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LINEAR SYSTEMS

Description of Limit Reachable and Null-Controllable Sets for Linear Discrete-Time Systems and Integral Second-Order Constraints

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Abstract—A discrete-time linear system with second-order integral control constraints is considered. The properties of the limit reachable and null-controllable sets of the system are investigated. Necessary and sufficient conditions for the boundedness of these sets are obtained, including for uncontrollable systems. It is proved that these sets are ellipsoids, when they are bounded, and their support function is constructed explicitly. Numerical simulations are performed.

Keywords: discrete-time systems, controllability, reachability, limit reachable set, limit controllable set, integral constraints on control

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1. INTRODUCTION

The study of reachable and null-controllable sets is an actual problem in mathematical control theory. From an applied perspective, these sets can be used to design controllers, test the solvability of boundary value problems, and analyze process quality. For example, this approach is used in the following modern works: [1–5]. When studying controllability and reachability, limit reachable and null-controllable sets are also important. This is because, for systems with constrained control, classical Kalman rank conditions are insufficient to judge the reachability of particular states in an arbitrary finite time.

It should be noted that, based on the mathematical apparatus and methods used to solve analysis and synthesis problems, dynamic systems in control theory are divided into two fundamental classes: discrete-time systems and continuous-time systems. However, due to the development of digital technologies, continuous systems are now often studied by discretization and reduction to discrete analogs [6]. This underscores the relevance of developing effective numerical methods for studying discrete-time systems. For the reasons described above, methods for constructing limit reachable and null-controllable sets are of interest. For discrete time, these methods reduce to calculating the limit of a sequence of compact sets.

For general systems, the only accessible approach to solving this problem is to examine individual states for reachability and controllability, which essentially amounts to a pointwise construction of the desired sets [7–9]. For the class of linear systems, various analytical methods are available, based on the application of convex and functional analysis. A well-established method involves considering the limit reachable and null-controllable sets as fixed points of contractive mappings in the Hausdorff space [10, 11]. This approach allows, on the one hand, to use the simple iteration

method to approach a fixed point, and on the other hand, to construct external estimates of the limit sets with an arbitrary order of accuracy.

The majority of the presented results concern systems with geometric constraints on control. In the case of integral constraints, the situation is more complex: the reachable and null-controllable sets are not polyhedra even under scalar control [12], and their description in recurrent form is impossible using standard mathematical constructions [13]. Currently, in [14], a generalization of the approach based on the principle of contraction mappings [10, 11] is proposed for systems with integral constraints on scalar control. Separately, in [14, Lemma 3], a description of the reachable and null-controllable sets in the form of ellipsoids is obtained for second-order constraints and a diagonalizable matrix of the system.

The goal of this paper is to generalize [14] to the case of linear systems with an arbitrary dynamic matrix and second-order integral constraints on vector control. Specifically, it is necessary to establish necessary and sufficient conditions for the boundedness of the limit reachable and null-controllable sets in terms of the system parameters and to obtain an explicit description of them by a support function or Minkowski functional.

The paper is structured as follows. Section 2 introduces the concepts of limit reachable and null-controllable sets and states the problem. Section 3 formulates necessary and sufficient conditions for the boundedness of these sets. Section 4 describes the structure of limit reachable and null-controllable sets for linear systems with second-order integral constraints. Section 5 demonstrates the theoretical results obtained by numerical simulation.

2. PROBLEM STATEMENT

A linear stationary system with discrete time and integral second-order control constraints (A, B, H, t) is considered:

$$\begin{aligned} &x(k+1) = Ax(k) + Bu(k),\ k \in \mathbb{N} \cup \{0\}, \\ &\sum_{k=0}^{\infty} u(k)^{\mathrm{T}} H u(k) \leqslant t^2, \end{aligned} \tag{1}$$

where $x(k) \in \mathbb{R}^n$ is the state vector, $u(k) \in \mathbb{R}^m$ is the control vector, $H \in \mathbb{R}^{m\times m}$ is a positive definite symmetric matrix, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ are the system matrices, $t > 0$ is the parameter defining the integral constraint.

For system (1), let $\mathcal{Y}_{t,2,\infty}$ denote the limit reachable set, which is the set of those states into which system (1) can be transferred from the origin in a finite number of steps:

$$\mathcal{Y}_{t,2,\infty} = \Bigg\{ x \in \mathbb{R}^n \colon \exists N \in \mathbb{N},\ u(0), \dots, u(N-1) \in \mathbb{R}^m, \sum_{k=0}^{N-1} u(k)^{\mathrm{T}} H u(k) \leqslant t^2,\ x(0) = 0,\ x(N) = x \Bigg\}. \tag{2}$$

Similarly, for system (1) we denote by $\mathcal{X}_{t,2,\infty}$ the limit null-controllable set, which is the set of those states from which system (1) can be transferred to the origin in a finite number of steps:

$$\mathcal{X}_{t,2,\infty} = \Bigg\{ x \in \mathbb{R}^n \colon \exists N \in \mathbb{N},\ u(0), \dots, u(N-1) \in \mathbb{R}^m, \sum_{k=0}^{N-1} u(k)^{\mathrm{T}} H u(k) \leqslant t^2,\ x(0) = x,\ x(N) = 0 \Bigg\}. \tag{3}$$

It is required to obtain an exact description of the sets $\mathcal{Y}_{t,2,\infty}$ and $\mathcal{X}_{t,2,\infty}$ in terms of their Minkowski functionals and support functions, and also to determine necessary and sufficient conditions for boundedness.

Remark 1. The notations for sets (2) and (3) are chosen in accordance with [12, 14]. In these papers, arbitrary ordering of constraints $r \in (1;+\infty)$ is allowed, and similar sets are denoted by $\mathcal{Y}_{t,r,\infty}$ and $\mathcal{X}_{t,r,\infty}$. This paper focuses on the study of an important special case of quadratic constraints defined by a positive-definite quadratic form, which determines the presence of the sign 2 in the indices $\mathcal{Y}_{t,2,\infty}$ and $\mathcal{X}_{t,2,\infty}$.

3. NECESSARY AND SUFFICIENT CONDITIONS FOR THE BOUNDEDNESS OF LIMIT SETS

As demonstrated in [12, Theorems 2, 3] and [15, Theorem 3], the conditions for the boundedness of the limit reachable and 0-controllable sets are determined by the relation between the matrices A and B. Denote by $h_1,\dots,h_n \in \mathbb{R}^n \setminus \{0\}$ the real Jordan basis of the matrix A [16, Section 3.4, Chapter 3], consisting of its eigenvectors and associated vectors or their imaginary and real parts. Consider the following invariant subspaces:

$$\mathbb{L}_{<1} = \mathrm{Lin}\{h\colon h \text{ corresponds to the eigenvalue } \lambda,\ |\lambda|<1\},$$
$$\mathbb{L}_{>1} = \mathrm{Lin}\,\{h\colon h \text{ corresponds to the eigenvalue } \lambda,\ |\lambda|>1\}.$$

Let us group the elements of the real Jordan basis as follows:

$$h_{1,1},\dots,h_{1,n_1},\dots,h_{M,1},\dots,h_{M,n_M}, \tag{4}$$

where $h_{i,j}\in\mathbb{R}^n$ is the adjoint vector of the matrix A of rank j corresponding to the eigenvalue $\lambda_i\in\mathbb{R}$, or $h_{i,j} = (h_{i,j}^{re}\ h_{i,j}^{im})\in\mathbb{R}^{n\times 2}$ are the real and imaginary parts of the adjoint vector of the matrix A of rank j corresponding to the eigenvalue $\lambda_i\in\mathbb{C}$. Then the real Jordan form of the matrix A consists of M cells, and n_i or $2n_i$ is the dimension of the ith cell, depending on the type of the eigenvalue. Moreover, by [16, Theorem 3.4.5], the following representation is true:

$$\begin{aligned} &A = S\Lambda S^{-1},\\ &S = \big(h_{1,1},\dots,h_{1,n_1},\dots,h_{M,1},\dots,h_{M,n_M}\big),\\ &\Lambda = \mathrm{diag}(J_1,\dots,J_M). \end{aligned} \tag{5}$$

For all $i=\overline{1,M}$, the Jordan cell J_i has one of the following two forms:

$$J_i = \begin{pmatrix} \lambda_i & 1 & \cdots & 0\\ 0 & \lambda_i & \ddots & 0\\ \vdots & \vdots & \ddots & 1\\ 0 & 0 & \cdots & \lambda_i \end{pmatrix} \in \mathbb{R}^{n_i\times n_i},\ \lambda_i\in\mathbb{R}, \tag{6}$$

$$J_i = \begin{pmatrix} r_iA_{\varphi_i} & I & \cdots & O\\ O & r_iA_{\varphi_i} & \ddots & O\\ \vdots & \vdots & \ddots & I\\ O & O & \cdots & r_iA_{\varphi_i} \end{pmatrix} \in \mathbb{R}^{2n_i\times 2n_i},\ \lambda_i = r_i\mathrm{e}^{\mathbf{i}\varphi_i}\in\mathbb{C}. \tag{7}$$

Here and below, A_φ and $I\in\mathbb{R}^{2\times 2}$ denote the rotation matrix and the identity matrix, respectively:

$$A_\varphi = \begin{pmatrix} \cos\varphi & \sin\varphi\\ -\sin\varphi & \cos\varphi \end{pmatrix},\ I = \begin{pmatrix} 1 & 0\\ 0 & 1 \end{pmatrix}.$$

The following criterion for the boundedness of the sets $\mathcal{Y}_{t,2,\infty}$ and $\mathcal{X}_{t,2,\infty}$ is valid.

Theorem 1. *Let the sets $\mathcal{Y}_{t,2,\infty}$ and $\mathcal{X}_{t,2,\infty}$ in system* (1) *be defined by relations* (2) *and* (3), *respectively, $B = (b_1, \dots, b_m)$. Then*

1) *$\mathcal{Y}_{t,2,\infty}$ is bounded if and only if $b_1, \dots, b_m \in \mathbb{L}_{<1}$;*

2) *$\mathcal{X}_{t,2,\infty}$ is bounded if and only if $\det A \neq 0$ and $b_1, \dots, b_m \in \mathbb{L}_{>1}$.*

The proof of Theorem 1 and all subsequent statements is given in the Appendix.

Example 1. Let us consider for $n = 3$ and $m = 2$ matrices

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0.9 & 1 \\ 0 & 0 & 0.9 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}^{-1}, \; B = \begin{pmatrix} 1 & -1 \\ 1 & 1 \\ 2 & 0 \end{pmatrix}, \; H = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Generally speaking, in the classical sense, for discrete-time system (1), the matrix A is not stable, since it has an eigenvalue $\lambda_1 = 2$, which is greater than 1 in absolute value. At the same time, the following representation is valid:

$$\mathbb{L}_{<1} = \operatorname{Lin}\left\{ \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}, \; b_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \in \mathbb{L}_{<1}, \; b_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \in \mathbb{L}_{<1}.$$

It follows that the condition of item 1 of Theorem 1 is satisfied and the limit reachable set $\mathcal{Y}_{t,2,\infty}$ of the system (A, B, H, t) is bounded.

