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STOCHASTIC SYSTEMS

The Use of Optimal Filtering Methods for Passive Monitoring of Available Bandwidth of a Network Channel

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Abstract—The article is devoted to the development of mathematical support for the solution to an applied problem of the available bandwidth estimation for a network data transmission channel based on the indirect observations of one of the transmitted flows. The problem is transformed to the state filtering of a Markov jump process given some indirect perfect (noiseless) and counting observations. The obtained estimates are represented as solutions to some coupled systems of ordinary differential equations and recursive relations. The performance of the proposed estimates is illustrated by a numerical example.

Keywords: available bandwidth of a channel, Markov jump process, perfect observations, martingale representation, optimal filtering equations

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1. INTRODUCTION

The problem of real-time available bandwidth (ABW) estimation in various telecommunication channels [1–4] is highly relevant for its results to be further used in

— computer network management systems to control the efficiency of network resources utilization,
— congestion control algorithms of the transport protocols,
— multimedia information streaming systems,
— algorithms for resource allocation of software defined networks, etc.

The way ABW and related numerical indicators are understood varies in different publications and may imply

— maximum residual capacity of the given channel at the current load by external flows,
— maximum data transmission rate (throughput) through the channel ensured using some fixed protocol (UDP, TCP, etc.) at the current load by external flows,
— maximum rate of useful data transmission (goodput) through the channel at the current load ensured using the selected protocol under additional requirements to the quality of service (QoS) such as maximum permissible delay, jitter, packet loss ratio, etc.

Currently, there is a whole palette of hardware and software tools for solving this problem. In terms of the statistical information involved, they are divided into active and passive. Active ones use additional service traffic, which is a sequence of small packets, possibly of variable size. The difference between the packets sent and received, including the gaps between them, serve as the basis for calculating the current ABW value. In passive tools, this value is calculated on the basis of information about the real current traffic in the given channel collected with the help of operating systems tools (such as the *tcpdump* utility). We should separately mention the ABW estimation tools based on channel models constructed mainly on queuing systems. It applies simulation modeling rather than real statistical information.

Processing of real data for ABW estimation relies on relatively simple probabilistic models, in particular, linear stochastic observation systems. These are the ones that allow applying the classical Kalman filter [5–7]. At present, the theory of state estimation of stochastic dynamic systems is sufficiently developed, within its framework one can select a model more similar to the operation of a network channel and construct a numerically efficient algorithm for it that estimates the state of the system based on the available data.

This work deals with using the mathematical framework of Markov jump processes (MJP) to construct mathematical models of packet data transmission channels. They are designed to solve the problem of real-time estimation of channel characteristics responsible for ABW from heterogeneous statistical information. The paper has the following structure. Section 2 introduces the class of network channels and transmitted data flows under study and the structure of available observations. The section presents arguments in favor of using the MJP concept to describe the evolution of channel characteristics.

Section 3 contains the theoretical framework for solving the applied ABW monitoring problem. Section 3.1 introduces the observation system under study. Its hidden state to be estimated is a homogeneous MJP with a finite set of states. Some of the observational components are MJP functions recorded without noise while some are Cox processes whose intensity depends on the state. The problem of filtering the MJP state using the available observations is proposed to be considered as a theoretical basis for solving the applied monitoring problem. Section 3.2 deals with solving it. The sought filtering estimate is described by a system of connected ordinary differential equations and recurrence relations.

Note that the proposed optimal filtering problem solved in this work differs from the problems studied in classical monographs [8–10]. In the mentioned works, the structure of observations is such that they can be transformed to a set of Wiener and Poisson processes by a suitable change of the probability measure. In that case, the obtained equations could be interpreted to some extent as different versions of the Kallianpur–Striebel formula [11]. This transition is possible if the condition of nondegeneracy of martingales in observations is fulfilled. By contrast, in the proposed stochastic system, however, some of the observations do not contain noise at all, which makes it impossible to apply the Girsanov transformation of the measure. At the same time, the equations describing the optimal filtering estimate can be treated as a special case of the abstract formula for the optimal filtering of a semimartingale given the observation of semimartingale [12].

Section 4 contains an illustrative example of solving the ABW channel monitoring problem. The channel processes two independent packet flows. The first one is described by a Poisson process with the known intensity. The second hidden flow is described by a Cox process whose intensity varies according to some MJP. The observations include the number of packets from the first flow present in the channel and the sequence of the packets lost from this flow due to congestion. The channel itself is a simple exponential service element with the known intensity combined with a pool of packets of the known capacity. Packets from the pool are randomly selected for transmission. The current ABW depends on its occupancy rate and the intensity of packets arriving from the second flow, so these are the characteristics that are proposed to be estimated. Since no service packet flows are used to obtain statistical information, the proposed monitoring algorithm is categorized as passive. The numerical experiment presented in the section illustrates the high quality of the proposed estimates.

Section 5 presents the analysis of the obtained results and directions for further research.

2. STATEMENT OF THE APPLIED PROBLEM OF AVAILABLE BANDWIDTH MONITORING

We describe the functioning of a network channel of packet data transmission in the form of a controlled stochastic observation system. The channel ensures data transmission of several data flows described by individual characteristics such as

— the intensity of packet arrival from the flow,
— the size of individual packets,
— the total amount of transmitted data,
— the data transmission control protocol, etc.

The channel itself is a set of telecommunication equipment and transmission lines characterized by

— the number of channel hops and their characteristics,
— characteristics of individual network devices (capacity, buffer sizes, internal software characteristics), etc.

Ideally, the channel state is a "snapshot" of the location and movement of the various target and service packets in all parts that make up the given channel, as well as all input and output packet flows, including lost packets.

The channel ABW estimation problem is to determine the maximum packet data flow that could be transmitted through the channel given its current load. In this statement, the problem is unlikely to have an exhaustive solution due to the following facts.

