ISSN 0005-1179 (print), ISSN 1608-3032 (online), Automation and Remote Control, 2026, Vol. 87, No. 10, pp. 885–900.
© The Author(s), 2026 published by Trapeznikov Institute of Control Sciences, Russian Academy of Sciences, 2026.
Russian Text © The Author(s), 2026, published in Avtomatika i Telemekhanika, 2026, No. 10, pp. 31–51.

LINEAR SYSTEMS

Optimization of Transients in a Three-Tempo Singularly Perturbed System

A. I. Kalinin[*,a] **and L. I. Lavrinovich**[*,b]

[*] *Belarusian State University, Minsk, Belarus*
e-mail: [a] *kalininai@bsu.by,* [b] *lavrinovich@bsu.by*

Received September 15, 2025
Revised April 20, 2026
Accepted April 28, 2026

Abstract—This paper considers the problem of minimum-energy control of a linear singularly perturbed system with three groups of variables changing at significantly different rates. Asymptotic approximations to the solution of this problem are constructed in the form of open-loop control and feedback control. The main advantage is that the proposed computational procedures decompose the original problem into three unperturbed optimal control problems of lower dimension.

Keywords: small parameter, singularly perturbed system, quadratic functional, optimal control, feedback, asymptotic approximations

DOI: 10.7868/S1608303226100022

1. INTRODUCTION

In the mathematical theory of optimal processes, considerable attention is given to optimization problems for singularly perturbed systems containing small parameters at some derivatives. This is due to the efficiency of asymptotic solution methods for such problems, which decompose the original problem into lower-dimensional optimal control ones. Moreover, the asymptotic approach avoids integrating singularly perturbed systems, which are stiff. Integration of such systems entails serious challenges manifested in high computational cost and the inevitable accumulation of computational errors. The first works on optimal control problems with singular perturbations addressed dynamic systems containing only two groups of variables with significantly different rates of change, the so-called slow and fast variables; see the reviews [1–4]. Subsequently, following the needs of applications, researchers considered optimization of systems with several groups of fast variables containing parameters of different orders of smallness at the derivatives. Such works were surveyed in [5]. This paper is devoted to constructing asymptotic approximations, in the form of open-loop control and feedback control, to the solution of the minimum-energy control problem for a singularly perturbed system with two groups of fast variables. The approach to constructing such approximations was first presented in [6]. Note that a similar problem for a two-tempo singularly perturbed system was studied in [7].

2. PROBLEM STATEMENT

In the class of r-dimensional control actions $u(t) = (u_1(t), \dots, u_r(t))$, $t \in T = [t_*, t^*]$, with piecewise continuous components, we consider a time-invariant optimal control problem of the form

$$\begin{aligned} \dot{x} &= A_1 x + B_1 y + C_1 z + D_1 u, \quad x(t_*) = x_*, \\ \mu \dot{y} &= A_2 x + B_2 y + C_2 z + D_2 u, \quad y(t_*) = y_*, \\ \mu^2 \dot{z} &= A_3 x + B_3 y + C_3 z + D_3 u, \quad z(t_*) = z_*, \end{aligned} \tag{1}$$

$$x(t^*) = 0, \quad y(t^*) = 0, \quad z(t^*) = 0, \tag{2}$$

$$J(u) = \frac{1}{2}\int\limits_{t_*}^{t^*} u^{\mathrm{T}} P u dt \to \min, \tag{3}$$

where μ is a small positive parameter; t_* and t^* are given time instants ($t_* < t^*$); x is the n_1-dimensional vector of slow variables; finally, y and z are the vectors of fast variables of dimensions n_2 and n_3, respectively. The other elements of the problem have compatible dimensions. In the performance criterion (3), P is a positive definite symmetric matrix.

Assumption 1. *The matrices C_3 and $B = B_2 - C_2 C_3^{-1} B_3$ are stable, i.e., the real parts of all their eigenvalues are negative.*

Let us introduce concepts to clarify the meaning of asymptotic approximations to the solution of this problem.

Definition 1. A control $u^{(N)}(t, \mu)$, $t \in T$, with piecewise continuous components is called an asymptotically suboptimal (open-loop) control of order N ($N = 0, 1, 2, \dots$) if it transfers the dynamic system (1) to a state of order $O(\mu^{N+1})$ and differs from the optimal control, in terms of the performance criterion (3), by a quantity of the same order of smallness.

Definition 2. A vector function $u^{(N)}(x, y, z, t, \mu)$ is called an asymptotically suboptimal feedback control of order N if, for any initial state (x_*, y_*, z_*, t_*), $t_* < t^*$, we have $u^{(N)}(x_*, y_*, z_*, t_*, \mu) = u^{(N)}(t_*, \mu)$, where $u^{(N)}(t, \mu)$, $t \in T$, is an asymptotically suboptimal control of order N in problem (1)–(3).

This paper is devoted to designing asymptotically suboptimal controls in problem (1)–(3). In addition, an asymptotically suboptimal feedback control of zero order is designed.

To shorten the notation, we will adopt the following designations:

$$A(\mu) = \begin{pmatrix} A_1 & B_1 & C_1 \\ A_2/\mu & B_2/\mu & C_2/\mu \\ A_3/\mu^2 & B_3/\mu^2 & C_3/\mu^2 \end{pmatrix}, \quad D(\mu) = \begin{pmatrix} D_1 \\ D_2/\mu \\ D_3/\mu^2 \end{pmatrix}.$$

3. FIRST BASIC PROBLEM

The essence of the asymptotically suboptimal control design approach proposed below lies in the asymptotic expansion of the initial values (at the time instant t^*) of the adjoint variables in integer powers of the small parameter. The leading coefficients of these expansions are determined by the initial values of the adjoint variables in three unperturbed optimal control problems of lower dimension. The first of these is the degenerate problem

$$\begin{gathered} \dot{x} = A_0 x + D_0 u, \ x(t_*) = x_*, \quad x(t^*) = 0, \\ J_1(u) = \frac{1}{2}\int\limits_{t_*}^{t^*} u^{\mathrm{T}} P u dt \to \min, \end{gathered} \tag{4}$$

where

$$\begin{gathered} A_0 = A_1 - C_1 C_3^{-1} A_3 - (B_1 - C_1 C_3^{-1} B_3) B^{-1} (A_2 - C_2 C_3^{-1} A_3), \\ D_0 = D_1 - C_1 C_3^{-1} D_3 - (B_1 - C_1 C_3^{-1} B_3) B^{-1} D_4, \quad D_4 = D_2 - C_2 C_3^{-1} D_3. \end{gathered} \tag{5}$$

It will be called the *first basic problem*.

Assumption 2. *The dynamic system in problem* (4) *is completely controllable* [8], *i.e.,*

$$\operatorname{rank}\left(D_0, A_0 D_0, A_0^2 D_0, \dots, A_0^{n_1 - 1} D_0\right) = n_1.$$

Under this assumption, admissible controls exist in the first basic problem, and hence it has a unique solution representing a normal extremal [9]. In other words, Pontryagin's maximum principle [10] in this case can be stated as follows: let $\overline{u}(t)$ and $x^0(t)$, $t \in T$, be an optimal control and an optimal trajectory, respectively, in problem (4); then there exists a solution $\psi^0(t)$, $t \in T$, of the adjoint system $\dot{\psi} = -A_0^{\mathrm{T}}\psi$ such that

$$\psi^{0\mathrm{T}}(t)D_0\overline{u}(t) - \frac{1}{2}\overline{u}^{\mathrm{T}}(t)P\overline{u}(t) = \max_{u\in R^r}\left(\psi^{0\mathrm{T}}(t)D_0u - \frac{1}{2}u^{\mathrm{T}}Pu\right), \quad t \in T.$$

This condition directly implies

$$\overline{u}(t) = P^{-1}D_0^{\mathrm{T}}\psi^0(t), \quad t \in T. \tag{6}$$

With $\lambda_0 = \psi^0(t^*)$, we have

$$\psi^{0\mathrm{T}}(t) = \lambda_0^{\mathrm{T}}F_0(t), \quad \psi^{0\mathrm{T}}(t)D_0 = \lambda_0^{\mathrm{T}}\Phi_0(t), \quad t \in T, \tag{7}$$

where

$$\Phi_0(t) = F_0(t)D_0, \quad t \in T, \tag{8}$$

and $F_0(t)$, $t \in T$, is a matrix function of dimensions $(n_1 \times n_1)$ representing the solution of the initial-value problem

$$\dot{F}_0 = -F_0A_0, \quad F_0(t^*) = E_{n_1}. \tag{9}$$

Due to (6), (7), and the Cauchy formula,

$$x^0(t^*) = F_0(t_*)x_* + S_1\lambda_0, \tag{10}$$

where

$$S_1 = \int_{t_*}^{t^*}\Phi_0(t)P^{-1}\Phi_0^{\mathrm{T}}(t)dt. \tag{11}$$

Under Assumption 2, this matrix is nonsingular, which can easily be verified using the implicit controllability criterion [11]. Since λ_0 is uniquely defined, according to the stated maximum principle, the optimal control in the first basic problem is associated with a unique vector of adjoint variables.