On the other hand, if for $m = 1$ we choose

$$B = (2, 2, 0)^{\mathrm{T}}, \; H = 1,$$

then the following formulas will be correct:

$$\mathbb{L}_{>1} = \operatorname{Lin}\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}, \; b_1 = 2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \in \mathbb{L}_{>1}.$$

Then the system (A, B, H, t) will satisfy the conditions of item 2 of Theorem 1, and the limit null-controllable set $\mathcal{X}_{t,2,\infty}$ will be bounded.

4. DESCRIPTION OF LIMIT REACHABLE AND NULL-CONTROLLABLE SETS IN THE FORM OF ELLIPSOIDS

Theorem 1 defines necessary and sufficient conditions for the boundedness of sets (2) and (3), but does not specify their structure. In the case of boundedness, the sets $\mathcal{Y}_{t,2,\infty}$ and $\mathcal{X}_{t,2,\infty}$ are ellipsoids. A similar result was obtained in [14, Lemma 3], but only in the case of scalar control and a diagonalizable matrix A. For system (1), the problem of constructing matrices of quadratic forms of the ellipsoids $\mathcal{Y}_{t,2,\infty}$ and $\mathcal{X}_{t,2,\infty}$ reduces to the summation of matrix series. Let us present the theoretical information, which is necessary for this.

Lemma 1. *Let in system* (1) $B = (b_1, \dots, b_m)$.

1) *If $b_1, \dots, b_m \in \mathbb{L}_{<1}$, then the following series converges:*

$$H_{\mathcal{Y}} \triangleq \sum_{k=0}^{\infty} A^k B H^{-1} B^{\mathrm{T}} (A^k)^{\mathrm{T}};$$

2) *if* $\det A \neq 0$ *and* $b_1, \ldots, b_m \in \mathbb{L}_{>1}$*, then the following series converges:*

$$H_{\mathcal{X}} \triangleq \sum_{k=1}^{\infty} A^{-k} B H^{-1} B^{\mathrm{T}} (A^{-k})^{\mathrm{T}}.$$

To describe the limit reachable and null-controllable sets of system (1), we generalize the concept of an ellipsoid. The classical definition assumes that for some positive definite symmetric $H \in \mathbb{R}^{m\times m}$ and $t > 0$, an ellipsoid is a set

$$\mathcal{U} = \left\{ u \in \mathbb{R}^m \colon u^{\mathrm{T}} H u \leqslant t^2 \right\}.$$

In this case $\mathcal{U}$ is determined by the Minkowski functional [17, §15, Chapter III].

However, this definition is not suitable for describing ellipsoidal sets whose dimension is less than the dimension of the space. In this case, it turns out to be more convenient to define the set $\mathcal{U}$ in terms of the support function [17, §4, Chapter I]. For a non-negative definite symmetric $\tilde{H} \in \mathbb{R}^{m\times m}$ and $t > 0$, an ellipsoid is defined as a set $\mathcal{U} \subset \mathbb{R}^m$ whose support function is given by the equality

$$\mathcal{S}(p,\mathcal{U}) \triangleq \sup_{u\in\mathcal{U}} p^{\mathrm{T}} u = t\sqrt{p^{\mathrm{T}} \tilde{H} p}. \tag{8}$$

If the matrix $\tilde{H}$ is positive definite and, as a consequence, non-singular, then the representation for the Minkowski functional is also true:

$$\mu(u,\mathcal{U}) \triangleq \inf\{\gamma > 0 \colon u \in \gamma\mathcal{U}\} = \frac{1}{t}\sqrt{u^{\mathrm{T}} \tilde{H}^{-1} u}. \tag{9}$$

Example 2. Consider a two-dimensional circle in three-dimensional space:

$$\mathcal{U} = \left\{ u \in \mathbb{R}^3 \colon u_1^2 + u_2^2 \leqslant 1,\ u_3 = 0 \right\}.$$

The set $\mathcal{U}$ can be considered as an ellipsoid in the general sense, although it is not described by a single inequality for a quadratic form. Nevertheless, the support function $\mathcal{U}$ satisfies the representation

$$\mathcal{S}(p,\mathcal{U}) = \sqrt{p_1^2 + p_2^2} = \sqrt{p^{\mathrm{T}} \tilde{H} p},\ \tilde{H} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Here $H \in \mathbb{R}^{3\times 3}$ is non-negative definite and degenerate, which does not allow us to use it to go to formula (9) for the Minkowski functional.

Theorem 2. *Let* $H_{\mathcal{Y}}, H_{\mathcal{X}} \in \mathbb{R}^{n\times n}$ *be taken from Lemma 1. Then*

1) *if* $\mathcal{Y}_{t,2,\infty}$ *is bounded, then the following representation for the support function is valid:*

$$\mathcal{S}(p, \mathcal{Y}_{t,2,\infty}) = t\sqrt{p^{\mathrm{T}} H_{\mathcal{Y}} p};$$

2) *if* $\mathcal{X}_{t,2,\infty}$ *is bounded, then the following representation for the support function is valid:*

$$\mathcal{S}(p, \mathcal{X}_{t,2,\infty}) = t\sqrt{p^{\mathrm{T}} H_{\mathcal{X}} p}.$$

Theorem 2 states that, in the case of boundedness, $\mathcal{Y}_{t,2,\infty}$ and $\mathcal{X}_{t,2,\infty}$ are ellipsoids. Transition to a description using the Minkowski functional is possible only if the matrices $H_{\mathcal{Y}}$ and $H_{\mathcal{X}}$ are invertible. This property can be guaranteed, for example, in the special case of control systems.

Lemma 2. *Let the controllability condition be satisfied for the system* (1):

$$\operatorname{rank}(B\ AB\ \dots\ A^{n-1}B) = n.$$

Then

1) *if* $b_1, \dots, b_m \in \mathbb{L}_{<1}$, *then* $\det H_{\mathcal{Y}} \neq 0$;
2) *if* $\det A \neq 0$ *and* $b_1, \dots, b_m \in \mathbb{L}_{>1}$, *then* $\det H_{\mathcal{X}} \neq 0$.

Due to Lemma 2, Theorem 2 and expressions (8)–(9), in the case of controlled system (1), the following representation for the Minkowski functional is valid:

$$\mu(x, \mathcal{Y}_{t,2,\infty}) = \frac{1}{t}\sqrt{x^{\mathrm{T}} H_{\mathcal{Y}}^{-1} x}, \quad \mu(x, \mathcal{X}_{t,2,\infty}) = \frac{1}{t}\sqrt{x^{\mathrm{T}} H_{\mathcal{X}}^{-1} x}.$$

Remark 2. The matrices $H_{\mathcal{Y}}$ and $H_{\mathcal{X}}$ are usually called the Gramians of a discrete-time linear system. If the matrix A is Schur, i.e., all its eigenvalues are less than 1 in absolute value, then $H_{\mathcal{Y}}$ is a solution to the discrete Lyapunov equation [18, Theorem K.2]:

$$P = APA^{\mathrm{T}} + BH^{-1}B^{\mathrm{T}}.$$

A large number of numerical methods are known for solving this equation [19, 20]. Also, one common analytical approach is to use the canonical Jordan decomposition of the matrix A [21]. Further in the article, we will refine this approach for a more general case, which is defined by Theorem 1 and allows for matrices A other than Schur matrices, and construct an explicit form for the Gramians $H_{\mathcal{Y}}$ and $H_{\mathcal{X}}$.

Although Theorem 2 defines the structure of sets (2) and (3) of system (1), their construction is reduced to calculating the matrix series from Lemma 1 for $H_{\mathcal{Y}}$ and $H_{\mathcal{X}}$. To calculate these series, we use the canonical Jordan form of the matrix A [16, Theorem 3.1.11]:

$$\begin{aligned}
&A = S\Lambda S^{-1},\\
&S \in \mathbb{C}^{n\times n},\ \det S \neq 0,\ \Lambda = \operatorname{diag}(J_1, \dots, J_M) \in \mathbb{C}^{n\times n},\\
&J_i = \lambda_i I + P_{n_i} \in \mathbb{C}^{n_i\times n_i},\ P_{n_i} = \begin{pmatrix} 0 & 1 & \cdots & 0\\ 0 & 0 & \ddots & 0\\ \vdots & \vdots & \ddots & 1\\ 0 & 0 & \cdots & 0 \end{pmatrix} \in \mathbb{R}^{n_i\times n_i},\ i = \overline{1, M},\\
&n_1 + \dots + n_M = n.
\end{aligned} \tag{10}$$

Here $\lambda_i \in \mathbb{C}$ is an eigenvalue of the matrix A of multiplicity n_i.

In notation (10) for all $i, j = \overline{1, M}$ we define the matrices $D_{ij} \in \mathbb{C}^{n_i\times n_j}$ from the following relations:

$$\begin{pmatrix} D_{11} & \dots & D_{1M}\\ \vdots & \ddots & \vdots\\ D_{M1} & \dots & D_{MM} \end{pmatrix} \triangleq S^{-1}BH^{-1}B^{\mathrm{T}}(S^{-1})^{\mathrm{T}},$$

for $\lambda_i\lambda_j \neq 1$ we introduce matrices

$$\Gamma_{ij} \triangleq \sum_{r=0}^{n_i-1}\sum_{q=0}^{n_j-1}\sum_{l=0}^{\min\{r,q\}} \frac{(r+q-l)!}{(r-l)!(q-l)!l!}\frac{\lambda_i^{r-l}\lambda_j^{q-l}}{(1-\lambda_i\lambda_j)^{r+q-l+1}} P_{n_i}^r D_{ij} (P_{n_j}^q)^{\mathrm{T}}.$$

Theorem 3. *Let $H_{\mathcal{Y}}$ and $H_{\mathcal{X}}$ be taken from Lemma 1. Then*
1) *if $b_1, \ldots, b_m \in \mathbb{L}_{<1}$, then*

$$H_{\mathcal{Y}} = S \begin{pmatrix} \Gamma^{\mathcal{Y}}_{11} & \ldots & \Gamma^{\mathcal{Y}}_{1M} \\ \vdots & \ddots & \vdots \\ \Gamma^{\mathcal{Y}}_{M1} & \ldots & \Gamma^{\mathcal{Y}}_{MM} \end{pmatrix} S^{\mathrm{T}},$$

where for all $i, j = \overline{1, M}$ in the case $|\lambda_i| \geqslant 1$ or $|\lambda_j| \geqslant 1$ it is true that $\Gamma^{\mathcal{Y}}_{ij} = O$, or in the case $|\lambda_i|, |\lambda_j| \in [0; 1)$ it is true that $\Gamma^{\mathcal{Y}}_{ij} = \Gamma_{ij}$;
2) *if $\det A \neq 0$ and $b_1, \ldots, b_m \in \mathbb{L}_{>1}$, then*

$$H_{\mathcal{X}} = S \begin{pmatrix} \Gamma^{\mathcal{X}}_{11} & \ldots & \Gamma^{\mathcal{X}}_{1M} \\ \vdots & \ddots & \vdots \\ \Gamma^{\mathcal{X}}_{M1} & \ldots & \Gamma^{\mathcal{X}}_{MM} \end{pmatrix} S^{\mathrm{T}},$$

where for all $i, j = \overline{1, M}$, in the case $|\lambda_i| \leqslant 1$ or $|\lambda_j| \leqslant 1$, $\Gamma^{\mathcal{X}}_{ij} = O$, or in the case $|\lambda_i|, |\lambda_j| \in (1; +\infty)$, $\Gamma^{\mathcal{X}}_{ij} = -\Gamma_{ij}$.

