right|\gamma_u \leq \delta\gamma_u.$$

On the other hand, by virtue of the Lipchitz property of the function $\sigma(\cdot)$ for some $L > \kappa$, when inequalities (4.7) and Assumptions 1 and 2 are met, owing to the above-given derivations, for all $\eta \in \Omega_1$ the following inequality holds:

$$\left|\sigma\left(\frac{1}{\gamma_u}\Phi\right)-\sigma\left(\frac{1}{\gamma_u}F\right)\right| \leq \frac{L}{\gamma_u}|\Phi-F| = \frac{L}{\gamma_u}\left|\tilde{\Phi}\right| \leq \frac{L\left(\tau+l^{-1}\right)\gamma_v\rho}{\gamma_u}.$$

On the third hand, by what has been proved above, when $|\eta_3(t_0)| \leq \lambda_2$ and inequalities (4.7), Assumptions 1 and 2 are met, for all $\eta \in \Omega_1$ the following holds:

$$\left|u_c+\sigma\left(\frac{1}{\gamma_u}\Phi\right)\right| = |\eta_3| \leq \lambda_2.$$

Then the upper bound for the derivative (A.8) for all $\eta \in \Omega_1$ is written as

$$\dot{V}_1 \leq |\eta_1|\left[(1-\gamma_u)\left(-\sigma\left(\frac{K}{1-\gamma_u}\eta_1\right)+|\eta_2|\right)+\gamma_u\delta+L\left(\tau+l^{-1}\right)\gamma_v\rho+\gamma_u\lambda_2\right].$$

If $|\eta_1| = \frac{1-\gamma_u}{K}$ and $|\eta_2| \leq \lambda_1$, then the condition $\dot{V}_1 < 0$ is met when

$$(1-\gamma_u)(-1+\lambda_1)+\delta\gamma_u+L\left(\tau+l^{-1}\right)\gamma_v\rho+\gamma_u\lambda_2 < 0. \tag{A.9}$$

Moreover, the following upper bound of $|\dot{\eta}_1|$ for all $\eta \in \Omega_1$holds:

$$|\dot{\eta}_1| \leq (1-\gamma_u)(1+\lambda_1)+\delta\gamma_u+L\left(\tau+l^{-1}\right)\gamma_v\rho+\gamma_u\lambda_2.$$

The derivative of the quadratic function $V_2 = \frac{1}{2}\eta_2^2$ for all $\eta \in \Omega_2$ is written as

$$\begin{aligned}\dot{V}_2 &= \eta_2\left[-\frac{(1-\mu\kappa\gamma_v)\rho}{1-\gamma_u}\sigma\left(\frac{1}{\lambda_1}\eta_2\right)+\frac{K}{1-\gamma_u}\sigma'\left(\frac{K}{1-\gamma_u}\eta_1\right)\dot{\eta}_1\right]\\ &\leq |\eta_2|\left[-\frac{(1-\mu\kappa\gamma_v)\rho}{1-\gamma_u}\left|\sigma\left(\frac{1}{\lambda_1}\eta_2\right)\right|+\left|\frac{K}{1-\gamma_u}\sigma'\left(\frac{K}{1-\gamma_u}\eta_1\right)\dot{\eta}_1\right|\right].\end{aligned} \tag{A.10}$$

If $|\eta_1| > \frac{1-\gamma_u}{K}$, than, following the property a) of the function $\sigma(.)$, it holds that $\sigma'\left(\frac{K}{1-\gamma_u}\eta_1\right) = 0$ and, when $|\eta_2| = \lambda_1$, we have $\dot{V}_2 < 0$. If $|\eta_1| \le \frac{1-\gamma_u}{K}$, then, by virtue of $\eta \in \Omega_2$, we also have that $\eta \in \Omega_1$ and

$$\left|\frac{K}{1-\gamma_u}\sigma'\left(\frac{K}{1-\gamma_u}\eta_1\right)\dot{\eta}_1\right| \le \frac{\kappa K}{1-\gamma_u}\left[(1-\gamma_u)(1+\lambda_1) + \delta\gamma_u + L\left(\tau + l^{-1}\right)\gamma_v\rho + \gamma_u\lambda_2\right],$$

therefore, when $|\eta_2| = \lambda_1$, the condition $\dot{V}_2 < 0$ is met if:

$$-\frac{(1-\mu\kappa\gamma_v)\rho}{1-\gamma_u} + \frac{\kappa K}{1-\gamma_u}\left[(1-\gamma_u)(1+\lambda_1) + \delta\gamma_u + L\left(\tau + l^{-1}\right)\gamma_v\rho + \gamma_u\lambda_2\right] < 0. \tag{A.11}$$

If $\eta \in \partial\Omega_i$, $i = 1, 2$, then $|\eta_1| = \frac{1-\gamma_u}{K}$, $|\eta_2| = \lambda_1$ for $i = 1$ or $|\eta_2| = \lambda_1$ for $i = 2$. As $\Omega_1 \subset \Omega_2$, then, owing to the proven above, $\dot{V}_2 < 0$, therefore, if $\eta(t_0) \in \Omega_1$, then $\eta(t) \in \Omega_1$ for all $t \ge t_0$, which, when (A.3) holds, means that $y(t) \in \mathcal{Y}$ for all $t \ge t_0$.