4. SECOND BASIC PROBLEM

At the second stage of computations, we solve an infinite-horizon optimal control problem of the form

$$\begin{gathered}\frac{dy}{ds} = By + D_4u, \quad y(0) = B^{-1}D_4\overline{u}(t^*),\\ y(-\infty) = 0, \quad J_2(u) = \frac{1}{2}\int_{-\infty}^{0}u^{\mathrm{T}}Puds \to \min,\end{gathered} \tag{12}$$

further called the *second basic problem.*

Assumption 3. *The controllability condition*

$$\operatorname{rank}\left(D_4, BD_4, B^2D_4, \ldots, B^{n_2-1}D_4\right) = n_2$$

holds.

This assumption ensures the existence of admissible controls in the second basic problem, implying in turn that problem (12) has a unique solution [9] and is normal. In this case, Pontryagin's maximum principle [10] for the problem can be stated as follows: let $u^*(s)$ and $y^*(s)$, $s \leqslant 0$, be

an optimal control and an optimal trajectory, respectively, in problem (12); then there exists a solution $\Pi\psi(s)$, $s \leqslant 0$, of the adjoint system

$$\frac{d}{ds}\Pi\psi = -B^{\mathrm{T}}\Pi\psi$$

such that

$$\Pi\psi^{\mathrm{T}}(s)D_4u^*(s) - \frac{1}{2}u^{*\mathrm{T}}(s)Pu^*(s) = \max_{u\in R^r}\left(\Pi\psi^{\mathrm{T}}(s)D_4u - \frac{1}{2}u^{\mathrm{T}}Pu\right), \quad s \leqslant 0.$$

This condition directly yields

$$u^*(s) = P^{-1}D_4^{\mathrm{T}}\Pi\psi(s), \quad s \leqslant 0. \tag{13}$$

With $\sigma_0 = \Pi\psi(0)$, we have

$$\Pi\psi^{\mathrm{T}}(s) = \sigma_0^{\mathrm{T}}G(s), \quad \Pi\psi^{\mathrm{T}}(s)D_4 = \sigma_0^{\mathrm{T}}\Pi\Phi(s), \quad s \leqslant 0, \tag{14}$$

where

$$\Pi\Phi(s) = G(s)D_4, \quad s \leqslant 0, \tag{15}$$

and $G(s)$, $s \leqslant 0$, is a matrix function of dimensions $(n_2 \times n_2)$ satisfying the differential equation

$$\frac{dG}{ds} = -GB, \quad G(0) = E_{n_2}. \tag{16}$$

By the implicit controllability criterion [11], under Assumption 3, the matrix

$$S_2 = \int\limits_{-\infty}^{0} \Pi\Phi(s)P^{-1}\Pi\Phi^{\mathrm{T}}(s)ds \tag{17}$$

is nonsingular. Based on (13), (14), and the Cauchy formula, we obtain

$$y^*(0) = S_2\sigma_0, \tag{18}$$

and, according to the above maximum principle, the optimal control in the second basic problem is associated with a unique vector of adjoint variables.

5. THIRD BASIC PROBLEM

The third basic problem has the form

$$\begin{gathered}\frac{dz}{d\tau} = C_3z + D_3u, \quad z(0) = C_3^{-1}D_3((\overline{u}(t^*) + u^*(0)),\\ z(-\infty) = 0, \quad J_3(u) = \frac{1}{2}\int\limits_{-\infty}^{0} u^{\mathrm{T}}Pud\tau \to \min.\end{gathered} \tag{19}$$

Assumption 4. *The controllability condition*

$$\operatorname{rank}\left(D_3, C_3D_3, C_3^2D_3, \ldots, C_3^{n_3-1}D_3\right) = n_3$$

holds.

This assumption ensures the existence of admissible controls in the third basic problem. Therefore, problem (19) has a unique solution [9] and is normal. In this case, Pontryagin's maximum principle [10] for the problem can be stated as follows: let $u_*(\tau)$ and $z_*(\tau)$, $\tau \leqslant 0$, be an optimal control and an optimal trajectory, respectively, in problem (19); then there exists a solution $Q\psi(\tau)$, $\tau \leqslant 0$, of the adjoint system

$$\frac{d}{d\tau}Q\psi = -C_3^{\mathrm{T}}Q\psi$$

such that

$$Q\psi^{\mathrm{T}}(\tau)D_3u_*(\tau)-\frac{1}{2}u_*^{\mathrm{T}}(\tau)Pu_*(\tau)=\max_{u\in R^r}\left(Q\psi^{\mathrm{T}}(\tau)D_3u-\frac{1}{2}u^{\mathrm{T}}Pu\right),\quad \tau\leqslant 0.$$

This condition directly implies

$$u_*(\tau)=P^{-1}D_3^{\mathrm{T}}Q\psi(\tau),\quad \tau\leqslant 0. \tag{20}$$

With $\chi_0=Q\psi(0)$, we have

$$Q\psi^{\mathrm{T}}(\tau)=\chi_0^{\mathrm{T}}G_1(\tau),\quad Q\psi^{\mathrm{T}}(\tau)D_3=\chi_0^{\mathrm{T}}Q\Phi(\tau),\quad \tau\leqslant 0, \tag{21}$$

where

$$Q\Phi(\tau)=G_1(\tau)D_3,\quad \tau\leqslant 0, \tag{22}$$

and $G_1(\tau)$, $\tau\leqslant 0$, is a matrix function of dimensions $(n_3\times n_3)$ satisfying the differential equation

$$\frac{dG_1}{d\tau}=-G_1C_3,\quad G_1(0)=E_{n_3}. \tag{23}$$

By the implicit controllability criterion [11], under Assumption 4, the matrix

$$S_3=\int\limits_{-\infty}^{0}Q\Phi(\tau)P^{-1}Q\Phi^{\mathrm{T}}(\tau)d\tau \tag{24}$$

is nonsingular. Based on (20), (21), and the Cauchy formula, we obtain

$$z_*(0)=S_3\chi_0, \tag{25}$$

and, according to the above maximum principle, the optimal control in the third basic problem is associated with a unique vector of adjoint variables.

Let us emphasize: the initial values σ_0 and χ_0 of the adjoint variable vectors are the only information about the solutions of the second and third basic problems used below to design asymptotically suboptimal controls. There is no need to construct the optimal controls $u^*(s)$, $s\leqslant 0$, and $u_*(\tau)$, $\tau\leqslant 0$, which (incidentally) is impossible during the numerical solution of the problem. In the second and third basic problems, the Hamiltonian along the optimal control is zero. By utilizing this property at $s=0$ and $\tau=0$, we establish the conditions for the vectors σ_0 and χ_0 :

$$\frac{1}{2}\sigma_0^{\mathrm{T}}D_4P^{-1}D_4^{\mathrm{T}}\sigma_0+\sigma_0D_4\overline{u}(t^*)=0,$$
$$\frac{1}{2}\chi_0^{\mathrm{T}}D_3P^{-1}D_3^{\mathrm{T}}\chi_0+\chi_0D_3(\overline{u}(t^*)+P^{-1}D_4^{\mathrm{T}}\sigma_0)=0.$$

After solving the basic problems, we find the vectors

$$\begin{aligned}\nu_0&=\sigma_0-\left((B_1-C_1C_3^{-1}B_3)B^{-1}\right)^{\mathrm{T}}\lambda_0,\\ \eta_0&=\chi_0-\left(C_1C_3^{-1}\right)^{\mathrm{T}}\lambda_0+\left(C_2C_3^{-1}\right)^{\mathrm{T}}\nu_0\end{aligned} \tag{26}$$

and compile the matrix

$$I_0=\begin{pmatrix}S_1 & 0_{n_1\times n_2} & 0_{n_1\times n_3}\\ I_{21} & S_2 & 0_{n_2\times n_3}\\ I_{31} & I_{32} & S_3\end{pmatrix} \tag{27}$$

of dimensions $(n_1+n_2+n_3)\times(n_1+n_2+n_3)$. The blocks S_1, S_2, and S_3 are given by formulas (11), (17), and (24), and

$$I_{21}=-B^{-1}\left(A_2-C_2C_3^{-1}A_3\right)S_1+\left(D_2-C_2C_3^{-1}D_3\right)+S_2\left(B^{-1}D_4\right)^{\mathrm{T}},$$
$$I_{31}=-C_3^{-1}\left(A_3S_1+B_2I_{21}+D_3P^{-1}D_0\right)+S_3\left(C_3^{-1}D_3\right)^{\mathrm{T}},$$
$$I_{32}=-C_3^{-1}\left(B_2S_2+D_3P^{-1}D_0\right)+S_3\left(C_3^{-1}D_3\right)^{T}.$$

As noted above, S_1, S_2, and S_3 are nonsingular matrices, and hence $\det I_0\neq 0$.