In general, the matrices in decomposition (10), and therefore the matrices Γ_{ij}, may be complex. However, as a result of the transformations, $H_{\mathcal{Y}}$ and $H_{\mathcal{X}}$ will turn out to be real.

Example 3. Consider the matrices

$$A = \begin{pmatrix} \sqrt{2} & \sqrt{2} \\ -\sqrt{2} & \sqrt{2} \end{pmatrix}, \; B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \; H = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

The matrix A has complex conjugate eigenvalues $\lambda_1 = \sqrt{2} + \mathbf{i}\sqrt{2}$, $\lambda_2 = \sqrt{2} - \mathbf{i}\sqrt{2}$ and the following expansion of form (10):

$$A = \begin{pmatrix} -\mathbf{i} & \mathbf{i} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{2} + \mathbf{i}\sqrt{2} & 0 \\ 0 & \sqrt{2} - \mathbf{i}\sqrt{2} \end{pmatrix} \begin{pmatrix} -\mathbf{i} & \mathbf{i} \\ 1 & 1 \end{pmatrix}^{-1}.$$

Due to Theorem 1, the limit null-controllable set $\mathcal{X}_{t,2,\infty}$ is bounded, and in the notation of Theorem 3 it is true that $M = 2$, $n_1 = n_2 = 1$ and

$$S^{-1} B H^{-1} B^{\mathrm{T}} \left(S^{-1}\right)^{\mathrm{T}} = \begin{pmatrix} \frac{\mathbf{i}}{2} & 1 \\ 1 & -\frac{\mathbf{i}}{2} \end{pmatrix}.$$

This leads to the following values:

$$-\Gamma^{\mathcal{X}}_{11} = \sum_{r=0}^{1-1} \sum_{q=0}^{1-1} \sum_{l=0}^{\min\{r,q\}} \frac{(r+q-l)!}{(r-l)!(q-l)!l!} \frac{\lambda_1^{r-l} \lambda_1^{q-l}}{(1-\lambda_1\lambda_1)^{r+q-l+1}} P_1^r D_{11} (P_1^q)^{\mathrm{T}} = \frac{1}{1-(\sqrt{2}+\mathbf{i}\sqrt{2})^2} \frac{\mathbf{i}}{2},$$

$$-\Gamma^{\mathcal{X}}_{22} = \frac{1}{1-(\sqrt{2}-\mathbf{i}\sqrt{2})^2} \left(-\frac{\mathbf{i}}{2}\right), \; -\Gamma^{\mathcal{X}}_{12} = -\Gamma^{\mathcal{X}}_{21} = \frac{1}{1-(\sqrt{2}+\mathbf{i}\sqrt{2})(\sqrt{2}-\mathbf{i}\sqrt{2})},$$

$$H_{\mathcal{X}} = \begin{pmatrix} \frac{22}{51} & -\frac{1}{17} \\ -\frac{1}{17} & \frac{46}{51} \end{pmatrix}, \; \mathcal{S}(p, \mathcal{X}_{t,2,\infty}) = \sqrt{\frac{22}{51} p_1^2 - \frac{2}{17} p_1 p_2 + \frac{46}{51} p_2^2}.$$

5. NUMERICAL EXPERIMENTS

Let us demonstrate the results of Theorems 1–3 by numerical simulation. Consider various discrete-time systems of form (1) with bounded reachable sets $\mathcal{Y}_{t,2,\infty}$. First, we construct these sets explicitly using Theorem 3. Then, we generate a set of admissible processes and plot their terminal states as elements of the limit reachable set. All numerical values will be rounded to four decimal places.

Example 4. Let us consider system (1) for $n = m = 2$ and the following parameter values:

$$A = \begin{pmatrix} 0.8791 & -1.1321 \\ 0.6429 & 0.0934 \end{pmatrix} = S \begin{pmatrix} 0.4863 + 0.7573\mathbf{i} & 0 \\ 0 & 0.4863 - 0.7573\mathbf{i} \end{pmatrix} S^{-1},$$

$$S = \begin{pmatrix} 0.7986 & 0.7986 \\ 0.2771 - 0.5342\mathbf{i} & 0.2771 + 0.5342\mathbf{i} \end{pmatrix},$$

$$B = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \ H = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \ t = 1.$$

The matrix A has a pair of complex conjugate eigenvalues $\lambda_{1,2} = 0.4863 \pm 0.7573\mathbf{i}$, whose absolute values are strictly less than 1. Therefore, by Theorem 1, the set $\mathcal{Y}_{t,2,\infty}$ is bounded, and by Theorem 2, it is an ellipsoid. The matrix $H_{\mathcal{Y}}$ can be calculated explicitly by Theorem 3. The following values are valid:

$$M = 2, \ n_1 = n_2 = 1,$$
$$D_{11} = -1.5609 - 0.2711\mathbf{i}, \ D_{12} = D_{21} = 2.0836, \ D_{22} = -1.5609 + 0.2711\mathbf{i}.$$

This leads to the equality

$$H_{\mathcal{Y}} = S \begin{pmatrix} \dfrac{D_{11}}{1 - \lambda_1\lambda_1} & \dfrac{D_{12}}{1 - \lambda_1\lambda_2} \\ \dfrac{D_{21}}{1 - \lambda_2\lambda_1} & \dfrac{D_{22}}{1 - \lambda_2\lambda_2} \end{pmatrix} S^{\mathrm{T}} = \begin{pmatrix} 12.9553 & 3.9416 \\ 3.9416 & 7.8973 \end{pmatrix}.$$

Limit reachable set (2) and terminal states of various processes are shown in Fig. 1.

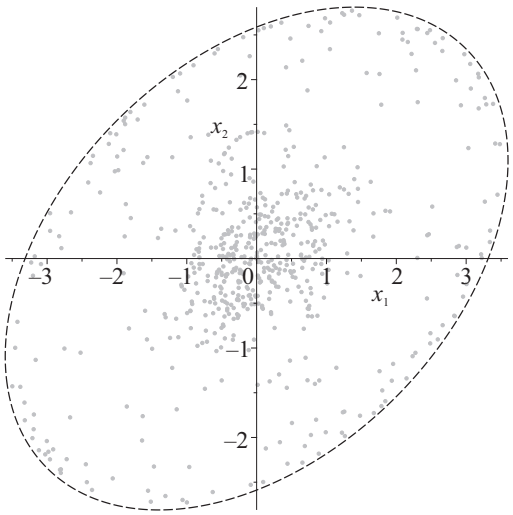


Fig. 1. The set $\mathcal{Y}_{t,2,\infty}$ and various terminal states for the system (A, B, H, t) from Example 4.

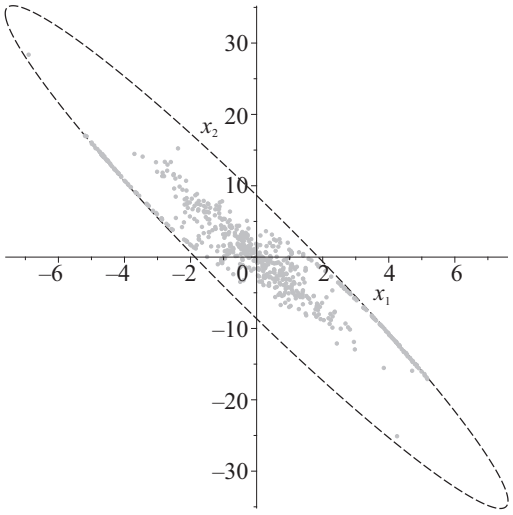


Fig. 2. The set $\mathcal{Y}_{t,2,\infty}$ and various terminal states for the system (A, B, H, t) from Example 5.

Example 5. Let us consider system (1) for $n = m = 2$ and the following values:

$$A = \begin{pmatrix} -1.3315 & -0.0769 \\ 2.7692 & -0.4085 \end{pmatrix} = S \begin{pmatrix} -0.87 & 1 \\ 0 & -0.87 \end{pmatrix} S^{-1},$$
$$S = \begin{pmatrix} -1 & 1 \\ 6 & 7 \end{pmatrix},\ B = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},\ H = \begin{pmatrix} 1 & 0.5 \\ 0.5 & 2 \end{pmatrix},\ t = 1.$$

The matrix A has one real eigenvalue $\lambda_1 = -0.87$, whose absolute value is strictly less than 1. Therefore, by Theorem 1, the set $\mathcal{Y}_{t,2,\infty}$ is bounded, and by Theorem 2, it is an ellipsoid. The matrix $H_{\mathcal{Y}}$ can be calculated explicitly using Theorem 3. The following values are valid:

$$M = 1,\ n_1 = 2,\ D_{11} = \begin{pmatrix} 0.2975 & -0.2739 \\ -0.2739 & 0.2975 \end{pmatrix}.$$

This leads to the equality

$$H_{\mathcal{Y}} = S \left(\frac{D_{11}}{1-\lambda_1^2} + \frac{\lambda_1 D_{11} P_2^{\mathrm{T}}}{(1-\lambda_1^2)^2} + \frac{\lambda_1 P_2 D_{11}}{(1-\lambda_1^2)^2} + \frac{P_2 D_{11} P_2^{\mathrm{T}}}{(1-\lambda_1^2)^2} + 2\frac{\lambda_1^2 P_2 D_{11} P_2^{\mathrm{T}}}{(1-\lambda_1^2)^3} \right) S^{\mathrm{T}}$$
$$= \begin{pmatrix} 57.9130 & -259.9758 \\ -259.9758 & 1241.6945 \end{pmatrix}.$$

Limit reachable set (2) and terminal states of various processes are shown in Fig. 2.