(1) Determining the maximum data flow that can be additionally transmitted through the channel in its current state depends on a number of additional characteristics such as the type of additional data (the protocol type), reliability of data transmission, etc. The point is that the additional bandwidth must be calculated taking into account all overheads and redundancy, including the transmission of service packets, retransmission of lost data and so on. For example, the bandwidth for the subsequent use of UDP traffic will be higher than for TCP since the latter involves resending packets which are not confirmed by the reciever via a special acknowledgement flow.

(2) The channel state mentioned above must have a huge dimension that prevents it from being used in any practical estimation tasks.

(3) The channel characteristics contain uncertainties of different nature, viz.

— the parameters of the individual transmission hops, which form the channel, are usually unknown,
— characteristics of communication devices (their transmission speed,, buffer/storage size) are partially or completely unknown,
— the firmware of the communication devices is proprietary with unknown performance and implemented algorithms,
— network channelling equipment may be simultaneously used by several channels, entailing additional uncertainty of its performance.

(4) Data flows transmitted by the channel also have properties that negatively affect the quality and the very possibility of solving the ABW monitoring problem as they are nonstationary, contain a priori uncertainty in their characteristics, and are partially or completely unobservable due to information security and access sharing restrictions.

In addition, the model is bulky for solving the mentioned practical problem. For this purpose, it is sufficient to consider only the "bottle neck" of the channel, viz. the section with the lowest performance. At the same time, relatively simple queuing systems consisting of service elements, queues or temporary packet storage buffers can be used to describe its operation.

Packet flows can be described by generalized renewal processes [13] — the latter both represent random event flows and may contain some additional packet header information important for subsequent estimation of the channel characteristics. Generally speaking, statistical information available for passive ABW monitoring can include

— part of input information flows,
— part of packet loss flows arising due to various reasons,
— part of service flows such as acknowledgements,
— characteristics of buffer occupancy with packets of the observable flows,
— additional numerical characteristics of individual packets of observable flows, (individual numbers, packet sending/receiving timestamps, etc.).

The state of the communication channel should determine the pair “bottleneck state — total load of the channel”. In their mathematical nature, the available observations can also be divided into two different categories, viz. counting processes with their intensity depending on the system state and some functions of the system state observed without additional noise.

As mentioned above, the ABW of a real channel depends on the type of additional load; however, in any case, it will be described by some function of the system state—the current total intensity of packet flows entering the channel and the degree of the channel occupancy. These are the ones that are proposed to be estimated using the available statistical information and then recalculated into ABW of the added flow of some type.

The additional assumption of the Markov property of the observation system under study is certainly a limitation. Nevertheless, it does not appear to be excessive. First, semi-Markov systems (Markov recovery processes) can be reduced to such systems by a suitable extension of the state vector [14–16]. Second, a wide class of non-Markov systems can be approximated using Markov systems [17]. Third, the mathematical framework of Markov processes supported by the theory of martingales allows us to solve a wide class of optimal state and parameter estimation problems. All these conclusions explain the subsequent choice of stochastic differential observation systems class describing the channel state and its filtering.

3. OPTIMAL FILTERING PROBLEM FOR THE STATE OF A MARKOV JUMP PROCESS BY A SET OF NOISELESS AND COUNTING OBSERVATIONS

In what follows, we use the following designations.

— $\mathbf{I}_{\mathcal{A}}(x)$ is the indicator function of the set $\mathcal{A}$,
— $\mathbb{S}^N = \{e_1, \dots, e_N\}$ is the set of coordinate unit vectors in $\mathbb{R}^N$,
— $\mathrm{col}(a^1, \dots, a^N)$ is the column vector composed of the components a^n, $n = \overline{1, N}$,
— $\mathrm{diag}(a)$ – is a diagonal matrix with the vector a as the diagonal,
— $a \wedge b \triangleq \min(a, b)$.

3.1. Statement of the Filtering Problem

On the complete probability space with filtration $(\Omega, \mathcal{F}, \mathsf{P}, \{\mathcal{F}_t\}_{t\geqslant 0})$ we consider the observation system

$$\theta_t = \theta_0 + \int_0^t A^\top \theta_s ds + M_t^\theta, \quad \theta_0 \sim \pi_0, \tag{1}$$

$$\xi_t = C\theta_t, \tag{2}$$

$$\eta_t = \int_0^t G\theta_s ds + M_t^\eta, \tag{3}$$

where

— $\theta_t = col(\theta_t^1, \dots, \theta_t^N) \in \mathbb{S}^N$ is an unobservable system state representing the $\mathcal{F}_t$-adapted homogeneous MJP with the values in $\mathbb{S}^N$, the transition intensity matrix (TIM) A, and the initial distribution π_0; $M_t^\theta = \mathrm{col}(M_t^{\theta,1}, \dots, M_t^{\theta,N})$ is an $\mathcal{F}_t$-adapted martingale,

— $\xi_t = col(\xi_t^1, \dots, \xi_t^M) \in \mathbb{R}^M$ is a noiseless (perfect) observation process; $C \in \mathbb{R}^{M\times N}$ is the observation plan matrix with the columns c^n, $n = \overline{1,N}$;

— $\eta_t = col(\eta_t^1, \dots, \eta_t^K) \in \mathbb{R}^K$ is an observable process with counting components: the matrix $G \in \mathbb{R}^{K\times N}$ determines conditional jump intensities of individual components η depending on the current state θ (G consists of the rows g^k, $k = \overline{1,K}$); $M_t^\eta = col(M_t^{\eta,1}, \dots, M_t^{\eta,K})$ – is an $\mathcal{F}_t$-adapted martingale.