6. ASYMPTOTIC ANALYSIS OF THE SOLUTION OF THE ORIGINAL PROBLEM

One can speak of asymptotically suboptimal controls only if problem (1)–(3) is solvable. We will show that under Assumptions 1–4, problem (1)–(3) with a sufficiently small number μ has a unique optimal control representing a normal extremal. In other words, Pontryagin's maximum principle [10] can be stated as follows: let $u^0(t,\mu)$, $t\in T$, be an optimal control in problem (1)–(3); then there exists a solution $\psi_1^0(t,\mu)$, $\psi_2^0(t,\mu)$, $\psi_3^0(t,\mu)$, $t\in T$, of the adjoint system

$$\begin{aligned}\dot\psi_1&=-A_1^{\mathrm{T}}\psi_1-A_2^{\mathrm{T}}\psi_2-A_3^{\mathrm{T}}\psi_3,\\ \mu\dot\psi_2&=-B_1^{\mathrm{T}}\psi_1-B_2^{\mathrm{T}}\psi_2-B_3^{\mathrm{T}}\psi_3,\\ \mu^2\dot\psi_3&=-C_1^{\mathrm{T}}\psi_1-C_2^{\mathrm{T}}\psi_2-C_3^{\mathrm{T}}\psi_3\end{aligned}\tag{28}$$

such that

$$u^0(t,\mu)=P^{-1}\left(D_1^{\mathrm{T}}\psi_1(t,\mu)+D_2^{\mathrm{T}}\psi_2(t,\mu)+D_3^{\mathrm{T}}\psi_3(t,\mu)\right),\quad t\in T.\tag{29}$$

Theorem 1. *Under Assumptions 1–4, problem* (1)–(3) *has a unique solution* $u^0(t,\mu)$, $t\in T$, *representing a normal extremal of the form* (29), *where* $(\psi_1^0(t,\mu),\psi_2^0(t,\mu),\psi_3^0(t,\mu))$, $t\in T$, *is the vector of adjoint variables corresponding, by the maximum principle, to the optimal control. The values* $\lambda(\mu)=\psi_1^0(t^*,\mu)$, $\nu(\mu)=\psi_2^0(t^*,\mu)$, *and* $\eta(\mu)=\psi_3^0(t^*,\mu)$ *admit the asymptotic expansions*

$$\lambda(\mu)\sim\lambda_0+\sum_{k=1}^{\infty}\mu^k\lambda_k,\ \ \nu(\mu)\sim\nu_0+\sum_{k=1}^{\infty}\mu^k\nu_k,\ \ \eta(\mu)\sim\eta_0+\sum_{k=1}^{\infty}\mu^k\eta_k,\tag{30}$$

in which ν_0 *and* η_0 *are given by formulas* (26) *while* λ_0, σ_0, *and* χ_0 *are the initial values of the adjoint variables in the basic problems.*

Proof. We introduce the vectors $\theta=(\lambda,\nu,\eta)$ and $\theta_0=(\lambda_0,\nu_0,\eta_0)$ and denote by $\psi_1(t,\theta,\mu)$, $\psi_2(t,\theta,\mu)$, $\psi_3(t,\theta,\mu)$, $t\in T$, $\theta\in R^{n_1+n_2+n_3}$, the solution of the adjoint system (28) with the initial conditions

$$\psi_1(t^*)=\lambda,\quad \psi_2(t^*)=\nu,\quad \psi_3(t^*)=\eta.\tag{31}$$

Let $x(t,\theta,\mu)$, $y(t,\theta,\mu)$, $z(t,\theta,\mu)$, $t\in T$, $\theta\in R^{n_1+n_2+n_3}$, be the trajectory of system (1) generated by the control

$$u(t,\theta,\mu)=P^{-1}\left(D_1^{\mathrm{T}}\psi_1(t,\theta,\mu)+D_2^{\mathrm{T}}\psi_2(t,\theta,\mu)+D_3^{\mathrm{T}}\psi_3(t,\theta,\mu)\right),\quad t\in T.\tag{32}$$

We will show that the left-hand sides of the equations

$$x(t^*,\theta,\mu)=0,\quad y(t^*,\theta,\mu)=0,\quad z(t^*,\theta,\mu)=0\tag{33}$$

can be asymptotically expanded in integer powers of the small parameter, and obtain formulas for the coefficients of these expansions.

Consider the fundamental matrix $F(t,\mu)$, $t\in T$, of dimensions $(n_1+n_2+n_3)\times(n_1+n_2+n_3)$ as the solution of the initial-value problem

$$\dot F=-FA(\mu),\quad F(t^*)=E_{n_1+n_2+n_3}. \tag{34}$$

It is convenient to represent the solution of this singularly perturbed equation in the block form

$$F(t,\mu)=\begin{pmatrix} F_1(t,\mu) & F_2(t,\mu) & F_3(t,\mu)\\ F_4(t,\mu) & F_5(t,\mu) & F_6(t,\mu)\\ F_7(t,\mu) & F_8(t,\mu) & F_9(t,\mu)\end{pmatrix},$$

where F_i, $i=\overline{1,9}$, are matrices of dimensions $n_1\times n_1$, $n_1\times n_2$, $n_1\times n_3$, $n_2\times n_1$, $n_2\times n_2$, $n_2\times n_3$, $n_3\times n_1$, $n_3\times n_2$, and $n_3\times n_3$, respectively. Using the method of boundary functions [12], they can be expanded into asymptotic series:

$$\begin{gathered} F_i(t,\mu)\sim\sum_{k=0}^{\infty}\mu^k(F_{ik}(t)+\Pi_kF_i(s)+Q_kF_i(\tau)),\\ s=(t-t^*)/\mu,\quad \tau=(t-t^*)/\mu^2,\quad t\in T,\quad i=\overline{1,9}.\end{gathered} \tag{35}$$

We emphasize that these are asymptotic expansions uniform with respect to $t\in T$. Note also that the matrix functions $\Pi_kF_i(s)$, $s\leqslant 0$, and $Q_kF_i(\tau)$, $\tau\leqslant 0$, called the boundary terms, have the upper bounds

$$\|\Pi_kF_i(s)\|\leqslant\beta_{1k}\exp(\gamma_{1k}s),\quad \|Q_kF_i(s)\|\leqslant\beta_{2k}\exp(\gamma_{2k}s) \tag{36}$$

for all $i=\overline{1,9}$, $k=0,1,\dots$, where β_{1k}, β_{2k}, γ_{1k}, and γ_{2k} are positive constants. Let us provide several leading coefficients of the expansions (35), needed to prove the theorem:

$$\begin{gathered} F_{10}=F_0,\quad F_{40}=KF_0,\quad F_{70}=MF_0,\quad F_{i0}=0,\quad i=2,3,5,6,8,9,\\ F_{21}=F_0L,\quad F_{51}=KF_0L,\quad F_{81}=MF_0L,\quad F_{i1}=0,\quad i=3,6,9,\\ F_{32}=-F_0(C_1+LC_2)C_3^{-1},\quad F_{62}=-KF_0(C_1+LC_2)C_3^{-1},\quad F_{92}=-MF_0(C_1+LC_2)C_3^{-1},\\ \Pi_0F_4=-GK,\quad \Pi_0F_5=G,\quad \Pi_0F_7=C_3^{-1}B_3GK,\\ \Pi_0F_8=-C_3^{-1}B_3G,\quad \Pi_0F_i=0,\quad i=1,2,3,6,9,\\ \Pi_1F_2=-LG,\quad \Pi_1F_3=0,\quad \Pi_1F_6=-GC_2C_3^{-1},\\ \Pi_1F_9=C_3^{-1}B_3GC_2C_3^{-1},\quad \Pi_2F_3=LGC_2C_3^{-1},\\ Q_0F_i=0,\quad i=\overline{1,6},\quad Q_0F_7=G_1C_3^{-1}A_3,\\ Q_0F_8=G_1C_3^{-1}B_3,\quad Q_9F_0=G_1,\\ Q_1F_2=0,Q_1F_3=0,\quad Q_1F_5=C_2C_3^{-1}G_1C_3^{-1}B_3,\\ Q_1F_6=C_2C_3^{-1}G_1,\quad Q_2F_3=C_1C_3^{-1}G_1.\end{gathered} \tag{37}$$