Example 6. Let us consider system (1) for $n = 3$, $m = 2$, and the following values:

$$A = \begin{pmatrix} -0.8421 & -0.0196 & 0.0976 \\ 0.3013 & -0.9058 & 0.1081 \\ -0.1239 & -0.0093 & -0.9540 \end{pmatrix} = S \begin{pmatrix} -0.8910 + 0.1270\mathbf{i} & 0 & 0 \\ 0 & -0.8910 - 0.1270\mathbf{i} & 0 \\ 0 & 0 & -0.92 \end{pmatrix} S^{-1},$$

$$S = \begin{pmatrix} 0.1703 + 0.3405\mathbf{i} & 0.1703 - 0.3405\mathbf{i} & -0.1562 \\ 0.8513 & 0.8513 & 0.9370 \\ -0.3575 + 0.0511\mathbf{i} & -0.3575 - 0.0511\mathbf{i} & 0.3123 \end{pmatrix},\ B = \begin{pmatrix} 1 & 1 \\ 2 & 0 \\ 1 & 1 \end{pmatrix},\ H = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

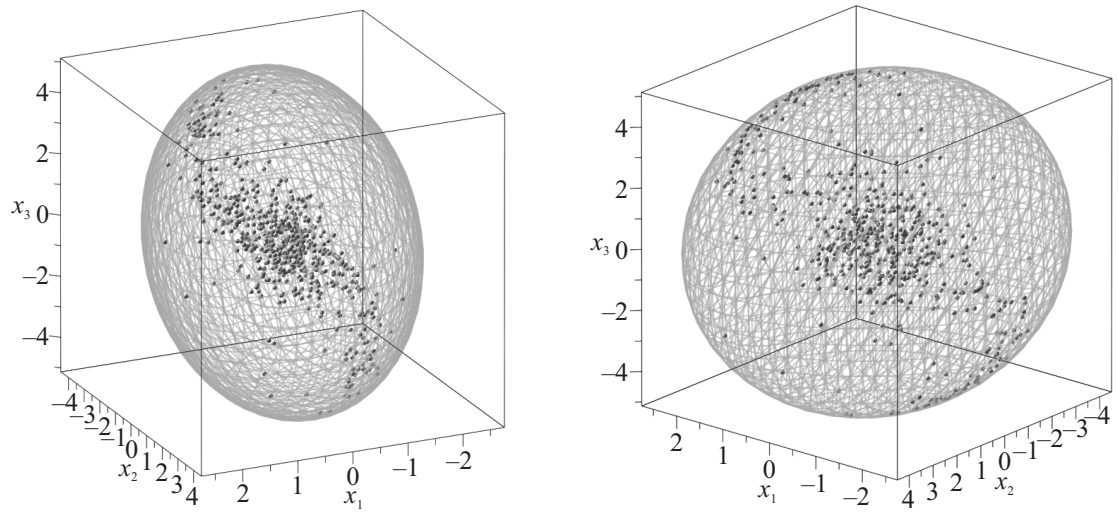


Fig. 3. The set $\mathcal{Y}_{t,2,\infty}$ and various terminal states for the system (A, B, H, t) from Example 6.

We also assume that $t = 1$. The matrix A has a pair of complex conjugate eigenvalues $\lambda_{1,2} = -0.8910 \pm 0.1270\mathbf{i}$ and one real eigenvalue $\lambda_3 = -0.92$, whose absolute values are strictly less than 1. Therefore, by Theorem 1, the set $\mathcal{Y}_{t,2,\infty}$ is bounded, and by Theorem 2, it is an ellipsoid. The matrix $H_{\mathcal{Y}}$ can be calculated explicitly using Theorem 3, similarly to Example 4:

$$M = 3,\ n_1 = n_2 = n_3 = 1,$$

$$H_{\mathcal{Y}} = \begin{pmatrix} 7.5669 & 4.6171 & 1.4047 \\ 4.6171 & 20.2330 & -5.4371 \\ 1.4047 & -5.4371 & 26.3126 \end{pmatrix}.$$

Limit reachable set (2) and terminal states of various processes are shown in Fig. 3.

Example 7. Let us consider system (1) for $n = 3$, $m = 2$, and the following values:

$$A = \begin{pmatrix} -0.4838 & -0.1586 & -0.2412 \\ 1.9715 & 0.2153 & 3.1897 \\ -1.1023 & 0.3068 & -0.5215 \end{pmatrix} = S \begin{pmatrix} 0.95 & 0 & 0 \\ 0 & -0.87 & 1 \\ 0 & 0 & -0.87 \end{pmatrix} S^{-1},$$

$$S = \begin{pmatrix} -1 & 1 & 3 \\ 6 & 7 & 1 \\ 2 & -3 & 0 \end{pmatrix},\ B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 2 \end{pmatrix},\ H = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix},\ t = 1.$$

The matrix A has a real non-multiple eigenvalue $\lambda_1 = 0.95$ and a real multiple eigenvalue $\lambda_2 = -0.87$, whose absolute values are strictly less than 1. Therefore, by Theorem 1, the set $\mathcal{Y}_{t,2,\infty}$ is bounded, and by Theorem 2, it is an ellipsoid. The matrix $H_{\mathcal{Y}}$ can be calculated explicitly using Theorem 3, similarly to Example 5:

$$M = 2,\ n_1 = 1,\ n_2 = 2,$$

$$H_{\mathcal{Y}} = \begin{pmatrix} 11.1345 & 52.5427 & -32.4125 \\ 52.5427 & 735.8157 & -266.4128 \\ -32.4125 & -266.4128 & 136.2212 \end{pmatrix}.$$

Limit reachable set (2) and terminal states of various processes are shown in Fig. 4.

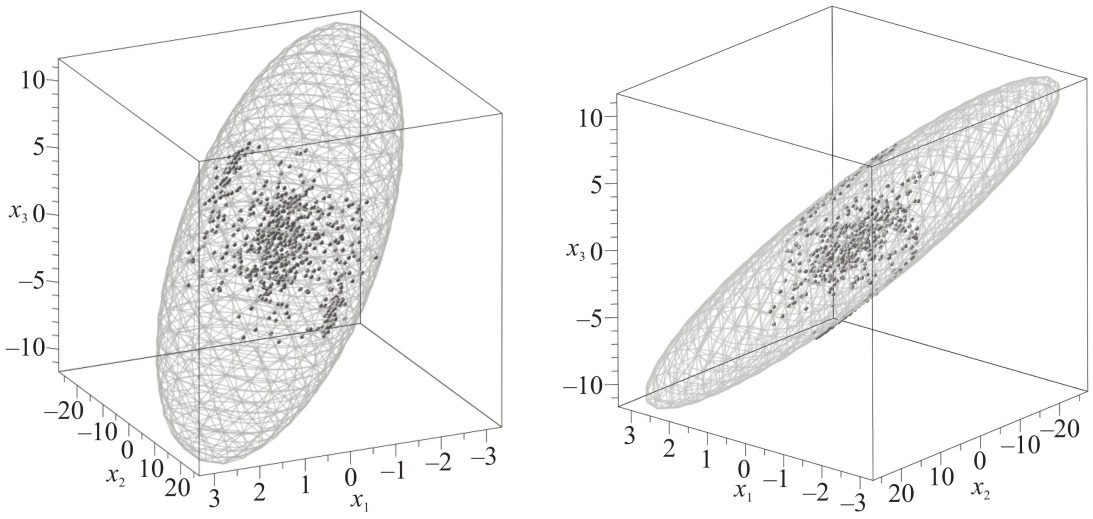


Fig. 4. The set $\mathcal{Y}_{t,2,\infty}$ and various terminal states for the system (A, B, H, t) from Example 7.

In Examples 4–7, only the construction of limit reachable sets (2) is considered, since the numerical modeling of its elements is simpler compared to the null-controllable sets (3). At the same time, according to Theorems 1–3, the construction of the set $\mathcal{X}_{t,2,\infty}$ is similar. It should also be noted that the case $m = 1$ is considered in detail in [14] and is therefore not presented here.

6. CONCLUSION

This paper solves the problem of constructing limit reachable and null-controllable sets for discrete-time linear systems with second-order integral constraints. Necessary and sufficient conditions for the boundedness of these sets are formulated in the form of Theorem 1, which reduces to the relationship between the matrix B and the eigensubspaces $\mathbb{L}_{<1}$ and $\mathbb{L}_{>1}$. For controllable systems, these conditions are equivalent to a constraint on the spectral radius of the matrix A for reachable sets and the matrix A^{-1} for null-controllable sets, respectively. It is also proved that, in the case of boundedness, the sets $\mathcal{Y}_{t,2,\infty}$ and $\mathcal{X}_{t,2,\infty}$ are ellipsoids. Their support function is constructed explicitly by the Jordan decomposition of the matrix A, which is formulated as Theorem 3. To confirm the obtained theoretical results, numerical experiments were conducted for systems of different dimensions, with different types of eigenvalues of the matrix A.

FUNDING

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APPENDIX

Let us introduce into $\mathbb{R}^n$ the dot product and the vector norm by the formulas

$$\langle x, y\rangle \triangleq x^{\mathrm{T}} y = \sum_{i=1}^{n} x_i y_i,\ \ \|x\| \triangleq \sqrt{\langle x, y\rangle} = \sqrt{\sum_{i=1}^{n} x_i^2}.$$

Proof of Theorem 1. Without loss of generality, we assume that the elements of basis (4) are grouped in such a way that for the class $M_0 \leqslant M$ the following representation is valid:

$$\mathbb{L}_{<1} = \operatorname{Lin}\left\{h_{i,j} \colon i = \overline{1, M_0},\ j = \overline{1, n_i}\right\}.$$

For each $u(k) \in \mathbb{R}^m$ we define the expansion in basis (4):

$$Bu(k) = \sum_{i=1}^{M} \sum_{j=1}^{n_i} h_{i,j} \alpha_{i,j}(k),$$

where $\alpha_{i,j}(k) \in \mathbb{R}$ if $\lambda_i \in \mathbb{R}$, and $\alpha_{i,j}(k) \in \mathbb{R}^2$ if $\lambda_i \in \mathbb{C}$. Taking into account representation (5), the following equalities hold:

$$S^{-1}Bu(k) = \operatorname{vec}\left(\alpha_{1,1}(k), \ldots, \alpha_{1,n_1}(k), \ldots, \alpha_{M,1}(k), \ldots, \alpha_{M,n_M}(k)\right),$$

$$\begin{pmatrix} \beta_1 \\ \vdots \\ \beta_M \end{pmatrix} \triangleq \sum_{k=0}^{N-1} \Lambda^k S^{-1} Bu(N-k-1)$$

$$= \begin{pmatrix} \sum\limits_{k=0}^{N-1} J_1^k \operatorname{vec}\left(\alpha_{1,1}(N-k-1), \ldots, \alpha_{1,n_1}(N-k-1)\right) \\ \vdots \\ \sum\limits_{k=0}^{N-1} J_M^k \operatorname{vec}\left(\alpha_{M,1}(N-k-1), \ldots, \alpha_{M,n_M}(N-k-1)\right) \end{pmatrix}, \ N \in \mathbb{N}.$$

Let $b_1, \ldots, b_m \in \mathbb{L}_{<1}$. Then for any $k \in \mathbb{N} \cup \{0\}$, $i = \overline{M_0+1, M}$ and all $j = \overline{1, n_i}$, $\alpha_{i,j}(k) = 0$, and therefore $\beta_i = 0$.