Suppose $\mathcal{O}_t \triangleq \sigma\{\xi_s, \eta_s :\ 0 \leqslant s \leqslant t\}$ be the natural flow of σ-algebras generated by observable processes. The optimal filtering problem for the state θ_t is to calculate the conditional mathematical expectation (CME) $\widehat{\theta}_t \triangleq \mathsf{E}\{\theta_t|\mathcal{O}_t\}$, $t \in [0,T]$; $T < \infty$ is some finite deterministic instant.

We assume that the considered probability triplet with filtration and the observation system satisfy the following conditions.

A) $\mathcal{F}_t \equiv \sigma\{\theta_s, \eta_s :\ 0 \leqslant s \leqslant t\}$ for $\forall\, t \in [0,T]$.

B) The martingale components $M_t^{\eta,k}$ of the counting observations η_t^k are strongly orthogonal to each other and also orthogonal to the martingale M_t^θ in MJP θ_t:

$$\langle \eta, \eta\rangle_t = \int_0^t \mathrm{diag}(G\theta_s)ds, \qquad \langle \eta, \theta\rangle_t \equiv 0.$$

C) Let $\{\tau_j\}_{j\in\mathbb{Z}_+}$ be the instants of jumps in the block process (θ_t, η_t), and $\{\zeta_j\}_{j\in\mathbb{Z}_+}$ be the instants of observation jumps (ξ_t, η_t), $\tau_0 = \zeta_0 \triangleq 0$. We assume that

$$\lim_{j\to+\infty} \tau_j = \lim_{j\to+\infty} \zeta_j = +\infty \qquad \mathsf{P}-\text{a.s.}$$

Then, Markov points $\tau_j' \triangleq \tau_j \wedge T$ and $\zeta_j' \triangleq \zeta_j \wedge T$ will be bounded by the constant T. In what follows, the primes in the designations of the Markov points are omitted for simplicity.

The intensity matrix G of counting observations can be an arbitrary matrix of a suitable dimension consisting of non-negative elements. There are no such restrictions on the matrix of exact observations C, and it only has to have a suitable dimension. Nevertheless, in practice, the matrix C consists of 0 and 1. Often, noiseless observations ξ_t are represented by information about what some set $\mathbb{S}' \subset \mathbb{S}^N$ contains at the current instant θ_t. In this case, the respective row of C will consist of the indicators $\mathbf{I}_{\mathbb{S}'}(e_n)$, $n = \overline{1,N}$.

3.2. Solving the Filtering Problem

Let $\mathcal{C}$ be the set of different columns of the matrix C. We construct the mapping $\Xi : \mathcal{C} \to \mathbb{R}^{1\times N}$ as follows:

$$\Xi(c) \triangleq \sum_{n:\, Ce_n = c} e_n^\top.$$

$\Xi(\cdot)$ characterizes the complete preimage of the mapping $e \to Ce$ in the following sense:

$$\mathrm{diag}(\Xi(c))e = \begin{cases} e, & \text{if } Ce = c,\\ 0 & \text{otherwise.}\end{cases}$$

We denote: $\overline{\theta}_\ell \triangleq \theta_{\zeta_\ell}$, $\overline{\xi}_\ell \triangleq \xi_{\zeta_\ell}$, $\overline{\eta}_\ell \triangleq \eta_{\zeta_\ell}$. We consider a non-decreasing sequence of σ-algebras $\mathfrak{O}_j \triangleq \sigma\{\zeta_\ell, \overline{\xi}_\ell, \overline{\eta}_\ell :\ 0 \leqslant \ell \leqslant j\}$. It is known [19] that $\mathfrak{O}_j \equiv \mathcal{O}_{\zeta_j}$ for all $j \in \mathbb{Z}_+$.

We also construct families of σ-algebras

$$\mathbf{O}_{j,t} \triangleq \sigma\{A \in \mathfrak{O}_j,\ \{\omega:\ t \in [\zeta_j(\omega), \zeta_{j+1}(\omega))\}\}.$$

Obviously, the σ-algebras $\mathbf{O}_{j,t}$ are richer than $\mathfrak{O}_j$, as they are augmented with random events of the form $\{\omega \in \Omega:\ \zeta_j(\omega) \leqslant t < \zeta_{j+1}(\omega)\}$, which carry the following meaning: there have been exactly j jumps of observations by the instant t.

To derive optimal filtering equations, the following auxiliary propositions are required.

Lemma 1. *Suppose* $\widehat{\pi}_j \triangleq \mathsf{E}\left\{\theta_{\zeta_j}|\mathfrak{O}_j\right\}$. *Then* P*-a.s. the following equalities are true*

$$\mathbf{I}_{[\zeta_j,+\infty)}(t)\mathsf{E}\left\{\theta_t \mathbf{I}_{[\zeta_j,\zeta_{j+1})}(t)|\mathfrak{O}_j\right\} = \mathbf{I}_{[\zeta_j,+\infty)}(t) m_t, \tag{4}$$

$$\mathbf{I}_{[\zeta_j,\zeta_{j+1})}(t)\mathsf{E}\left\{\theta_t|\mathbf{O}_{j,t}\right\} = \mathbf{I}_{[\zeta_j,\zeta_{j+1})}(t)\mu_t, \tag{5}$$

where the functions m_t *and*

$$\mu_t = (\mathbf{1} m_t)^{-1} m_t \tag{6}$$

are the solutions to the following systems of ordinary differential equations:

$$\begin{cases} \dot{m}_t = \left[\operatorname{diag}(\Xi(\overline{\xi}_j))A^\top - \sum_{k=1}^K \operatorname{diag}(g^k)\right] m_t, \quad t > \zeta_j, \\ m_{\zeta_j} = \widehat{\pi}_j, \end{cases} \tag{7}$$

$$\begin{cases} \dot{\mu}_t = \left[\operatorname{diag}(\Xi(\overline{\xi}_j))A^\top - \sum_{k=1}^K \operatorname{diag}(g^k)\right] \mu_t - \mu_t \left[\Xi(\overline{\xi}_j)A^\top - \sum_{k=1}^K g^k\right] \mu_t, \quad t > \zeta_j, \\ \mu_{\zeta_j} = \widehat{\pi}_j. \end{cases} \tag{8}$$

Proof of Lemma 1 is given in Appendix.