Here, $K=-B^{-1}(A_2-C_2C_3^{-1}A_3)$, $M=-C_3^{-1}(A_3-B_3K)$, and $L=-(B_1-C_1C_3^{-1}B_3)B^{-1}$, while F_0, G, and G_1 are the solutions of equations (9), (16), and (23), respectively.

Note that

$$\begin{aligned}\psi_1^{\mathrm T}(t,\theta,\mu)&=\lambda^{\mathrm T}F_1(t,\mu)+\mu\nu^{\mathrm T}F_4(t,\mu)+\mu^2\eta^{\mathrm T}F_7(t,\mu),\\ \mu\psi_2^{\mathrm T}(t,\theta,\mu)&=\lambda^{\mathrm T}F_2(t,\mu)+\mu\nu^{\mathrm T}F_5(t,\mu)+\mu^2\eta^{\mathrm T}F_8(t,\mu),\\ \mu^2\psi_3^{\mathrm T}(t,\theta,\mu)&=\lambda^{\mathrm T}F_3(t,\mu)+\mu\nu^{\mathrm T}F_6(t,\mu)+\mu^2\eta^{\mathrm T}F_9(t,\mu).\end{aligned}$$

Therefore, due to (35) and (37), we arrive at the asymptotic expansions

$$\begin{gathered}\psi_i(t,\theta,\mu)\sim\sum_{k=0}^{\infty}\mu^k\left(\psi_{ik}(t,\theta)+\Pi_k\psi_i(s,\theta)+Q_k\psi_i(\tau,\theta)\right),\\ s=(t-t^*)/\mu,\quad \tau=(t-t^*)/\mu^2,\quad t\in T,\ i=1,2,3,\end{gathered}\tag{38}$$

where

$$\begin{gathered}\psi_{10}^{\mathrm{T}}(t,\theta)=\lambda^{\mathrm{T}}F_0(t),\quad \psi_{20}^{\mathrm{T}}(t,\theta)=\lambda^{\mathrm{T}}F_0(t)L,\\ \psi_{30}^{\mathrm{T}}(t,\theta)=-\lambda^{\mathrm{T}}F_0(t)(C_1-LC_2)C_3^{-1},\\ \Pi_0^{\mathrm{T}}\psi_1(s,\theta)=0,\quad \Pi_0^{\mathrm{T}}\psi_2(s,\theta)=-\left(\lambda^{\mathrm{T}}L-\nu^{\mathrm{T}}\right)G(s),\\ \Pi_0^{\mathrm{T}}\psi_3(s,\theta)=\left(\lambda^{\mathrm{T}}L-\nu^{\mathrm{T}}\right)G(s)C_2C_3^{-1},\\ Q_0^{\mathrm{T}}\psi_1(\tau,\theta)=0,\quad Q_0^{\mathrm{T}}\psi_2(\tau,\theta)=0,\\ Q_0^{\mathrm{T}}\psi_3(\tau,\theta)=\left(\lambda^{\mathrm{T}}C_1C_3^{-1}+\nu^{\mathrm{T}}C_2C_3^{-1}+\eta^{\mathrm{T}}\right)G_1(\tau),\\ \psi_{1k}^{\mathrm{T}}(t,\theta)=\lambda^{\mathrm{T}}F_{1k}(t)+\nu^{\mathrm{T}}F_{4,k-1}(t)+\eta^{\mathrm{T}}F_{7,k-2}(t),\\ \psi_{2k}^{\mathrm{T}}(t,\theta)=\lambda^{\mathrm{T}}F_{2,k+1}(t)+\nu^{\mathrm{T}}F_{5,k}(t)+\eta^{T}F_{8,k-1}(t),\\ \psi_{3k}^{\mathrm{T}}(t,\theta)=\lambda^{\mathrm{T}}F_{3,k+2}(t)+\nu^{\mathrm{T}}F_{6,k+1}(t)+\eta^{T}F_{9,k}(t),\\ \Pi_k^{\mathrm{T}}\psi_1(s,\theta)=\lambda^{\mathrm{T}}\Pi_kF_1(s)+\nu\Pi_{k-1}F_4(s)+\eta^{\mathrm{T}}\Pi_{k-2}F_7(s),\\ \Pi_k^{\mathrm{T}}\psi_2(s,\theta)=\lambda^{\mathrm{T}}\Pi_{k+1}F_2(s)+\nu\Pi_kF_5(s)+\eta^{\mathrm{T}}\Pi_{k-1}F_8(s),\\ \Pi_k^{\mathrm{T}}\psi_3(s,\theta)=\lambda^{\mathrm{T}}\Pi_{k+2}F_3(s)+\nu\Pi_{k+1}F_6(s)+\eta^{\mathrm{T}}\Pi_kF_9(s),\\ Q_k^{\mathrm{T}}\psi_1(\tau,\theta)=\lambda^{\mathrm{T}}Q_kF_1(\tau)+\nu^{\mathrm{T}}Q_{k-1}F_4(\tau)+\eta^{\mathrm{T}}Q_{k-2}F_7(\tau),\\ Q_k^{\mathrm{T}}\psi_2(\tau,\theta)=\lambda^{\mathrm{T}}Q_{k+1}F_2(\tau)+\nu^{\mathrm{T}}Q_kF_5(\tau)+\eta^{\mathrm{T}}Q_{k-1}F_8(\tau),\\ Q_k^{\mathrm{T}}\psi_3(\tau,\theta)=\lambda^{\mathrm{T}}Q_{k+2}F_3(\tau)+\nu^{\mathrm{T}}Q_{k+1}F_6(\tau)+\eta^{\mathrm{T}}Q_kF_9(\tau),\\ F_{7,-1}(t)=0,\quad \Pi_{-1}F_7(s)=0,Q_{-1}F_7(\tau)=0,\\ k\geqslant 1,\quad t\in T,\quad s\leqslant 0,\quad \tau\leqslant 0.\end{gathered}\tag{39}$$

These expansions are uniform in the domain $\|\theta-\theta_0\|\leqslant\epsilon_0$, $t\in T$, with a positive number ϵ_0. Note that from (7), (14), (21), (26), and (39) it follows that

$$\begin{gathered}\psi_{10}(t,\theta_0)=\psi^0(t),\quad \psi_{20}(t,\theta_0)=L^{\mathrm{T}}\psi^0(t),\\ \psi_{30}(t,\theta_0)=-\left((C_1-LC_2)C_3^{-1}\right)^{\mathrm{T}}\psi^0(t),\quad t\in T,\\ \Pi_0\psi_2(s,\theta_0)=\Pi\psi(s),\quad \Pi_0\psi_3(s,\theta_0)=-\left(C_2C_3^{-1}\right)^{\mathrm{T}}\Pi\psi(s),\quad s\leqslant 0,\\ Q_0\psi_3(\tau,\theta_0)=Q\psi(\tau),\quad \tau\leqslant 0.\end{gathered}\tag{40}$$