Let $i = \overline{1, M_0}$. Let representation (6) holds for the Jordan cell J_i. Let us introduce the matrix

$$P = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \ddots & 0 \\ \vdots & \vdots & \ddots & 1 \\ 0 & 0 & \cdots & 0 \end{pmatrix} \in \mathbb{R}^{n_i \times n_i}.$$

Note the following property of the matrix P:

$$(P^l x)_j = \begin{cases} x_{j+l}, & j+l \leqslant n_i, \\ 0, & j+l > n_i, \end{cases} = \begin{cases} x_{j+l}, & l \leqslant n_i - j, \\ 0, & l > n_i - j. \end{cases}$$

Let C_k^l denotes the binomial coefficients. According to Newton's binomial theorem

$$J_i^k = (\lambda_i I + P)^k = \sum_{l=0}^{k} C_k^l \lambda_i^{k-l} P^l,$$

$$\begin{pmatrix} \beta_{i,1} \\ \vdots \\ \beta_{i,n_i} \end{pmatrix} \triangleq \beta_i = \sum_{k=0}^{N-1} J_i^k \begin{pmatrix} \alpha_{i,1}(N-k-1) \\ \vdots \\ \alpha_{i,n_i}(N-k-1) \end{pmatrix} = \sum_{k=0}^{N-1} \sum_{l=0}^{k} C_k^l \lambda_i^{k-l} P^l \begin{pmatrix} \alpha_{i,1}(N-k-1) \\ \vdots \\ \alpha_{i,n_i}(N-k-1) \end{pmatrix}.$$

Taking into account the properties of the matrix P, for all $j = \overline{1, n_i}$ the following representation is valid:

$$\beta_{i,j} = \sum_{k=0}^{N-1} \sum_{l=0}^{\min\{k, n_i - j\}} C_k^l \lambda_i^{k-l} \alpha_{i,j+l}(N-k-1). \tag{A.1}$$

Assuming $N - 1 \geqslant n_i - j$, we change the order of summation:

$$\beta_{i,j} = \sum_{l=0}^{n_i-j} \sum_{k=l}^{N-1} C_k^l \lambda_i^{k-l} \alpha_{i,j+l}(N-k-1) = \sum_{l=0}^{n_i-j} \sum_{k=0}^{N-l-1} C_{k+l}^l \lambda_i^k \alpha_{i,j+l}(N-k-l-1). \tag{A.2}$$

We take into account that $u(k)$, by (1), belongs to the ellipsoidal set $\mathcal{U} = \{u \in \mathbb{R}^m \colon u^{\mathrm{T}} H u \leqslant t^2\}$, which is compact. Then, by the Weierstrass theorem [22, Chapter VII, §2, Section 2] and the continuous dependence of the numbers $\alpha_{i,j}(k)$ on $u(k)$, there exists a number $\alpha_{\max} > 0$ such that for all $k \in \mathbb{N} \cup \{0\}$, $i = \overline{1, M}$, $j = \overline{1, n_i}$, the constraint $|\alpha_{i,j}(k)| \leqslant \alpha_{\max}$ holds. The following estimate is valid:

$$|\beta_{i,j}| \leqslant \sum_{l=0}^{n_i - j} \sum_{k=0}^{N-l-1} C_{k+l}^l |\lambda_i|^k |\alpha_{i,j+l}(N-k-l-1)| \leqslant \alpha_{\max} \sum_{l=0}^{n_i-j} \sum_{k=0}^{\infty} C_{k+l}^l |\lambda_i|^k$$

$$= \sum_{l=0}^{n_i-j} \frac{\alpha_{\max}}{(1-|\lambda_i|)^{l+1}} = \frac{(n_i - j + 1)\alpha_{\max}}{(1-|\lambda_i|)^{n_i - j + 1}} \leqslant \frac{n_i \alpha_{\max}}{(1-|\lambda_i|)^{n_i}}.$$

We will finally get

$$\|\beta_i\|^2 = \sum_{j=1}^{n_i} |\beta_{i,j}|^2 \leqslant \frac{n_i^3 \alpha_{\max}^2}{(1-|\lambda_i|)^{2n_i}}. \tag{A.3}$$

Let representation (7) be valid for the Jordan cell J_i. Setting $P \in \mathbb{R}^{2n_i \times 2n_i}$, we obtain from Newton's binomial theorem

$$J_i^k = \left(\operatorname{diag}(r_i A_{\varphi_i}, \ldots, r_i A_{\varphi_i}) + P^2\right)^k = \sum_{l=0}^{k} C_k^l \operatorname{diag}\left(r_i^{k-l} A_{(k-l)\varphi_i}, \ldots, r_i^{k-l} A_{(k-l)\varphi_i}\right) P^{2l},$$

$$\begin{pmatrix} \beta_{i,1} \\ \vdots \\ \beta_{i,n_i} \end{pmatrix} \triangleq \beta_i = \sum_{k=0}^{N-1} J_i^k \begin{pmatrix} \alpha_{i,1}(N-k-1) \\ \vdots \\ \alpha_{i,n_i}(N-k-1) \end{pmatrix}$$

$$= \sum_{k=0}^{N-1} \sum_{l=0}^{k} \operatorname{diag}\left(r_i^{k-l} A_{(k-l)\varphi_i}, \ldots, r_i^{k-l} A_{(k-l)\varphi_i}\right) P^{2l} \begin{pmatrix} \alpha_{i,1}(N-k-1) \\ \vdots \\ \alpha_{i,n_i}(N-k-1) \end{pmatrix}.$$

Taking into account the properties of the matrix P, for all $j = \overline{1, n_i}$ the following representation is valid:

$$\beta_{i,j} = \sum_{k=0}^{N-1} \sum_{l=0}^{\min\{k, n_i - j\}} C_k^l r_i^{k-l} A_{(k-l)\varphi_i} \alpha_{i,j+l}(N-k-1). \tag{A.4}$$

Assuming $N - 1 \geqslant n_i - j$, we change the order of summation:

$$\beta_{i,j} = \sum_{l=0}^{n_i-j} \sum_{k=l}^{N-1} C_k^l r_i^{k-l} A_{(k-l)\varphi_i} \alpha_{i,j+l}(N-k-1) \tag{A.5}$$

$$= \sum_{l=0}^{n_i-j} \sum_{k=0}^{N-l-1} C_{k+l}^l r_i^{k-l} A_{(k-l)\varphi_i} \alpha_{i,j+l}(N-k-l-1).$$

We take into account that $u(k)$, by (1), belongs to the ellipsoidal set $\mathcal{U} = \{u \in \mathbb{R}^m \colon u^{\mathrm{T}} H u \leqslant t^2\}$, which is compact. Then, by Weierstrass's theorem [22, Chapter VII, §2, Section 2] and the continuous dependence of the two-dimensional vectors $\alpha_{i,j}(k)$ on $u(k)$, there exists a number $\alpha_{\max} > 0$

such that for all $k \in \mathbb{N} \cup \{0\}$, $i = \overline{1, M}$, $j = \overline{1, n_i}$, the constraint $\|\alpha_{i,j}(k)\| \leqslant \alpha_{\max}$ holds. The following estimate is valid:

$$\|\beta_{i,j}\| \leqslant \sum_{l=0}^{n_i-j} \sum_{k=0}^{N-l-1} C_{k+l}^{l} r_i^k \|A_{k\varphi_i} \alpha_{i,j+l}(N-k-l-1)\|$$

$$= \sum_{l=0}^{n_i-j} \sum_{k=0}^{N-l-1} C_{k+l}^{l} r_i^k \|\alpha_{i,j+l}(N-k-l-1)\| \leqslant \alpha_{\max} \sum_{l=0}^{n_i-j} \sum_{k=0}^{\infty} C_{k+l}^{l} r_i^k$$

$$= \sum_{l=0}^{n_i-j} \frac{\alpha_{\max}}{(1-r_i)^{l+1}} = \frac{(n_i-j+1)\alpha_{\max}}{(1-r_i)^{n_i-j+1}} \leqslant \frac{n_i \alpha_{\max}}{(1-r_i)^{n_i}}.$$

We will finally get

$$\|\beta_i\|^2 = \sum_{j=1}^{n_i} \|\beta_{i,j}\|^2 \leqslant \frac{n_i^3 \alpha_{\max}^2}{(1-r_i)^{2n_i}}. \tag{A.6}$$

We take into account the inequalities (A.3) and (A.6) and obtain for all $N \geqslant n$ and the condition $x(0) = 0$ the estimate

$$\|x(N)\| \overset{(1)}{=} \left\| S \sum_{k=0}^{N-1} \Lambda^k S^{-1} B u(k) \right\| \leqslant \|S\| \left(\sum_{i=1}^{M_0} \|\beta_i\|^2 \right)^{\frac{1}{2}} \leqslant \frac{\|S\| \left(M_0 n^3 \alpha_{\max}^2\right)^{\frac{1}{2}}}{\left(1 - \max\limits_{i=\overline{1,M_0}} |\lambda_i|\right)^n},$$

i.e., by definition $\mathcal{Y}_{t,2,\infty}$ is bounded.