Lemma 2. *The estimate* $\widehat{\pi}_{j+1} \triangleq \mathsf{E}\left\{\theta_{\zeta_{j+1}}|\mathfrak{O}_{j+1}\right\}$ *is specified by the formula*

$$\begin{aligned} \widehat{\pi}_{j+1} = \sum_{k=1}^K \left(g^k \mu_{\zeta_{j+1}}\right)^{-1} \operatorname{diag}(g^k)\mu_{\zeta_{j+1}}(\overline{\eta}^k_{j+1} - \overline{\eta}^k_j) \\ + \left(\Xi(\overline{\xi}_{j+1})A^\top \mu_{\zeta_{j+1}}\right)^{-1} \operatorname{diag}\left(\Xi(\overline{\xi}_{j+1})\right)\left(I - \operatorname{diag}\left(\Xi(\overline{\xi}_j)\right)\right) A^\top \mu_{\zeta_{j+1}}, \end{aligned} \tag{9}$$

where the vector $\mu_{\zeta_{j+1}}$ *is the solution to* (8) *taken at the instant* ζ_{j+1}.

The proof of Lemma 2 is given in the Appendix.

Lemmas 1 and 2 allow us to prove the main proposition of this work.

Theorem 1. *The optimal filtering estimate* $\widehat{\theta}_t$ *can be represented as*

$$\widehat{\theta}_t = \mathsf{E}\left\{\theta_t|\mathcal{O}_t\right\} = \sum_{j\geqslant 0} \mathbf{I}_{[\zeta_j,\zeta_{j+1})}(t)\mu_t, \tag{10}$$

where the functions μ_t *are specified by the solution of* (8) *on the intervals* $[\zeta_j, \zeta_{j+1})$. *At the instants* ζ_{j+1} *of jumps of observations* (ξ_t, η_t), *the estimate* $\widehat{\theta}_{j+1} = \widehat{\pi}_{j+1}$ *is calculated using recurrence relation* (9); *the filtering estimate at the initial instant is*

$$\widehat{\theta}_0 = (\Xi(\xi_0)\pi_0)^{-1} \operatorname{diag}\left(\Xi(\xi_0)\right)\pi_0. \tag{11}$$

The estimate $\widehat{\theta}_t$ *is the solution to the stochastic system*

$$\begin{aligned}
\widehat{\theta}_t &= (\Xi(\xi_0)\pi_0)^{-1}\operatorname{diag}(\Xi(\xi_0))\,\pi_0 \\
&+\int_{\zeta_j}^{t}\left[\left(\operatorname{diag}(\Xi(\xi_s))A^\top-\sum_{k=1}^{K}\operatorname{diag}(g^k)\right)\widehat{\theta}_s-\widehat{\theta}_s\left(\Xi(\xi_s)A^\top-\sum_{k=1}^{K}g^k\right)\widehat{\theta}_s\right]ds \\
&+\sum_{\zeta_j:\zeta_j\leqslant t}\left[\sum_{k=1}^{K}\left(g^k\widehat{\theta}_{\zeta_j-}\right)^{-1}\operatorname{diag}(g^k)\widehat{\theta}_{\zeta_j-}\Delta\eta^k_{\zeta_j}\right. \\
&\left.+\left(\Xi(\xi_{\zeta_j})A^\top\widehat{\theta}_{\zeta_j-}\right)^{-1}\operatorname{diag}\left(\Xi(\xi_{\zeta_j})\right)\left(I-\operatorname{diag}\left(\Xi(\xi_{\zeta_j-})\right)\right)A^\top\widehat{\theta}_{\zeta_j-}-\widehat{\theta}_{\zeta_j-}\right].
\end{aligned}\tag{12}$$

Theorem 1 is proved in the Appendix.

Remark 1. Although the integral part of final equation (12) is nonlinear and corresponds to (8), linear system (7) also plays an important role in the numerical implementation of the filtering algorithm. Note that (8) represents a system of Riccati differential equations, numerical solution of which may be difficult for some sets of parameters. The point is that the exact solution μ satisfies the conditions of non-negativity and normalization, and the approximate solution must satisfy the same conditions. Otherwise it loses the probabilistic sense of the conditional distribution, and the approximation itself diverges. To neutralize this disadvantage, more complicated numerical solution algorithms can be used or the time step can be reduced. In contrast to direct numerical solution of (8), μ_t can be computed using (7) *exactly* for any value of the time step h. For this purpose, it is sufficient to compute once the exponential $Q=\exp\left[h\left(\operatorname{diag}(\Xi(\overline{\xi}_j))A^\top-\sum_{k=1}^{K}\operatorname{diag}(g^k)\right)\right]$ and the sum of its rows $q=\mathbf{1}Q$. Then the exact values of the conditional distribution μ_{ζ_j+ih} on a uniform time grid with step h, starting at ζ_j, can be calculated by the simple recurrence

$$\mu_{\zeta_j+(i+1)h}=\frac{1}{q\mu_{\zeta_j+ih}}Q\mu_{\zeta_j+ih},\quad i\in\mathbb{N}.$$

4. NUMERICAL EXAMPLE OF AVAILABLE BANDWIDTH ESTIMATION

We present the channel structure and the structure of information transmitted through it in more detail. Figure 1 shows the channel operating scheme.