As a result, combining (A.3), (A.9) and (A.11), we have that, when Assumptions 1–3 are met and, under given parameters

$$\gamma_u \in (0, 1), \gamma_v \in (0, 1), bu^{\mathrm{U}} > 0, \delta \in (0, 1), \kappa \in (1, 2), L > \kappa, \rho > 0, \varepsilon > 0$$

and some scalar $1 < \mu^* < \frac{1}{\kappa\gamma_v}$, inequalities (4.9) are feasible with respect to

$$\Psi_1 = \frac{1}{K}, \Psi_2 = \lambda_1, \Psi_3 = \lambda_2, \Psi_4 = l^{-1}, \Psi_5 = \tau,$$

then the law (4.4), (4.5) with parameters (4.10), initial conditions (4.7) and (4.11) for all $t \ge t_0$ ensures $u(t) \in \mathcal{U}$ and $y(t) \in \mathcal{Y}$.

Proof of Lemma 1. Equation (4.9) is rewritten in the following form:

$$\begin{gathered}
(1-\mu^*)\Psi_4 + \Psi_5 < 0, \\
(1-\gamma_u)\Psi_2 + \gamma_u\Psi_3 + L\gamma_v\rho(\Psi_4 + \Psi_5) + (1+\delta)\gamma_u - 1 < 0, \\
-\frac{(1-\mu^*\kappa\gamma_v)\rho}{\kappa}\Psi_1 + (1-\gamma_u)\Psi_2 + \gamma_u\Psi_3 + L\gamma_v\rho(\Psi_4 + \Psi_5) + 1 - (1-\delta)\gamma_u < 0, \\
(1-\gamma_u)bu^{\mathrm{U}}\Psi_1 < \varepsilon.
\end{gathered} \tag{A.12}$$

Considering sufficiently small Ψ_2, Ψ_3, Ψ_4, Ψ_5, inequalities (A.12) are satisfied if the following inequalities hold:

$$\begin{gathered}
\Psi_4 > \frac{\alpha\delta\gamma_u}{\mu - 1}, \quad \gamma_u < \frac{1}{1 + (1+\alpha)\delta}, \\
\frac{1 - (1 - (1+\alpha)\delta)\gamma_u}{(1-\mu\kappa\gamma_v)\rho}\kappa < \Psi_1 < \frac{\varepsilon}{(1-\gamma_u)bu^{\mathrm{U}}}
\end{gathered} \tag{A.13}$$

for $1 > \alpha > 0$.

The third inequality from (A.13) can be met, if it holds for $\varepsilon > 0$ that

$$\varepsilon > \frac{1 - (1 - (1+\alpha)\delta)\gamma_u}{(1-\mu\kappa\gamma_v)\rho}(1-\gamma_u)bu^{\mathrm{U}}\kappa = \underline{\varepsilon}, \tag{A.14}$$

which completes the proof of lemma.

Proof of Proposition 1. To prove the proposition, the upper bound of the solution of equation (4.16) is written as

$$\left|\tilde{F}(t)\right| = \left|-e^{-kt}F(0) - \int_0^t e^{-k(t-\tau)}\frac{\dot{f}(t,\ x,\ u)}{bu^{\mathrm{U}}}d\tau\right| \le e^{-kt}F(0) + \frac{\gamma_v\rho}{k}. \tag{A.15}$$

Then, using the initialization algorithm (4.13), (4.15), the satisfaction of inequality (4.11b) will follow from the satisfaction of the inequality

$$e^{-kt_0}F(0) + \frac{\gamma_v\rho}{k} \le \tau\gamma_v\rho. \tag{A.16}$$

Conducting simple reasoning, we obtain that inequality (A.16) holds for all

$$k > k^* = \max\left\{-t_0^{-1}\ln\left(\frac{\vartheta\tau\gamma_v\rho}{F(0)}\right), \frac{1}{(1-\vartheta)\tau}\right\}, \vartheta \in (0,\ 1),$$

which completes the proof of proposition.

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