Now suppose that for some $\tilde{m} = \overline{1, m}$ the condition $b_{\tilde{m}} \notin \mathbb{L}_{<1}$ is satisfied. I.e., there exists $u \in \mathbb{R}^m$ such that in the expansion over basis (4)

$$Bu = \sum_{i=1}^{M} \sum_{j=1}^{n_i} h_{i,j} \alpha_{i,j}$$

For some $i = \overline{M_0 + 1, M}$ and $j = \overline{1, n_i}$, it is true that $\alpha_{i,j} \neq 0$. Fix these i, j, and $N \geqslant n$. Without loss of generality, we assume that either $j = n_i$ or $\alpha_{i,j+1} = \ldots = \alpha_{i,n_i} = 0$. Let J_i satisfy representation (6). Choose

$$u(N-k-1) = \frac{\sqrt{6} t \operatorname{sgn}(\lambda_i^k)}{(k+1)\pi} \frac{u}{\sqrt{u^{\mathrm{T}} H u}}, \ k = \overline{0, N-1}.$$

Then, on the one hand,

$$\sum_{k=0}^{N-1} u(k)^{\mathrm{T}} H u(k) = \sum_{k=0}^{N-1} \left(\frac{\sqrt{6} t \operatorname{sgn}(\lambda_i^k)}{(k+1)\pi} \right)^2 \frac{u^{\mathrm{T}} H u}{\sqrt{u^{\mathrm{T}} H u} \sqrt{u^{\mathrm{T}} H u}} < \frac{6t^2}{\pi^2} \sum_{k=0}^{\infty} \frac{1}{(k+1)^2} = t^2,$$

i.e., the selected control actions are admissible for system (1). On the other hand, for this control, according to (A.2), $\beta_{i,j}$ satisfies the relation

$$\beta_{i,j} = \sum_{l=0}^{n_i-j} \sum_{k=0}^{N-l-1} C_{k+l}^{l} \lambda_i^k \alpha_{i,j+l}(N-k-l-1) = \sum_{k=0}^{N-1} \lambda_i^k \alpha_{i,j}(N-k-1)$$

$$= \sum_{k=0}^{N-1} |\lambda_i|^k \frac{\sqrt{6} t}{(k+1)\pi} \frac{\alpha_{i,j}}{\sqrt{u^{\mathrm{T}} H u}} \geqslant \frac{\sqrt{6} t \alpha_{i,j}}{\pi \sqrt{u^{\mathrm{T}} H u}} \sum_{k=0}^{N-1} \frac{1}{k+1} \xrightarrow{N \to \infty} +\infty.$$

Let J_i satisfy representation (7). Choose an increasing sequence $k_1, k_2, \ldots \in \mathbb{N} \cup \{0\}$ such that the condition $|\varphi_i k_p| \mod 2\pi < \frac{\pi}{6}$ holds for all $p \in \mathbb{N}$. Such a sequence exists. For the case $\varphi_i = 2\pi\frac{K'}{K''}$, it suffices to set $k_p = pK''$. For the opposite case, by Weyl's theorem, the set $\{k\varphi_i \mod 2\pi\colon k \in \mathbb{N} \cup \{0\}\}$ is dense in $[0; 2\pi)$. Set for all $k = \overline{0, k_N}$

$$u(k_N - k) = \begin{cases} 0, & k \notin \{k_1, \ldots, k_N\}, \\ \dfrac{\sqrt{6}t}{p\pi}\dfrac{u}{\sqrt{u^{\mathrm{T}}Hu}}, & k = k_p \in \{k_1, \ldots, k_N\}. \end{cases}$$

Then

$$\sum_{k=0}^{k_N} u(k)^{\mathrm{T}} H u(k) = \sum_{p=1}^{N} \left(\frac{\sqrt{6}t}{p\pi}\right)^2 \frac{u^{\mathrm{T}}Hu}{\sqrt{u^{\mathrm{T}}Hu}\sqrt{u^{\mathrm{T}}Hu}} < \frac{6t^2}{\pi^2}\sum_{p=1}^{\infty}\frac{1}{p^2} = t^2,$$

i.e., the selected control actions are admissible for system (1). On the other hand, for this control, according to (A.5), $\beta_{i,j}$ satisfies the relation

$$\begin{aligned}
\|\beta_{i,j}\|^2 &= \left\langle \sum_{l=0}^{n_i-j}\sum_{k=0}^{k_N-l} C_{k+l}^{l} r_i^k A_{k\varphi_i}\alpha_{i,j+l}(k_N-k-l), \sum_{l=0}^{n_i-j}\sum_{k=0}^{k_N-l} C_{k+l}^{l} r_i^k A_{k\varphi_i}\alpha_{i,j+l}(k_N-k-l)\right\rangle \\
&= \left\langle \sum_{k=0}^{k_N} r_i^k A_{k\varphi_i}\alpha_{i,j}(k_N-k), \sum_{k=0}^{k_N} r_i^k A_{k\varphi_i}\alpha_{i,j}(k_N-k)\right\rangle \\
&= \left\langle \sum_{p=1}^{N} r_i^{k_p} A_{k_p\varphi_i}\alpha_{i,j}(k_N-k_p), \sum_{q=1}^{N} r_i^{k_q} A_{k_q\varphi_i}\alpha_{i,j}(k_N-k_q)\right\rangle \\
&= \sum_{p=1}^{N}\sum_{q=1}^{N} r_i^{k_p+k_q}\left\langle A_{(k_p-k_q)\varphi_i}\alpha_{i,j}(k_N-k_p), \alpha_{i,j}(k_N-k_q)\right\rangle \\
&= \sum_{p=1}^{N}\sum_{q=1}^{N} r_i^{k_p+k_q}\frac{6t^2}{\pi^2 pq}\frac{\|\alpha_{i,j}\|^2}{u^{\mathrm{T}}Hu}\cos(\varphi_i(k_p-k_q)) \geqslant \sum_{p=1}^{N}\sum_{q=1}^{N}\frac{6t^2}{\pi^2 pq}\frac{\|\alpha_{i,j}\|^2}{2u^{\mathrm{T}}Hu} \\
&= \frac{\|\alpha_{i,j}\|^2}{u^{\mathrm{T}}Hu}\frac{3t^2}{\pi^2}\left(\sum_{p=1}^{N}\frac{1}{p}\right)\left(\sum_{q=1}^{N}\frac{1}{q}\right) \xrightarrow{N\to\infty} +\infty.
\end{aligned}$$

Thus, since

$$\left\|\sum_{k=0}^{k_N}\Lambda^k S^{-1}Bu(k_N-k)\right\| = \|S^{-1}x(k_N+1)\| \leqslant \|S^{-1}\|\|x(k_N+1)\|,$$

then

$$\|x(k_N+1)\| \geqslant \frac{1}{\|S^{-1}\|}\left\|\sum_{k=0}^{k_N}\Lambda^k S^{-1}Bu(k_N-k)\right\| = \frac{\left(\sum\limits_{l=1}^{M}\|\beta_l\|^2\right)^{\frac{1}{2}}}{\|S^{-1}\|} \geqslant \frac{\|\beta_{i,j}\|}{\|S^{-1}\|} \xrightarrow{N\to\infty} +\infty.$$

Therefore, $\mathcal{Y}_{t,2,\infty}$ is unbounded by definition.

Now we prove item 2. If $\det A = 0$, then there exists an element of the real Jordan basis $h_{i,j} \in \mathbb{R}^n \setminus \{0\}$ corresponding to the eigenvalue $\lambda_i = 0$. Then, by (1), the inclusion $\mathrm{Lin}\{h_{i,j}\} \subset \mathcal{X}_{t,2,\infty}$ holds, which means that set (3) is unbounded. If $\det A \neq 0$, then by the definitions of (2) and (3),

the set $\mathcal{Y}_{t,2,\infty}$ for the system (A, B, H, t) and the set $\mathcal{X}_{t,2,\infty}$ for the system $(A^{-1}, A^{-1}B, H, t)$ coincide. This implies item 2.

Theorem 1 is proved.

Proof of Lemma 1. Without loss of generality, we will assume that the elements of basis (4) are grouped in such a way that for some $M_0 \leqslant M$ the following representation is true:

$$\mathbb{L}_{<1} = \operatorname{Lin}\left\{h_{i,j}\colon i = \overline{1, M_0},\ j = \overline{1, n_i}\right\}.$$

Since H^{-1} is positive definite and symmetric, it satisfies the Cholesky decomposition [16, Corollary 7.2.9], i.e., there exists a lower triangular non-singular matrix $L \in \mathbb{R}^{m\times m}$ such that $H^{-1} = LL^{\mathrm{T}}$. In terms of (5), we introduce the notation

$$\begin{pmatrix} \gamma_{1,1}(k) & \dots & \gamma_{1,m}(k) \\ \vdots & \ddots & \vdots \\ \gamma_{M,1}(k) & \dots & \gamma_{M,m}(k) \end{pmatrix} \stackrel{\triangle}{=} \Lambda^k S^{-1} BL,$$

where $\gamma_{i,1}(k), \dots, \gamma_{i,m}(k)$ are vectors whose dimension coincides with the dimension of the Jordan block J_i. Then, from the inclusion $b_1, \dots, b_m \in \mathbb{L}_{<1}$, it follows that $\gamma_{i,l}(k) = 0$ if $i = \overline{M_0 + 1, M}$, $l = \overline{1, m}$. In the case $i = \overline{1, M_0}$, the vectors $\gamma_{i,l}(k)$ are the product of the powers of the Jordan blocks J_i^k and the corresponding blocks of the matrix $S^{-1}BL$. Consequently, the order of smallness of the elements of the vectors $\gamma_{i,l}(k)$ coincides with the order of smallness of the elements J_i^k, which, according to the reasoning from [16, Section 3.2.5], can be estimated from above by the quantity

$$k^{n_i}|\lambda_i|^k \leqslant k^n \lambda_{\max}^k, \text{ where } \lambda_{\max} \stackrel{\triangle}{=} \max\{|\lambda|\colon \lambda \text{ is eigenvalue of } A,\ |\lambda| < 1\}.$$

From here, by the sufficient condition for convergence [22, Chapter III, §1, Section 4, Corollary 2], it follows that the matrix series $\sum\limits_{k=0}^{\infty} \gamma_{i,l}(k)\gamma_{j,l}(k)^{\mathrm{T}}$ converges elementwise for all $i, j = \overline{1, M}$.

Consider the chain of equalities

$$H_{\mathcal{Y}} = \sum_{k=0}^{\infty} A^k B H^{-1} B^{\mathrm{T}} (A^k)^{\mathrm{T}} = S \left(\sum_{k=0}^{\infty} \Lambda^k S^{-1} B H^{-1} B^{\mathrm{T}} (S^{-1})^{\mathrm{T}} (\Lambda^k)^{\mathrm{T}} \right) S^{\mathrm{T}}$$

$$= S \sum_{k=0}^{\infty} \sum_{l=1}^{m} \begin{pmatrix} \gamma_{1,l}(k)\gamma_{1,l}(k)^{\mathrm{T}} & \dots & \gamma_{1,l}(k)\gamma_{M,l}(k)^{\mathrm{T}} \\ \vdots & \ddots & \vdots \\ \gamma_{M,l}(k)\gamma_{1,l}(k)^{\mathrm{T}} & \dots & \gamma_{M,l}(k)\gamma_{M,l}(k)^{\mathrm{T}} \end{pmatrix} S^{-1}$$

$$= \sum_{l=1}^{m} S \begin{pmatrix} \sum\limits_{k=0}^{\infty} \gamma_{1,l}(k)\gamma_{1,l}(k)^{\mathrm{T}} & \dots & \sum\limits_{k=0}^{\infty} \gamma_{1,l}(k)\gamma_{M,l}(k)^{\mathrm{T}} \\ \vdots & \ddots & \vdots \\ \sum\limits_{k=0}^{\infty} \gamma_{M,l}(k)\gamma_{1,l}(k)^{\mathrm{T}} & \dots & \sum\limits_{k=0}^{\infty} \gamma_{M,l}(k)\gamma_{M,l}(k)^{\mathrm{T}} \end{pmatrix} S^{-1}.$$

From the convergence of the matrix series in the extreme right-hand side of these relations follows clause 1 of the Lemma 1.