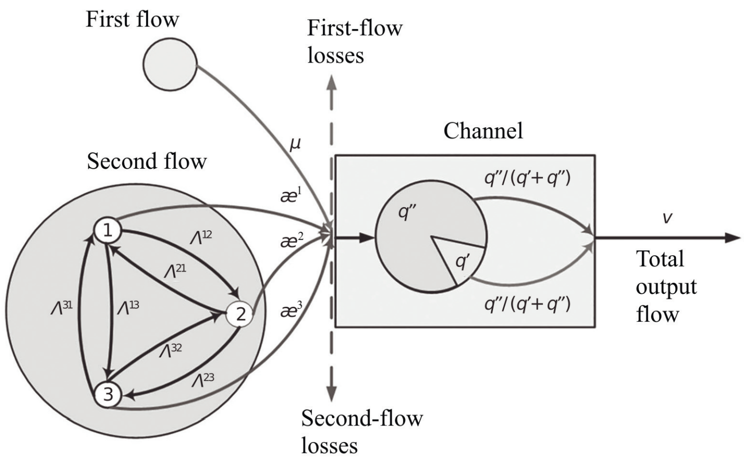

Fig. 1. Scheme of operation of the network channel.

Data in the form of packets arrive in the channel from two independent flows. The first flow—the simplest one with the intensity μ—is partially observable. The second, completely unobservable, is described by a Cox process with the intensity $\varkappa_t$ taking values from the set $\{\varkappa^s\}_{s=\overline{1,S}}$ and varies according to a hidden homogeneous MJP with the known transition intensity matrix $\Lambda = \|\Lambda^{ij}\|_{ij=\overline{1,S}}$. In fact, the second flow is an external integral non-stationary load.

A transmission channel is a service element that can simultaneously contain no more than N^p transmitted packets. Packets arriving in a completely occupied channel are lost. When the channel is not empty, it transmits a packet, spending a random time on it that has an exponential distribution with the constant parameter ν. The transmitted packet is chosen randomly from packets of both flows—if there are q' packets of the first flow and q'' packets of the second flow in the channel at the given instant, then the probabilities that a packet from the first or the second flow will be transmitted are $\frac{q'}{q'+q''}$ and $\frac{q''}{q'+q''}$, respectively. Thus, the considered model implements the *Active Queueing Management* mechanism [19], which provides different flows with fair access to resources in proportion to the number of packets of each flow that are in the channel.

Obviously, the current channel bandwidth is determined by two variables hidden from direct observation, viz. The amount of packets in the server $q_t^\Sigma \triangleq q_t' + q_t''$ and the total packet arrival intensity from the two flows $\varkappa_t^\Sigma \triangleq \mu + \varkappa_t$. These two processes are being monitored.

The arrival processes of packets of both flows into the channel and their processing are described by a unified MJP with a finite set of states $\theta_t = (s_t, q_t', q_t'')$, where s_t is the current state of the second flow ($s = \overline{1,S}$), q_t' is the number of packets of the first flow in the server, and q_t'' is the number of packets of the second flow in the server ($0 \leqslant q', q'' : q_t' + q_t'' \leqslant N^p$). One can easily check that the total number of possible MJP states is $N = \frac{S(N^p+1)(N^p+2)}{2}$.

The matrix A of MJP transition intensities X_t is defined element by element as follows:

— $(i, q', q'') \xrightarrow{\Lambda^{ij}} (j, q', q'')$, $(i, j = \overline{1,S},\ i \neq j,\ q', q'' \geqslant 0:\ q' + q'' \leqslant N^p)$ — change of intensity of the second flow from $\varkappa^i$ to $\varkappa^i$;

— $(s, q', q'') \xrightarrow{\mu} (s, q'+1, q'')$, $(s = \overline{1,S}, q', q'' \geqslant 0:\ q' + q'' \leqslant N^p - 1)$ — arrival of a new packet of the first flow into the channel;

— $(s, q', q'') \xrightarrow{\varkappa^s} (s, q', q''+1)$, $(s = \overline{1,S}, q', q'' \geqslant 0:\ q' + q'' \leqslant N^p - 1)$ — arrival of a new packet of the second flow into the channel;

— $(s, q', q'') \xrightarrow{\frac{q'}{q'+q''}\nu} (s, q'-1, q'')$, $(s = \overline{1,S},\ q' > 0,\ q'' \geqslant 0:\ q' + q'' \leqslant N^p)$ — transmission of the first flow packet through the channel;

— $(s, q', q'') \xrightarrow{\frac{q''}{q'+q''}\nu} (s, q', q''-1)$, $(s = \overline{1,S},\ q' \geqslant 0,\ q'' > 0:\ q' + q'' \leqslant N^p)$ — transmission of the second flow packet through the channel.

To estimate the characteristics q_t^Σ and $\varkappa_t^\Sigma$, one can use the following statistical information:

— continuous observations of the number of packets of the first flow currently in the channel: $\xi_t = q_t'$,

— the process that counts packet losses of the first flow caused by channel overflow: $\eta_t = \int_0^t \mathbf{I}_{\{N^p\}}(q_u^\Sigma)\mu du + M_t^\eta$.

We performed numerical experiments for the following parameter values: $N^p = 32$, $S = 3$, $N = 1683$, $\mu = 1$, $\nu = 13$, $T = 2000$,

$$\Lambda = \begin{bmatrix} -0.002 & 0.001 & 0.001 \\ 0.001 & -0.002 & 0.001 \\ 0.001 & 0.001 & -0.002 \end{bmatrix}, \qquad \varkappa = \begin{bmatrix} 1 \\ 5 \\ 11 \end{bmatrix}.$$