In view of (38), the control (29) also admits the asymptotic expansion

$$\begin{gathered}u(t,\theta,\mu)\sim\sum_{k=0}^{\infty}\mu^k\left(u_k(t,\theta)+\Pi_ku(s,\theta)+Q_ku(\tau,\theta)\right),\\ s=(t-t^*)/\mu,\quad \tau=(t-t^*)/\mu^2,\quad t\in T,\end{gathered}\tag{41}$$

where

$$\begin{gathered}u_k(t,\theta)=P^{-1}(D_1^{\mathrm{T}}\psi_{1k}(t,\theta)+D_2^{\mathrm{T}}\psi_{2k}(t,\theta)+D_3^{\mathrm{T}}\psi_{3k}(t,\theta)),\\ \Pi_k u(s,\theta)=P^{-1}(D_1^{\mathrm{T}}\Pi_k\psi_1(s,\theta)+D_2^{\mathrm{T}}\Pi_k\psi_2(s,\theta)+D_3^{\mathrm{T}}\Pi_k\psi_3(s,\theta)),\\ Q_k u(\tau,\theta)=P^{-1}(D_1^{\mathrm{T}}Q_k\psi_1(\tau,\theta)+D_2^{\mathrm{T}}Q_k\psi_2(\tau,\theta)+D_3^{\mathrm{T}}Q_k\psi_3(\tau,\theta)),\\ t\in T,\quad s\leqslant 0,\quad \tau\leqslant 0,\quad k=0,1,2,\dots.\end{gathered}\tag{42}$$

Along with formulas (5), (6), (13), (20), and (40), the conditions $\Pi_0\psi_1(s,\theta)=0$, $Q_0\psi_1(\tau,\theta)=0$, and $Q_0\psi_2(\tau,\theta)=0$ hold. Consequently,

$$\begin{gathered}u_0(t,\theta_0)=\overline{u}(t),\quad t\in T,\quad \Pi_0 u(s,\theta_0)=u^*(s),\quad s\leqslant 0,\\ Q_0u(\tau,\theta_0)=u_*(\tau),\quad \tau\leqslant 0.\end{gathered}\tag{43}$$

Based on the Cauchy formula, we have

$$\begin{pmatrix}x(t^*,\theta,\mu)\\ y(t^*,\theta,\mu)\\ z(t^*,\theta,\mu)\end{pmatrix}=F(t_*,\mu)\begin{pmatrix}x_*\\ y_*\\ z_*\end{pmatrix}+\int\limits_{t_*}^{t^*}F(t,\mu)D(\mu)u(t,\theta,\mu)dt.$$

According to (5), (35)–(39), (41), and (42), this vector function admits the asymptotic expansion

$$\begin{gathered}x(t^*,\theta,\mu)\sim\sum_{k=0}^{\infty}\mu^k x_k(\theta),\quad y(t^*,\theta,\mu)\sim\sum_{k=0}^{\infty}\mu^k y_k(\theta),\\ z(t^*,\theta,\mu)\sim\sum_{k=0}^{\infty}\mu^k z_k(\theta),\end{gathered}\tag{44}$$

where

$$\begin{gathered}x_0(\theta)=F_0(t_*)x_*+\int\limits_{t_*}^{t^*}F_0(t)D_0u_0(t,\theta)dt,\\ y_0(\theta)=Kx_0(\theta)+\int\limits_{-\infty}^{0}(G(s)D_4(u_0(t^*,\theta)+\Pi_0u(s,\theta))ds,\\ z_0(\theta)=Mx_0(\theta)+\int\limits_{-\infty}^{0}-C_3^{-1}B_3G(s)D_4(u_0(t^*,\theta)+\Pi_0u(s,\theta))ds\\ +\int\limits_{-\infty}^{0}G_1(\tau)D_3(u_0(t^*,\theta)+\Pi_0u(0,\theta)+Q_0u(\tau,\theta))d\tau,\\ x_k(\theta)=F_{1k}(t^*)x_*+F_{2k}(t^*)y_*+F_{3k}(t^*)z_*+\int\limits_{t_*}^{t^*}\sum_{p=0}^{k}(F_{1,k-p}(t)D_1u_p(t,\theta))dt\\ +\int\limits_{t_*}^{t^*}\sum_{p=0}^{k}(F_{2,k-p}(t)D_2u_p(t,\theta))dt+\int\limits_{t_*}^{t^*}\sum_{p=2}^{k+2}(F_{3,k+2-p}(t)D_3u_p(t,\theta))dt\\ +\int\limits_{-\infty}^{0}\sum_{p=0}^{k-1}((F_{1,k-1-p}(t^*)+\Pi_{k-1-p}F_1(s))D_1(u_p(t^*,\theta)+\Pi_pu(s,\theta)))ds\\ +\int\limits_{-\infty}^{0}\sum_{p=0}^{k}((F_{2,k-p}(t^*)+\Pi_{k-p}F_2(s))D_2(u_p(t^*,\theta)+\Pi_pu(s,\theta)))ds\end{gathered}\tag{45}$$

$$+\int_{-\infty}^{0}\sum_{p=0}^{k+1}((F_{3,k-p+1}(t^*)+\Pi_{k+1-p}F_3s))D_3(u_p(t^*,\theta)+\Pi_pu(s,\theta)))ds$$
$$+\int_{-\infty}^{0}\sum_{p=0}^{k-2}((F_{1,k-2-p}(t^*)+\Pi_{k-2-p}F_1(0)+Q_{k-2-p}F_1(\tau))D_1(u_p(t^*,\theta)$$
$$+\Pi_pu(0,\theta))+Q_pu(\tau,\theta))d\tau$$
$$+\int_{-\infty}^{0}\sum_{p=0}^{k-1}((F_{2,k-1-p}(t^*)+\Pi_{k-1-p}F_2(0)+Q_{k-1-p}F_2(\tau))D_2u_p(t^*,\theta)$$
$$+\Pi_pu(0,\theta))+Q_pu(\tau,\theta))d\tau$$
$$\int_{-\infty}^{0}\sum_{p=0}^{k}((F_{3,k-p}(t^*)+\Pi_{k-p}F_3(0)+Q_{k-p}F_3(\tau))D_3(u_p(t^*,\theta)$$
$$+\Pi_pu(0,\theta))+Q_pu(\tau,\theta))d\tau,\quad k\geqslant 1.$$

For $y_k(\theta)$, $z_k(\theta)$, $k\geqslant 1$, analogous formulas hold with the only difference being that the matrices F_{1i}, F_{2i}, F_{3i}, Π_iF_1, Π_iF_2, Π_iF_3, Q_iF_1, Q_iF_2, and Q_iF_3 are replaced by F_{4i}, F_{5i}, F_{6i}, Π_iF_4, Π_iF_5, Π_iF_6, Q_iF_4, Q_iF_5, and Q_iF_6 (in the first case) and by F_{7i}, F_{8i}, F_{9i}, Π_iF_7, Π_iF_8, Π_iF_9, Q_iF_7, Q_iF_8, and Q_iF_9 (in the second case).

Using the implicit function theorem, we verify that system (33) is uniquely solvable for θ under sufficiently small μ. It can be written in the form

$$R(\theta,\mu)=0. \tag{46}$$

According to (44), the uniform asymptotic expansion

$$R(\theta,\mu)\sim\sum_{k=0}^{\infty}\mu^kR_k(\theta) \tag{47}$$

holds in the domain $\|\theta-\theta_0\|\leqslant\epsilon_0$, where $R_k(\theta)=(x_k(\theta),y_k(\theta),z_k(\theta))$.

With $R(\theta,0)=R_0(\theta)$, the vector function $R(\theta,\mu)$ becomes continuous together with its partial derivatives with respect to the components of the vector θ in the domain $\|\theta-\theta_0\|<\epsilon_0$, $0\leqslant\mu<\mu_0$, where μ_0 is a sufficiently small positive number.

Due to (43), (45), and the Cauchy formula, we have $x_0(\theta_0)=x^0(t^*)=0$. The matrix function $G(s)$, $s\leqslant 0$, is a solution of the initial-value problem (16), and since $G(s)\to 0$ as $s\to-\infty$,

$$\int_{-\infty}^{0}G(s)D_4u_0(t^*,\theta_0)ds=\int_{-\infty}^{0}\left(-\frac{dG(s)}{ds}B^{-1}D_4(u_0(t^*,\theta_0)\right)ds=-B^{-1}D_4(u_0(t^*,\theta_0).$$

At the same time, conditions (43) and the Cauchy formula give

$$\int_{-\infty}^{0}G(s)D_4\Pi_0u(s,\theta_0)ds=-y^*(0)=B^{-1}D_4u_0(t^*,\theta_0).$$

Hence, from the second formula of (45), we obtain $y_0(\theta_0)=0$.