Item 2 of the lemma follows from item 1 due to the fact that the eigenvalues of A are mutually inverse to the eigenvalues of A^{-1}.

Lemma 1 is proved.

Proof of Theorem 2. Let us prove item 1. For arbitrary $N \in \mathbb{N}$, $u(0), \dots, u(N-1) \in \mathbb{R}^m$, $p \in \mathbb{R}^n \setminus \{0\}$, we introduce the notations

$$p_0 = \begin{pmatrix} \left(A^{N-1}B\right)^{\mathrm{T}} p \\ \vdots \\ B^{\mathrm{T}}p \end{pmatrix} \in \mathbb{R}^{mN},\ u_0 = \begin{pmatrix} u(0) \\ \vdots \\ u(N-1) \end{pmatrix} \in \mathbb{R}^{mN},\ H_0 = \begin{pmatrix} H & \dots & O \\ \vdots & \ddots & \vdots \\ O & \dots & H \end{pmatrix} \in \mathbb{R}^{mN\times mN}.$$

Then by definition (2) we get

$$\mathcal{S}(p, \mathcal{Y}_{t,2,\infty}) = \sup_{x\in\mathcal{Y}_{t,2,\infty}} \langle p, x\rangle = \sup_{N\in\mathbb{N}} \max_{\sum\limits_{k=0}^{N-1} u(k)^{\mathrm{T}}Hu(k)\leqslant t^2} \left\langle p, \sum_{k=0}^{N-1} A^k B u(N-k-1)\right\rangle$$

$$= \sup_{N\in\mathbb{N}} \max_{\sum\limits_{k=0}^{N-1} u(k)^{\mathrm{T}}Hu(k)\leqslant t^2} \sum_{k=0}^{N-1} \left\langle \left(A^kB\right)^{\mathrm{T}} p, u(N-k-1)\right\rangle = \sup_{N\in\mathbb{N}} \max_{u_0^{\mathrm{T}}H_0u_0\leqslant t^2} \langle p_0, u_0\rangle .$$

Taking into account the invertibility of the matrix H_0 and definitions (8), (9), it is true that

$$\mathcal{S}(p, \mathcal{Y}_{t,2,\infty}) = \sup_{N\in\mathbb{N}} t\sqrt{p_0^{\mathrm{T}}H_0^{-1}p_0} = t \sup_{N\in\mathbb{N}} \sqrt{p^{\mathrm{T}} \sum_{k=0}^{N-1} A^kBH^{-1}B^{\mathrm{T}}(A^k)^{\mathrm{T}}p}$$

$$= t\sqrt{p^{\mathrm{T}} \sum_{k=0}^{\infty} A^kBH^{-1}B^{\mathrm{T}}(A^k)^{\mathrm{T}}p} = t\sqrt{p^{\mathrm{T}}H_{\mathcal{Y}}p}.$$

The last transition is correct, since by Lemma 1 the corresponding series converges to $H_{\mathcal{Y}}$.

Item 2 of the theorem follows from the fact that if $\det A \neq 0$, then due to definitions (2) and (3) the set $\mathcal{Y}_{t,2,\infty}$ for the system (A, B, H, t) and the set $\mathcal{X}_{t,2,\infty}$ for the system $(A^{-1}, A^{-1}B, H, t)$ coincide.

Theorem 2 is proved.

Proof of Lemma 2. Let us prove item 1. The series $\sum\limits_{k=0}^{\infty} A^kBH^{-1}B^{\mathrm{T}}(A^k)^{\mathrm{T}}$, due to Lemma 1, converges to a non-negative definite symmetric matrix $H_{\mathcal{Y}}$. The following representation holds:

$$\sum_{k=0}^{n-1} A^kBH^{-1}B^{\mathrm{T}}(A^k)^{\mathrm{T}} = (B\ \dots\ A^{n-1}B) \begin{pmatrix} H^{-1} & \dots & O \\ \vdots & \ddots & \vdots \\ O & \dots & H^{-1} \end{pmatrix} \begin{pmatrix} B^{\mathrm{T}} \\ \vdots \\ (A^{n-1}B)^{\mathrm{T}} \end{pmatrix}.$$

Moreover, according to the controllability condition, each of the factors is a matrix of full rank. Consequently, their product is also a matrix of full rank [16, Section 0.4.6], which means that $\det \sum\limits_{k=0}^{n-1} A^kBH^{-1}B^{\mathrm{T}}(A^k)^{\mathrm{T}} \neq 0$. Since $\sum\limits_{k=0}^{n-1} A^kBH^{-1}B^{\mathrm{T}}(A^k)^{\mathrm{T}}$ is also non-negative definite and symmetric by construction, this matrix turns out to be positive definite. Then $H_{\mathcal{Y}}$ is also positive definite, since it can be represented as the sum of positive definite and non-negative definite matrices:

$$H_{\mathcal{Y}} = \sum_{k=n}^{\infty} A^kBH^{-1}B^{\mathrm{T}}(A^k)^{\mathrm{T}} + \sum_{k=0}^{n-1} A^kBH^{-1}B^{\mathrm{T}}(A^k)^{\mathrm{T}}.$$

Item 2 of the lemma follows from the fact that the eigenvalues of A are mutually inverse to the eigenvalues of A^{-1}.

Lemma 2 is proved.

Lemma 3. *Let $r, q \in \mathbb{N} \cup \{0\}$, and let $\lambda, \mu \in \mathbb{C}$ be numbers such that $|\lambda| < 1$, $|\mu| < 1$. Then*

$$\sum_{k=\max\{r,q\}}^{\infty} C_k^r C_k^q \lambda^{k-r} \mu^{k-q} = \sum_{l=0}^{\min\{r,q\}} \frac{(r+q-l)!}{(r-l)!(q-l)!l!} \frac{\lambda^{q-l}\mu^{r-l}}{(1-\lambda\mu)^{r+q-l+1}}.$$

Proof of Lemma 3. To prove the lemma, we use the well-known formula for $|\alpha| < 1$:

$$\sum_{k=0}^{\infty} C_{k+r}^r \alpha^k = \frac{1}{(1-\alpha)^{r+1}}. \tag{A.7}$$

We choose neighborhoods of zero in the complex plane $\mathcal{O}_{\delta_1}(0), \mathcal{O}_{\delta_2}(0) \subset \mathbb{C}$ so that for any $x \in \mathcal{O}_{\delta_1}(0)$ and $y \in \mathcal{O}_{\delta_2}(0)$ the following inequalities hold:

$$|(x+\lambda)(y+\mu)| < 1,$$
$$\left|\frac{x(y+\mu)}{1-\lambda(y+\mu)}\right| < 1,$$
$$\left|\frac{y\lambda}{1-\lambda\mu}\right| < 1.$$

Such neighborhoods exist because $|\lambda| < 1$, $|\mu| < 1$. Consider the function $G\colon \mathcal{O}_{\delta_1}(0) \times \mathcal{O}_{\delta_2}(0) \to \mathbb{C}$ of the following form:

$$G(x,y) = \frac{1}{1-(x+\lambda)(y+\mu)}.$$

On the one side,

$$G(x,y) \overset{\text{(A.7)}}{=} \sum_{k=0}^{\infty} (x+\lambda)^k (y+\mu)^k = \sum_{k=0}^{\infty} \sum_{r=0}^{k} \sum_{q=0}^{k} C_k^q C_k^r \lambda^{k-r} \mu^{k-q} x^r y^q$$
$$= \sum_{r=0}^{\infty} \sum_{q=0}^{\infty} \left(\sum_{k=\max\{r,q\}}^{\infty} C_k^q C_k^r \lambda^{k-r} \mu^{k-q} \right) x^r y^q. \tag{A.8}$$

On the other hand, from the definitions of δ_1 and δ_2 we obtain

$$G(x,y) = \frac{1}{1-\lambda(y+\mu)-x(y+\mu)} = \frac{1}{1-\lambda(y+\mu)} \frac{1}{1-\frac{x(y+\mu)}{1-\lambda(y+\mu)}}$$
$$\overset{\text{(A.7)}}{=} \frac{1}{1-\lambda(y+\mu)} \sum_{r=0}^{\infty} \left(\frac{x(y+\mu)}{1-\lambda(y+\mu)} \right)^r = \sum_{r=0}^{\infty} x^r (y+\mu)^r \frac{1}{(1-\lambda(y+\mu))^{r+1}}$$
$$= \sum_{r=0}^{\infty} \sum_{l=0}^{r} x^r C_r^l y^l \mu^{r-l} \frac{1}{(1-\lambda\mu)^{r+1}} \frac{1}{\left(1-\frac{y\lambda}{1-\lambda\mu}\right)^{r+1}}$$
$$\overset{\text{(A.7)}}{=} \sum_{r=0}^{\infty} \sum_{l=0}^{r} \sum_{q=0}^{\infty} x^r C_r^l y^l \mu^{r-l} \frac{1}{(1-\lambda\mu)^{r+1}} C_{r+q}^r \left(\frac{y\lambda}{1-\mu\lambda} \right)^q$$
$$= \sum_{r=0}^{\infty} \sum_{l=0}^{r} \sum_{q=l}^{\infty} x^r C_r^l y^l \mu^{r-l} \frac{1}{(1-\lambda\mu)^{r+1}} C_{r+q-l}^r \left(\frac{y\lambda}{1-\mu\lambda} \right)^{q-l}$$
$$= \sum_{r=0}^{\infty} \sum_{l=0}^{r} \sum_{q=l}^{\infty} \frac{C_r^l C_{r+q-l}^r \mu^{r-l} \lambda^{q-l}}{(1-\lambda\mu)^{q+r-l+1}} x^r y^q = \sum_{r=0}^{\infty} \sum_{q=0}^{\infty} \left(\sum_{l=0}^{\min\{r,q\}} \frac{C_r^l C_{r+q-l}^r \mu^{r-l} \lambda^{q-l}}{(1-\lambda\mu)^{q+r-l+1}} \right) x^r y^q. \tag{A.9}$$

Since $G(x,y)$ is analytic on its domain by construction, it can be represented as a power series in a unique way [23, Section A, Chapter I]. Therefore, from the representations (A.8) and (A.9) it follows that Lemma 3 is true.