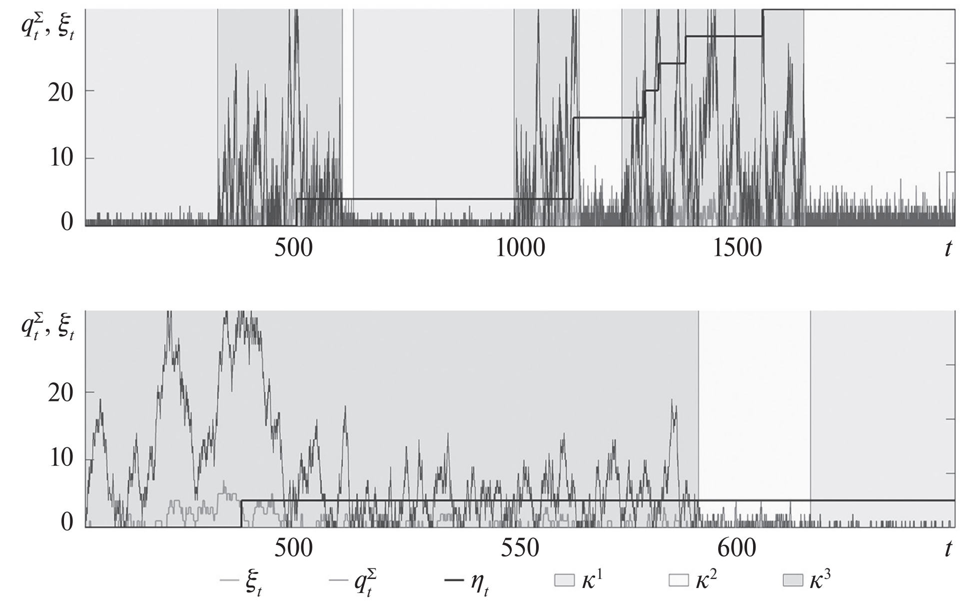

Fig. 2. Evolution of channel load and available observations.

The initial distribution of the MJP describing packet transmission coincides with a stationary one. Simulation of all processes and search for the numerical solution to the optimal filtering problem was performed with the time step $h = 0.01$.

Figure 2 gives information about the hidden state of the channel and available observations:

— the hidden intensity state of the second flow $\varkappa_t$ (displayed as the background filling),

— the hidden channel load q_t^Σ,

— the observed number of packets of the first flow ξ_t that are in the channel,

— the observable counting process of packet losses of the first flow η_t (the values are displayed on the right ordinate axis).

The MJP state filtering estimate $\widehat{\theta}_t$ obtained by solving (12) is a vector whose components are conditional probabilities $\mathsf{P}\left\{s_t = S, q'_t = Q', q''_t = Q'' | \mathcal{O}_t\right\}$. Using the vector $\widehat{\theta}_t$, we can calculate the estimates of the current total channel load $\widehat{q}_t^\Sigma$:

$$\widehat{q}_t^\Sigma = \sum_{s,q',q''} (q' + q'')\mathsf{P}\left\{s_t = s, q'_t = q', q''_t = q'' | \mathcal{O}_t\right\}, \tag{13}$$

and the estimates $\widehat{\varkappa}_t^\Sigma$ of the current total intensity of the packets arriving in the channel:

$$\widehat{\varkappa}_t^\Sigma = \sum_{s,q',q''} \varkappa^s \mathsf{P}\left\{s_t = s, q'_t = q', q''_t = q'' | \mathcal{O}_t\right\}. \tag{14}$$

These characteristics, in turn, allow for real-time ABW estimation under different QoS conditions. We consider the channel operating with the assumption that the second flow is also simple with the constant intensity $\varkappa$. Depending on this parameter, we calculate the average number of packets in the channel $\mathsf{E}\left\{q^\Sigma\right\} = E(\varkappa)$ and the probability $\mathsf{P}\left\{q^\Sigma = N^p\right\} = P_\ell(\varkappa)$ of the packet loss in the stationary mode. Figure 3 shows the dependences $E(\varkappa)$ and $P_\ell(\varkappa)$ (on the auxiliary ordinate axis).

Suppose that the QoS requirement is fixed in the form of an upper bound for the packet loss probability $\overline{\mathsf{P}}_\ell$. We assume that the maximum bandwidth of this channel $\overline{B}$ equals the total intensity

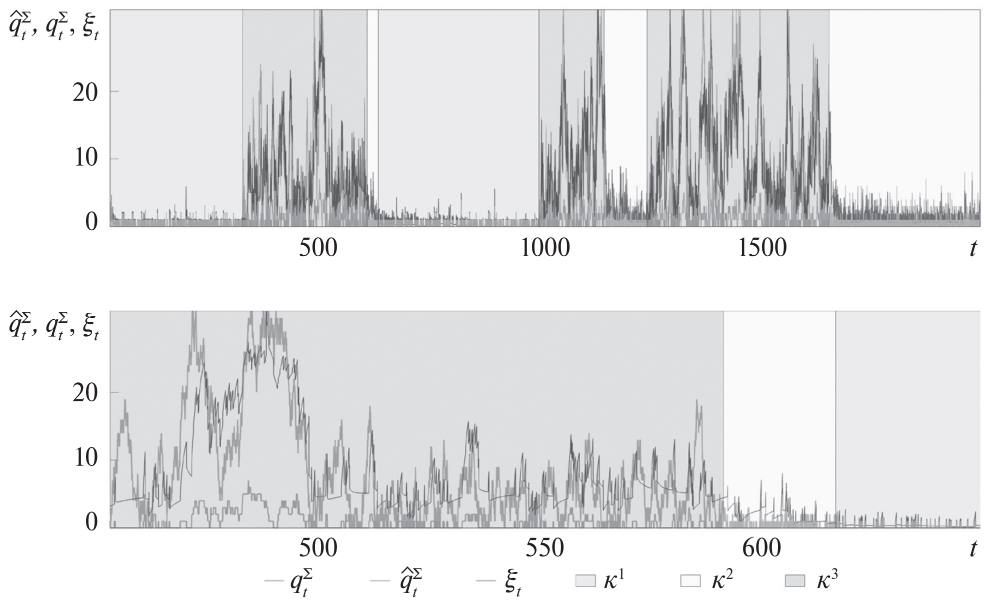

Fig. 3. Average number $E(\varkappa)$ of the packets in the channel and probability of packet loss $P(\varkappa)$.