Similarly, since the matrix function $G_1(\tau)$, $\tau \leqslant 0$, is a solution of the initial-value problem (23) and $G_1(\tau) \to 0$ as $\tau \to -\infty$,

$$\int_{-\infty}^{0} G_1(\tau) D_3 \left(u_0(t^*, \theta_0) + \Pi_0 u(0, \theta_0)\right) d\tau$$
$$= \int_{-\infty}^{0} \left(-\frac{dG_1(\tau)}{d\tau} C_3^{-1} D_3 (u_0(t^*, \theta_0) + \Pi_0 u(0, \theta_0))\right) d\tau$$
$$= -C_3^{-1} D_3 (u_0(t^*, \theta_0) + \Pi_0 u(0, \theta_0)).$$

At the same time, from (43) and the Cauchy formula it follows that

$$\int_{-\infty}^{0} G_1(\tau) D_3 Q_0 u(\tau, \theta_0) d\tau = -z^*(0) = C_3^{-1} D_3 (u_0(t^*, \theta_0) + \Pi_0 u(0, \theta_0)).$$

Consequently, in view of the third formula of (45) and

$$\int_{-\infty}^{0} -C_3^{-1} B_3 G(s) D_4 (u_0(t^*, \theta) + \Pi_0(s, \theta)) ds = C_3^{-1} B_3 (K x_0(\theta_0) - y_0(\theta_0)) = 0,$$

we obtain $z_0(\theta_0) = 0$. Thus, $R(\theta_0, 0) = R_0(\theta_0) = 0$.

Taking into account (8), (15), (22), (37), (39), and (42), direct differentiation of the vector function (45) yields $\partial R(\theta_0, 0)/\partial\theta = \partial R_0(\theta_0)/\partial\theta = I_0$ (see (27)). Since this Jacobian matrix is nonsingular, the conditions of the implicit function theorem hold for system (46) or, equivalently, for system (33). According to this theorem, a vector function $\theta(\mu) = (\lambda(\mu), \nu(\mu), \eta(\mu))$ satisfying equations (33) is uniquely defined in some right-hand neighborhood $0 \leqslant \mu < \mu_1$ of zero; it is continuous, and $\theta(0) = \theta_0 = (\lambda_0, \nu_0, \eta_0)$. Therefore, the control $u^0(t, \mu) = u(t, \theta, \mu)$, $t \in T$, is admissible in problem (1)–(3) for sufficiently small μ, and it satisfies the maximum principle with the vector of adjoint variables $\psi_1^0(t, \mu) = \psi_1(t, \theta(\mu), \mu)$, $\psi_2^0(t, \mu) = \psi_2(t, \theta(\mu), \mu)$, $\psi_3^0(t, \mu) = \psi_3(t, \theta(\mu), \mu)$, $t \in T$; moreover, it represents a normal extremal. Hence, $u^0(t, \mu)$, $t \in T$, is the unique optimal control in problem (1)–(3).

As mentioned, the expansion (47) is uniform in the domain $\|\theta - \theta_0\| < \epsilon_0$, and its coefficients are linear functions. Then the solution $\theta(\mu) = (\lambda(\mu), \nu(\mu), \eta(\mu))$ of system (33) admits the asymptotic expansions (30). The proof of Theorem 1 is complete.

7. CONSTRUCTION OF ASYMPTOTICALLY SUBOPTIMAL CONTROLS

Using the theorem and related formulas, we present the algorithm for constructing asymptotic approximations to the solution of problem (1)–(3).

The vector function

$$u^{(0)}(t, \mu) = P^{-1}(D_1^{\mathrm{T}} \psi_1(t, \theta_0, \mu) + D_2^{\mathrm{T}} \psi_2(t, \theta_0, \mu) + D_3^{\mathrm{T}} \psi_3(t, \theta_0, \mu)), \quad t \in T,$$

is an asymptotically suboptimal control of zero order in the original problem. Note that it can be formed immediately after solving the basic problems. An asymptotically suboptimal control of order $N \geqslant 1$ has the form

$$u^{(N)}(t, \mu) = P^{-1}(D_1^{\mathrm{T}} \psi_1(t, \theta^{(N)}(\mu), \mu) + D_2^{\mathrm{T}} \psi_2(t, \theta^{(N)}(\mu), \mu) + D_3^{\mathrm{T}} \psi_3(t, \theta^{(N)}(\mu, \mu)), \quad t \in T, \quad (48)$$

where

$$\theta^{(N)}(\mu) = \sum_{k=0}^{N} \mu^k \theta_k, \quad \theta_k = (\lambda_k, \nu_k, \eta_k). \tag{49}$$

To construct the control (48), it is necessary to find the coefficients of the asymptotic series (30). This can be done by the method of undetermined coefficients with the expansion (47). More precisely, we expand, using Taylor's formula, the vector function

$$\sum_{k=0}^{N} \mu^k R_k(\theta^{(N)}(\mu))$$

in the powers of μ up to order N inclusive and equate the coefficients to zero, starting with that of μ. The resulting nonsingular systems of linear algebraic equations,

$$I_0\theta_1 = -R_1(\theta_0), \quad I_0\theta_k = -R_k(\theta_0) - \sum_{j=1}^{k-1} \frac{\partial R_j(\theta_0)}{\partial \theta}\theta_{k-j}, \quad k \geqslant 2, \tag{50}$$

serve to successively find the vectors θ_k, $k = \overline{1, N}$. (Here, we take into account that the coefficients of the expansions (47) are linear vector functions.) Note that, due to the structure (27) of the Jacobian matrix I_0, systems (50) decouple. Solving these systems successively yields the vectors θ_k, $k = \overline{1, N}$, and the polynomials (49). As mentioned above, the control (48) is asymptotically suboptimal of order N in the original problem. To construct this control, it is necessary to solve the Cauchy problem for the adjoint system (28) with the initial conditions

$$\psi_1(t^*) = \sum_{k=0}^{N} \mu^k \lambda_k, \quad \psi_2(t^*) = \sum_{k=0}^{N} \mu^k \nu_k, \quad \psi_3(t^*) = \sum_{k=0}^{N} \mu^k \eta_k.$$

The adjoint system is singularly perturbed and, consequently, stiff. Integration of stiff systems can be avoided by replacing the vector functions $\psi_i(t, \theta, \mu)$, $i = \overline{1, 3}$, in (48) by their asymptotic approximations

$$\psi_i^{(N)}(t, \theta, \mu) = \sum_{k=0}^{N} \mu^k (\psi_{ik}(t, \theta) + \Pi_k \psi_i(s, \theta) + Q_k \psi_i(\tau, \theta)),$$
$$s = (t - t*)/\mu, \quad \tau = (t - t^*)/\mu^2, \quad t \in T, \quad i = \overline{1, 3}.$$

The vector function

$$\overline{u}^{(N)}(t, \mu) = P^{-1}(D_1^{\mathrm{T}} \psi_1^{(N)}(t, \theta^{(N)}(\mu), \mu) + D_2^{\mathrm{T}} \psi_2^{(N)}(t, \theta^{(N)}(\mu), \mu) + D_3^{\mathrm{T}} \psi_3^{(N)}(t, \theta^{(N)}(\mu,)\mu)), \quad t \in T,$$

along with (48), is an asymptotically suboptimal control of order N in problem (1)–(3). In particular, it follows from (5), (6), (13), (20), (39), and (40) that

$$\begin{aligned} \overline{u}^{(0)}(t, \mu) &= P^{-1}\left(D_0^{\mathrm{T}} \psi^{(0)}(t) + D_4^{\mathrm{T}} \Pi\psi\left((t - t^*)/\mu\right) + D_3^{\mathrm{T}} Q\psi\left((t - t)/\mu^2\right)\right) \\ &= \overline{u}(t) + u^*\left((t - t^*)/\mu\right) + u_*\left((t - t)/\mu^2\right), \quad t \in T. \end{aligned} \tag{51}$$

Formulas (41) and (43) also indicate that this is an asymptotically suboptimal control of zero order. Note that the control (51) is independent of the initial states y_*, z_* of the fast variable vectors and, for small μ, substantially differs from the solution $\overline{u}(t)$, $t \in T$, of the first basic problem only in the boundary layer (i.e., in some left-hand neighborhood of the point t^*).