Lemma 4. *Let $r,q \in \mathbb{N}\cup\{0\}$, and let $\lambda,\mu\in\mathbb{C}$ be numbers such that $|\lambda|>1$, $|\mu|>1$. Then*

$$\sum_{k=1}^{\infty} C_{k+r-1}^{r} C_{k+q-1}^{q} \lambda^{-k-r}\mu^{-k-q} = \sum_{l=0}^{\min\{r,q\}} \frac{(r+q-l)!}{(r-l)!(q-l)!l!}\frac{(-1)^l\lambda^{q-l}\mu^{r-l}}{(\lambda\mu-1)^{r+q-l+1}}.$$

Proof of Lemma 4. We choose neighborhoods of zero in the complex plane $\mathcal{O}_{\delta_1}(0),\mathcal{O}_{\delta_2}(0)\subset\mathbb{C}$ so that for any $x\in\mathcal{O}_{\delta_1}(0)$ and $y\in\mathcal{O}_{\delta_2}(0)$ the following inequalities hold:

$$|(\lambda-x)|>1,\ |(\mu-y)|>1,$$

$$\left|\frac{x(\mu-y)}{\lambda(\mu-y)-1}\right|<1,$$

$$\left|\frac{y\lambda}{\lambda\mu-1}\right|<1.$$

Such neighborhoods exist because $|\lambda|>1$, $|\mu|>1$. Consider the function $F\colon \mathcal{O}_{\delta_1}(0)\times\mathcal{O}_{\delta_2}(0)\to\mathbb{C}$ of the following form:

$$F(x,y)=\frac{1}{(\lambda-x)(\mu-y)-1}.$$

On the one side,

$$F(x,y)=\frac{1}{(\lambda-x)(\mu-y)}\frac{1}{1-(\lambda-x)^{-1}(\mu-y)^{-1}}$$

$$\overset{(A.7)}{=}\frac{1}{(\lambda-x)(\mu-y)}\sum_{k=0}^{\infty}(\lambda-x)^{-k}(\mu-y)^{-k}=\sum_{k=0}^{\infty}\frac{1}{(\lambda-x)^{k+1}}\frac{1}{(\mu-y)^{k+1}}$$

$$=\sum_{k=0}^{\infty}\frac{1}{(\lambda\mu)^{k+1}}\frac{1}{\left(1-\frac{x}{\lambda}\right)^{k+1}}\frac{1}{\left(1-\frac{y}{\mu}\right)^{k+1}}$$

$$\overset{(A.7)}{=}\sum_{k=0}^{\infty}\frac{1}{(\lambda\mu)^{k+1}}\sum_{r=0}^{\infty}C_{k+r}^{r}\left(\frac{x}{\lambda}\right)^r\sum_{q=0}^{\infty}C_{k+q}^{q}\left(\frac{y}{\mu}\right)^q$$

$$=\sum_{r=0}^{\infty}\sum_{q=0}^{\infty}\sum_{k=0}^{\infty}C_{k+r}^{r}C_{k+q}^{q}\lambda^{-r-k-1}\mu^{-q-k-1}x^ry^q$$

$$=\sum_{r=0}^{\infty}\sum_{q=0}^{\infty}\left(\sum_{k=1}^{\infty}C_{k+r-1}^{r}C_{k+q-1}^{q}\lambda^{-r-k}\mu^{-q-k}\right)x^ry^q. \tag{A.10}$$

On the other hand,

$$F(x,y)=\frac{1}{\lambda(\mu-y)-1-x(\mu-y)}=\frac{1}{\lambda(\mu-y)-1}\frac{1}{1-\frac{x(\mu-y)}{\lambda(\mu-y)-1}}$$

$$\overset{(A.7)}{=}\frac{1}{\lambda(\mu-y)-1}\sum_{r=0}^{\infty}\left(\frac{x(\mu-y)}{\lambda(\mu-y)-1}\right)^r=\sum_{r=0}^{\infty}\frac{x^r(\mu-y)^r}{(\lambda\mu-1-\lambda y)^{r+1}}$$

$$=\sum_{r=0}^{\infty}\sum_{l=0}^{r}x^rC_r^l(-y)^l\mu^{r-l}\frac{1}{(\lambda\mu-1)^{r+1}}\frac{1}{\left(1-\frac{y\lambda}{\lambda\mu-1}\right)^{r+1}}$$

$$\overset{(A.7)}{=}\sum_{r=0}^{\infty}\sum_{l=0}^{r}\sum_{q=0}^{\infty}x^r C_r^l(-y)^l\mu^{r-l}\frac{1}{(\lambda\mu-1)^{r+1}}C_{r+q}^r\left(\frac{y\lambda}{\lambda\mu-1}\right)^q$$

$$=\sum_{r=0}^{\infty}\sum_{l=0}^{r}\sum_{q=l}^{\infty}x^r C_r^l(-y)^l\mu^{r-l}\frac{1}{(\lambda\mu-1)^{r+1}}C_{r+q-l}^r\left(\frac{y\lambda}{\lambda\mu-1}\right)^{q-l}$$

$$=\sum_{r=0}^{\infty}\sum_{l=0}^{r}\sum_{q=l}^{\infty}\frac{(-1)^l C_r^l C_{r+q-l}^r\mu^{r-l}\lambda^{q-l}}{(\lambda\mu-1)^{q+r-l+1}}x^r y^q$$

$$=\sum_{r=0}^{\infty}\sum_{q=0}^{\infty}\left(\sum_{l=0}^{\min\{r,q\}}\frac{(-1)^l C_r^l C_{r+q-l}^r\mu^{r-l}\lambda^{q-l}}{(1-\lambda\mu)^{q+r-l+1}}\right)x^r y^q. \tag{A.11}$$

Since $F(x,y)$ is analytic on its domain by construction, it can be represented as a power series in a unique way [23, Section A, Chapter I]. Therefore, from the representations (A.10) and (A.11) it follows that Lemma 4 is true.

Proof of Theorem 3. Let us prove item 1. Due to (10) and the definition of $H_{\mathcal{Y}}$

$$H_{\mathcal{Y}}=S\left(\sum_{k=0}^{\infty}\Lambda^k S^{-1}BH^{-1}B^{\mathrm{T}}(S^{-1})^{\mathrm{T}}(\Lambda^k)^{\mathrm{T}}\right)S^{\mathrm{T}}$$

$$=S\left(\sum_{k=0}^{\infty}\begin{pmatrix}J_1^k D_{11}(J_1^k)^{\mathrm{T}} & \dots & J_1^k D_{1M}(J_M^k)^{\mathrm{T}}\\ \vdots & \ddots & \vdots\\ (J_M^k)^{\mathrm{T}}D_{M1}(J_1^k)^{\mathrm{T}} & \dots & J_M^k D_{MM}(J_M^k)^{\mathrm{T}}\end{pmatrix}\right)S^{\mathrm{T}},$$

i.e., it is true that

$$H_{ij}^{\mathcal{Y}}=\sum_{k=0}^{\infty}J_i^k D_{ij}(J_j^k)^{\mathrm{T}}.$$

From the inclusion $b_1,\dots,b_m\in\mathbb{L}_{<1}$ and (10) it follows that $D_{ij}=O$ if $|\lambda_i|\geqslant 1$ or $|\lambda_j|\geqslant 1$, i.e., $H_{ij}^{\mathcal{Y}}=O$. Otherwise, we take into account that for $k\geqslant n_0$ the equality $P_{n_0}^k=O$ holds, and consider the series

$$\sum_{k=0}^{\infty}J_i^k D_{ij}(J_j^k)^{\mathrm{T}}=\sum_{k=0}^{\infty}\left(\lambda_i I+P_{n_i}\right)^k D_{ij}\left(\lambda_j I+P_{n_j}^{\mathrm{T}}\right)^k$$

$$=\sum_{k=0}^{\infty}\sum_{r=0}^{\min\{k,n_i-1\}}\sum_{q=0}^{\min\{k,n_j-1\}}C_k^r\lambda_i^{k-r}P_{n_i}^r D_{ij}C_k^q\lambda_j^{k-q}(P_{n_j}^q)^{\mathrm{T}}$$

$$=\sum_{r=0}^{n_i-1}\sum_{q=0}^{n_j-1}\sum_{k=\max\{r,q\}}^{\infty}C_k^r C_k^q\lambda_i^{k-r}\lambda_j^{k-q}P_{n_i}^r D_{ij}(P_{n_j}^q)^{\mathrm{T}}.$$

Then, taking into account Lemma 3, item 1 is proved.

Now we prove item 2. Due to (10) and the definition of $H_{\mathcal{X}}$

$$H_{\mathcal{X}}=S\left(\sum_{k=1}^{\infty}\Lambda^{-k}S^{-1}BH^{-1}B^{\mathrm{T}}(S^{-1})^{\mathrm{T}}(\Lambda^{-k})^{\mathrm{T}}\right)S^{\mathrm{T}}$$

$$=S\left(\sum_{k=1}^{\infty}\begin{pmatrix}J_1^{-k} D_{11}(J_1^{-k})^{\mathrm{T}} & \dots & J_1^{-k} D_{1M}(J_M^{-k})^{\mathrm{T}}\\ \vdots & \ddots & \vdots\\ (J_M^{-k})^{\mathrm{T}}D_{M1}(J_1^{-k})^{\mathrm{T}} & \dots & J_M^{-k} D_{MM}(J_M^{-k})^{\mathrm{T}}\end{pmatrix}\right)S^{\mathrm{T}},$$

i.e., it is true that

$$H_{ij}^{\mathcal{X}} = \sum_{k=1}^{\infty} J_i^{-k} D_{ij} (J_j^{-k})^{\mathrm{T}}.$$

According to [16, Section 3.2.5] the following representation is valid for the Jordan cell:

$$J_i^{-k} = \sum_{r=0}^{n_i-1} (-1)^r C_{k+r-1}^r \frac{1}{\lambda_i^{k+r}} P_{n_i}^r. \tag{A.12}$$

From the inclusion $b_1, \ldots, b_m \in \mathbb{L}_{>1}$ and (10) it follows that $D_{ij} = O$ if $|\lambda_i| \leqslant 1$ or $|\lambda_j| \leqslant 1$, i.e., $H_{ij}^{\mathcal{X}} = O$. Otherwise, we take into account that for $k \geqslant n_0$ the equality $P_{n_0}^k = O$ holds, and consider the series

$$\sum_{k=1}^{\infty} J_i^k D_{ij} (J_j^k)^{\mathrm{T}} = \sum_{k=1}^{\infty} (\lambda_i I + P_{n_i})^{-k} D_{ij} \left(\lambda_j I + P_{n_j}^{\mathrm{T}}\right)^{-k}$$

$$\overset{(A.12)}{=} \sum_{k=0}^{\infty} \sum_{r=0}^{n_i-1} \sum_{q=0}^{n_j-1} (-1)^{r+q} C_{k+r-1}^r C_{k+q-1}^q \frac{1}{\lambda_i^{k+r}} \frac{1}{\lambda_j^{k+q}} P_{n_i}^r D_{ij} (P_{n_j}^q)^{\mathrm{T}}$$

$$= \sum_{r=0}^{n_i-1} \sum_{q=0}^{n_j-1} \sum_{k=1}^{\infty} C_{k+r-1}^r C_{k+q-1}^q (-\lambda_i)^{-k-r} (-\lambda_j)^{-k-q} P_{n_i}^r D_{ij} (P_{n_j}^q)^{\mathrm{T}}.$$

Then, taking into account Lemma 4, item 2 is proved.

Theorem 3 is proved.

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