of both flows, provided that the packet loss probability does not exceed $\overline{\mathsf{P}}_\ell : \overline{B} \triangleq P^{-1}(\overline{\mathsf{P}}_\ell) + \mu$. For example, if we choose $\overline{\mathsf{P}} = 0.05$ the respective maximum bandwidth is $\overline{B} = 12.45$. Then, we propose to take the difference $B_t^a \triangleq \overline{B} - \varkappa_t^\Sigma$, i.e., such a maximum addition to the current intensity of the second flow that does not violate the packet loss probability constraint, as the ABW at the instant t. However, the variable $\varkappa_t^\Sigma$ cannot be observed directly, so we propose to use the variable $\widehat{B}_t^a \triangleq \max(\overline{B} - \widehat{\varkappa}_t^\Sigma, 0)$, which is a function of the obtained estimate $\widehat{\varkappa}_t^\Sigma$, as an ABW estimate.

We consider another type of QoS requirement in the form of an upper bound $\overline{T}$ for the average packet transmission time. If a packet is currently on the server with the current total number of packets q_t^Σ and the channel is in the stationary mode, the average transmission time can be characterized by $\frac{q_t^\Sigma}{\nu}$. Thus, for this QoS requirement to be met, the maximum allowable number of packets being simultaneously at the server should not exceed $\overline{Q} = \overline{T}\nu$. For example, if we choose $\overline{T} = 1$ the upper value $\overline{Q} = 13$ and the respective maximum bandwidth is $\overline{B} = E^{-1}(13) = 11.55$. As an ABW estimate, we propose to use the variable $\widehat{B}_t^a \triangleq \max(\overline{B} - E^{-1}(\widehat{q}_t^\Sigma), 0)$ which is a function of the obtained estimate $\widehat{q}_t^\Sigma$.

Figure 4 shows the evolution of the channel load and its estimate:

— the hidden intensity state of the second flow $\varkappa_t$ (displayed as the background filling),

— the total hidden channel load q_t^Σ,

— the estimate of the total channel load $\widehat{q}_t^\Sigma$,

— the observed number of packets of the first flow ξ_t that are in the channel.

The upper graph shows the trajectories over the entire estimation interval [0; 2000], the lower one shows the interval [450; 650]. Note that a more detailed graph shows the piecewise continuous nature of the estimate: a continuous trajectory on the intervals of no jumps in observations and its jump change corresponding to a jump in observations. The registration of packet loss of the first flow unambiguously signals that the channel is full at the moment, i.e., $q_t^\Sigma = N^p$. The presented filtering estimate behaves in full accordance with this conclusion—at the instant $t = 485.91$, there is a packet loss, and the estimate $\widehat{q}_t^\Sigma$ coincides with the real channel load q_t^Σ, which is N^p, at this instant.

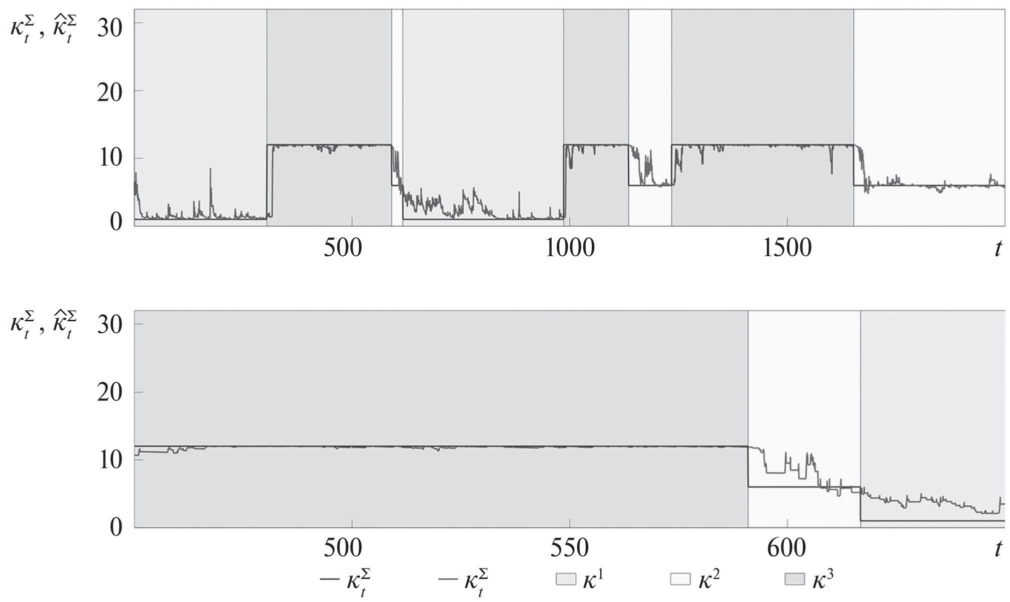

Fig. 4. Channel load and its estimate.