The constructed asymptotics can be used to find the optimal control $u^0(t, \mu)$, $t \in T$, in problem (1)–(3) for a given value of μ. For this purpose, one should follow the finishing procedure [13], i.e., find the roots of system (46) by Newton's method with $\theta^{(N)}(\mu)$ as an initial approximation. To avoid integrating stiff systems, one can take the approximation I_0 instead of the exact matrix $\partial R(\theta, \mu)/\partial\theta$.

8. ASYMPTOTICALLY SUBOPTIMAL DESIGN

Asymptotically suboptimal open-loop controls, of course, depend on the initial state (x_*, y_*, z_*, t_*) of the dynamic system. Previously, this dependence has been neglected by supposing a given initial state. In the current section, devoted to the asymptotically suboptimal design of a feedback control of zero order, we will be concerned with this dependence. From now on, let the matrix C be a function of t_*, $t_* < t^*$. The time instant t^* is still assumed given.

From equality (10) we have

$$\psi^0(t^*) = \lambda_0 = -S_1^{-1}(t_*)F_0(t_*)x_*, \tag{52}$$

which implies, together with (7),

$$\psi^0(t_*) = -F_0^{\mathrm{T}}(t_*)S_1^{-1}(t_*)F_0(t_*)x_*. \tag{53}$$

Due to (6) and (18),

$$\Pi\psi(0) = \sigma_0 = S_2^{-1}y^*(0) = -S_2^{-1}B^{-1}D_4\overline{u}(t^*) = -S_2^{-1}B^{-1}D_4P^{-1}D_0^{\mathrm{T}}\psi^0(t^*). \tag{54}$$

Hence, taking (14) and (52) into account, we obtain

$$\Pi\psi(s) = G^{\mathrm{T}}(s)S_2^{-1}B^{-1}D_4P^{-1}D_0^{\mathrm{T}}F_0^{\mathrm{T}}(t_*)S_1^{-1}(t_*)F_0(t_*)x_*, \quad s \leqslant 0. \tag{55}$$

In view of (6), (13), and (25),

$$\begin{aligned} Q\psi(0) = \chi_0 = S_3^{-1}z^*(0) &= -S_3^{-1}C_3^{-1}D_3(\overline{u}(t^*) + u^*(0)) \\ &= -S_3^{-1}C_3^{-1}D_3P^{-1}(D_0^{\mathrm{T}}\psi(t^*) + D_4^{\mathrm{T}}\mathrm{T}\Pi\psi(0)). \end{aligned}$$

Considering (21), (52), and (54), the above expression yields

$$Q\psi(\tau) = G_1^{\mathrm{T}}(\tau)S_3^{-1}C_3^{-1}D_3P^{-1}(D_0^{\mathrm{T}} + D_4^{\mathrm{T}}S_2^{-1}B^{-1}D_4P^{-1}D_0^{\mathrm{T}})C^{-1}(t_*)F_0(t_*)x_*.$$

According to formulas (51), (53), and (55), the asymptotically suboptimal control of zero order at the initial time instant can be represented as

$$\begin{aligned} \overline{u}^{(0)}(t_*, \mu) = P^{-1}\left(D_0^{\mathrm{T}} - D_4^{\mathrm{T}}G^{\mathrm{T}}((t_* - t*)/\mu\right)S_2^{-1}B^{-1}D_4P^{-1}D_0^{\mathrm{T}}F_0^{\mathrm{T}}(t_*) \\ -D_3^{\mathrm{T}}G_1\left((t_* - t^*)/\mu^2\right)S_3^{-1}C_3^{-1}D_3P^{-1}\left(D_0^{\mathrm{T}} + D_4^{\mathrm{T}}S_2^{-1}B^{-1}D_4P^{-1}D_0^{\mathrm{T}})\right)S_1^{-1}(t_*)F_0(t_*)x_*. \end{aligned}$$

Since (x_*, y_*, z_*, t_*) is an arbitrary initial state of the dynamic system, by Definition 2, the vector function

$$\begin{aligned} \overline{u}^{(0)}(x, y, z, t, \mu) = P^{-1}\Big(D_0^{\mathrm{T}} - D_4^{\mathrm{T}}G^{\mathrm{T}}\left((t - t*)/\mu\right)S_2^{-1}B^{-1}D_4P^{-1}D_0^{\mathrm{T}}F_0^{\mathrm{T}}(t) \\ - D_3^{\mathrm{T}}G_1\left((t - t^*)/\mu^2\right)S_3^{-1}C_3^{-1}D_3P^{-1}\left(D_0^{\mathrm{T}} + D_4^{\mathrm{T}}C_0^{-1}B^{-1}D_4P^{-1}D_0^{\mathrm{T}}\right)\Big)S_1^{-1}(t)F_0(t)x \end{aligned} \tag{56}$$

represents an asymptotically suboptimal feedback control of zero order in the original problem. Note that this feedback is independent of the current position of the fast variables.

9. EXAMPLE

As an illustrative example, we consider a control model for the horizontal rectilinear motion of two bodies with significantly different small masses and an elastic coupling:

$$\begin{gathered}\dot{x}_1 = y, \quad \dot{x}_2 = z,\\ \mu\dot{y} = -kx_1 + kx_2 - c_1 y + d_1 u_1, \quad \mu^2\dot{z} = kx_1 - kx_2 - c_2 z + d_2 u_2,\\ x_1(t_*) = x_{*1}, \quad x_2(t_*) = x_{*2}, \quad y(t_*) = y_*, \quad z(t_*) = z_*,\\ x_1(t^*) = x_2(t^*) = y(t^*) = z(t^*) = 0,\\ J(u) = \frac{1}{2}\int\limits_{t_*}^{t^*}(u_1^2 + u_2^2)dt \to \min.\end{gathered} \tag{57}$$

All constants in this problem are positive, and $\mu \ll 1$. Assumption 1 holds in this case.

The first basic problem has the form

$$\begin{gathered}\dot{x}_1 = -\frac{k}{c_1}x_1 + \frac{k}{c_1}x_2 + \frac{d_1}{c_1}u_1, \quad \dot{x}_2 = \frac{k}{c_2}x_1 - \frac{k}{c_2}x_2 + \frac{d_2}{c_2}u_2,\\ x_1(t_*) = x_{*1}, \quad x_2(t_*) = x_{*2},\\ x_1(t^*) = x_2(t^*) = 0, \quad J_1(u) = \frac{1}{2}\int\limits_{t_*}^{t^*}(u_1^2 + u_2^2)dt \to \min.\end{gathered} \tag{58}$$

As is easily verified, the dynamic system in the first basic problem is completely controllable. According to the maximum principle, the optimal control in problem (58) has the form

$$\begin{gathered}\overline{u}_1(t) = \frac{d_1}{c_1}\left(\frac{c_1(\lambda_{01}+\lambda_{02}) - (c_1\lambda_{02} - c_2\lambda_{01})\exp((k(c_1+c_2)(t-t_*))/c_1c_2)}{c_1+c_2}\right),\\ \overline{u}_2(t) = \frac{d_2}{c_2}\left(\frac{c_2(\lambda_{01}+\lambda_{02}) + (c_1\lambda_{02} - c_2\lambda_{01})\exp((k(c_1+c_2)(t-t_*))/c_1c_2)}{c_1+c_2}\right),\\ t \in [t_*, t^*].\end{gathered} \tag{59}$$

The vector $\lambda_0 = (\lambda_{01}, \lambda_{02})$ is calculated by

$$\lambda_0 = S_1^{-1}F_0(t^*)x_*, \tag{60}$$

where

$$\begin{gathered}F_0(t) = \begin{pmatrix} \dfrac{c_2\exp((k(c_1+c_2)(t^*-t))/c_1c_2) - c_1}{c_2-c_1} & \dfrac{c_2\exp((k(c_1+c_2)(t^*-t))/c_1c_2) - c_2}{c_1-c_2}\\ \dfrac{c_1\exp((k(c_1+c_2)(t^*-t))/c_1c_2) - c_1}{c_2-c_1} & \dfrac{c_1\exp((k(c_1+c_2)(t^*-t))/c_1c_2) - c_2}{c_1-c_2}\end{pmatrix},\\ t \in [t_*, t^*],\\ S_1 = \int\limits_{t_*}^{t^*} F_0(t)\begin{pmatrix} d_1^2/c_1^2 & 0\\ 0 & d_2^2/c_2^2\end{pmatrix}F_0^T(t)dt, \quad x_* = \begin{pmatrix} x_{*1}\\ x_{*2}\end{pmatrix}.\end{gathered}$$

After solving problem (58), we form the second basic problem

$$\begin{gathered}\frac{dy}{ds} = -c_1 y + d_1 u_1, \quad y(0) = -\frac{d_1}{c_1}\overline{u}(t^*),\\ y(-\infty) = 0, \quad J_2(u) = \frac{1}{2}\int\limits_{-\infty}^{0}(u_1^2 + u_2^2)ds \to \min.\end{gathered} \tag{61}$$

Here, $\overline{u}(t)$, $t \in [t_*, t^*]$, is the solution of problem (58). The dynamic system in the second basic problem is completely controllable. By the maximum principle, the solution of problem (61) is as follows:

$$u_1^*(s) = -2\overline{u}_1(t^*)\exp(c_1 s), \quad u_2^*(s) = 0, \quad s \leqslant 0.$$

The third basic problem has the form

$$\frac{dz}{d\tau} = -c_2 z + d_2 u_2, \quad z(0) = -\frac{d_2}{c_2}\overline{u}_2(t^*), \quad z(-\infty) = 0, \quad J_2(u) = \frac{1}{2}\int\limits_{-\infty}^{0}(u_1^2 + u_2^2)d\tau \to \min.$$

The dynamic system in this problem is also completely controllable. Based on the maximum principle, the optimal control in the third basic problem can be represented as

$$u_{*1}(\tau) = 0, \quad u_{*2}(\tau) = -2\overline{u}_2(t^*)\exp(c_2\tau), \quad \tau \leqslant 0.$$

According to (51), the control

$$\begin{gathered}\overline{u}_1^0(t,\mu) = \overline{u}_1(t) - 2\overline{u}_1(t^*)\exp(c_1(t-t^*)/\mu),\\ \overline{u}_2^0(t,\mu) = \overline{u}_2(t) - 2\overline{u}_2(t^*)\exp(c_2(t-t^*)/\mu^2), \quad t \in [t_*, t^*],\end{gathered} \tag{62}$$

is asymptotically suboptimal of zero order in problem (57).

When constructing an asymptotically suboptimal feedback control of zero order, we consider the matrix C in (60) as a function of t_* : $C = C(t_*)$. In the example,

$$D_0 = \begin{pmatrix} d_1/c_1 & 0 \\ 0 & d_2/c_2 \end{pmatrix}.$$

Then, due to (56), the vector function

$$\overline{u}^{(0)}(x,t,\mu) = D_0(E_2(1 - 2\exp((t-t^*)/\mu^2)) + 2\exp((t-t^*)/\mu)F_0^{\mathrm{T}}(t))C^{-1}(t)F_0(t)x$$

represents an asymptotically suboptimal feedback control of zero order in problem (57).

To estimate the quality of the above asymptotic approximations to the solution of problem (57), we computed the resulting states $x_1(t^*,\mu)$, $x_2(t^*,\mu)$, $y(t^*,\mu)$, and $z(t^*,\mu)$ of the dynamic system with the control (62) for particular values of the small parameter in the case where $k = 1$, $c_1 = 3$, $c_2 = 1$, $d_1 = 1$, $d_2 = 2$, $t_* = 0$, $t^* = 5$, $x_{*1} = 5$, $x_{*2} = -3.0$, $y_* = 2.0$, and $z_* = 1$. As it turned out,

$$\begin{gathered}x_1(5,\ 0.1) = 0.068161 \qquad x_2(5,\ 0.1) = 0.033727\\ y(5,\ 0.1) = -0.023323 \qquad z(5,\ 0.1) = -0.011998\\ x_1(5,\ 0.025) = 0.016755 \quad x_2(5,\ 0.025) = 0.014029\\ y(5,\ 0.025) = -0.004879 \quad z(5,\ 0.025) = -0.000235.\end{gathered}$$

The numerical results are provided with an accuracy of 10^{-6}.

10. CONCLUSIONS

This paper has proposed computational procedures for constructing asymptotic approximations to the solution of the problem in the form of open-loop control and feedback control without the need to integrate singularly perturbed systems. With the proposed algorithms, the original optimal control problem decomposes into three unperturbed ones of lower dimension. Such a decomposition enables efficient solution of optimization problems for dynamic systems with a large number of state variables. Also, the constructed asymptotic approximations have been used to solve the problem for given values of the small parameter.

According to the proof of Theorem 1, under Assumptions 2–4, there exists an admissible control in the original problem for any initial state of the dynamic system. Therefore, these assumptions ensure the complete controllability of system (1) for sufficiently small values of μ. This result was first established in [14].

The method serves to study problems containing several groups of fast variables with a hierarchy of rates of change in integer powers of the small parameter. In this case, the number of problems in the decomposition of the original problem is equal to that of groups of multi-tempo variables. Such a generalization introduces nonfundamental modifications into the algorithm, which are easily traced using the example of the considered problem. At the same time, it leads to cumbersome formulas. Note also that the above results can be extended to time-varying systems with sufficiently smooth coefficients without fundamental difficulties.

REFERENCES

1. Dmitriev, M.G. and Kurina, G.A., Singular Perturbations in Control Problems, *Autom. Remote Control*, 2006, vol. 67, no. 1, pp. 1–43.
2. Kalinin, A.I., Asymptotics of Solutions of Perturbed Optimal Control Problems, *J. Comput. Syst. Sci. Int.*, 1995, vol. 33, no. 6, pp. 75–84.
3. Zhang, Y., Naidu, D.S., Cai, C., and Zou, Y., Singular Perturbation and Time Scales in Control Theories and Applications. An Overview 2002–2012, *Int. J. Information and Systems Sciences*, 2014, vol. 9, no. 1, pp. 1–36.
4. Kokotovic, P.V. and Khalil, H.K., *Singular Perturbations in Systems and Control*, New York: IEEE Press, 1986.
5. Kurina, G.A. and Kalashnikova, M.A., Singularly Perturbed Problems with Multi-Tempo Fast Variables, *Autom. Remote Control*, 2022, vol. 83, no. 11, pp. 1679–1723.
6. Kalinin, A.I. and Lavrinovich, L.I., Asymptotics of the Solution of the Transient Optimization Problem in a Three-Tempo Singularly Perturbed System, *Trudy XIV Belorusskoi matematicheskoi konferentsii, posvyashchennoi 65-letiyu Instituta matematiki NAN Belarusi* (Proceedings of XIV Belarusian Mathematical Conference Dedicated to the 65th Anniversary of the Institute of Mathematics of the NAS of Belarus), Minsk, October 28–November 1, 2024, Minsk: Belaruskaya Navuka, 2024, part 1, pp. 116–117.
7. Kalinin, A.I. and Lavrinovich, L.I., Application of the Small Parameter Method to the Singularly Perturbed Linear-Quadratic Optimal Control Problem, *Autom. Remote Control*, 2016, vol. 77, no. 5, pp. 751–763.
8. Krasovskii, N.N., *Teoriya upravleniya dvizheniem: lineinye sistemy* (Theory of Control of Motion), Moscow: Nauka, 1968.
9. Mordukhovich, B.Sh., Existence of Optimal Controls, *Journal of Soviet Mathematics*, 1977, vol. 7, no. 5, pp. 795–849.
10. Pontryagin, L.S., Boltyanskii, V.G., Gamkrelidze, R.V., and Mishchenko, E.F., *The Mathematical Theory of Optimal Processes*, New York: Interscience Publishers, 1962.
11. Gabasov, R. and Kirillova, F.M., *Optimization of Linear Systems*, New York: Plenum Press, 1979.
12. Vasil'eva, A.B., Asymptotic Methods in the Theory of Ordinary Differential Equations with Small Parameters Multiplying the Highest Derivatives, *USSR. Comput. Math. Math. Phys.*, 1963, vol. 3, no. 4, pp. 611–642.
13. Gabasov, R. and Kirillova, F.M., *Konstruktivnye metody optimizatsii* (Constructive Methods of Optimization), part 2, Minsk: Universitetskoe, 1984.
14. Kurina, G.A., Complete Controllability of Various-Speed Singularly Perturbed Systems, *Math. Notes*, 1992, vol. 52, no. 4, pp. 1029–1033.

This paper was recommended for publication by M.V. Khlebnikov, a member of the Editorial Board