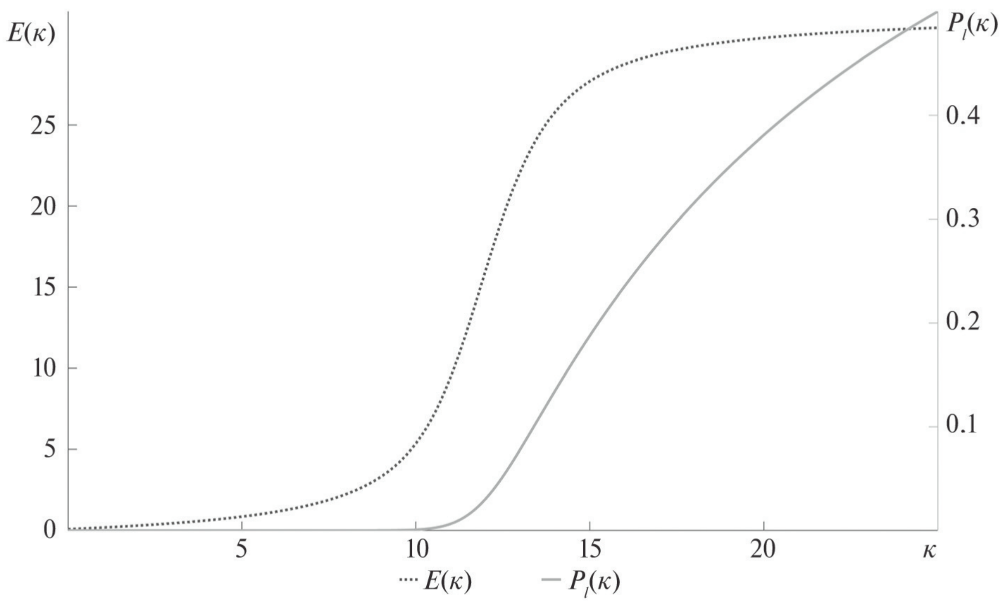

Fig. 5. Total packet arrival intensity and its estimate.

Figure 5 shows the evolution of the total packet arrival intensity in the channel and its estimate:

— the hidden intensity state of the second flow $\varkappa_t$ (displayed as the background filling),

— the packet arrival intensity $\varkappa_t^{\Sigma}$,

— the intensity estimate $\widehat{\varkappa}_t^{\Sigma}$.

The upper graph shows the trajectories over the entire estimation interval [0; 2000], the lower one shows the interval [450; 650]. Note that the more detailed graph also shows the piecewise continuous nature of the estimate.

Analyzing the graphs, we can conclude that the proposed estimates of the current characteristics of the channel bandwidth have high accuracy. We compare it with the accuracy of the trivial estimation, viz. unconditional mathematical expectation of the processes q_t^Σ and $\varkappa_t^\Sigma$ calculated for the MJP stationary distribution X. The accuracy of the trivial estimates $\mathsf{E}\left\{q^\Sigma\right\}$ and $\mathsf{E}\left\{\varkappa^\Sigma\right\}$ is characterized by the variances $\mathsf{D}\{q^\Sigma\}$ and $\mathsf{D}\{\varkappa^\Sigma\}$. As accuracy metrics for the proposed estimates we employ the following indices

$$\varepsilon^q = 1 - \frac{\int\limits_0^T \mathsf{E}\left\{(\widehat{q}_t^\Sigma - q_t^\Sigma)^2 dt\right\}}{T\mathsf{D}\{q^\Sigma\}} \quad \text{and} \quad \varepsilon^\varkappa = 1 - \frac{\int\limits_0^T \mathsf{E}\left\{(\widehat{\varkappa}_t^\Sigma - \varkappa_t^\Sigma)^2 dt\right\}}{T\mathsf{D}\{\varkappa^\Sigma\}},$$

which can be considered as analogues of the determination coefficients accepted in mathematical statistics [20]. In this example, the numerators of both indicators are obtained by the Monte Carlo method using a bundle of trajectories $N^{MC} = 10\,000$: $\varepsilon^q = 0.76$ and $\varepsilon^\varkappa = 0.94$.

5. CONCLUSIONS

In this work, we study the applied problem of real-time ABW estimation for a packet transmission channel using observations of one of the data flows served. The available observations include information on the number of packets of the flow currently in it, as well as the counting process of the packet losses. Since the proposed estimation procedure does not need to generate additional service flows through the channel that drain its resources, the proposed monitoring algorithm belongs to the passive class.

The principal idea that allowed us to construct an efficient numerical estimation algorithm is to use a partially observable MJP to describe the channel operating and incoming flows. The statistical information includes a set of some state functions observed without noise and counting processes whose intensity depends on the estimated state. The obtained filtering estimate is given by a sequence of recurrently connected ordinary differential equations calculated in the intervals between jumps of observations and discrete transformations that update the estimates at instants of changes in the observations. The work gives numerical experimental results illustrating the high quality of the presented estimates.

The research in the field of constructing efficient algorithms for ABW channel estimation can be continued in the following directions. First, it is of practical interest to solve the ABW estimation problem for an exponential element with a limited queue for the case of the non-stationary flow of incoming packets described here.

Second, it is important for telecommunication applications to complicate the model of channel and incoming flow operating by switching from Markov to semi-Markov processes.

Third, the ABW estimation problem was solved under conditions of full a priori information about the channel and flows transmitted through it. The construction of procedures for adaptive estimation of probabilistic parameters of the "channel-flows" pair and robust upgrading of the proposed monitoring algorithm also seems promising.

Fourth, the available statistical information in real data transmission networks is much richer than that used in this paper. For example, there are data linking the packet flows at the channel input and output, there is information about the individual transmission time of each packet, and so on. All this information included in the observation system may cause the extended stochastic observation system to cease to be Markov, which radically complicates the ABW estimation algorithms. Therefore, it seems promising to extend the class of observation systems in a way that, on the one hand, preserves the Markov property of the suitably extended system state and, on the other hand, allows us to use some of the additional statistical information similarly to [21, 22].

Fifth, using MJP with a finite set of states to solve applied problems involves very serious complexity. It consists of the rapid growth of the MJP dimension. Indeed, even in the considered numerical example with the channel capacity $K = 3$ and three possible variants of external load, the total number of MJP states is 48. We should also take into account that different states are described by vectors of dimension 3 rather than by scalar values, which leads to an additional increase in the amount of RAM required to implement the filtering algorithm. These circumstances make it topical to develop special efficient software that implements the estimation algorithms in stochastic observation systems with MJP.

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APPENDIX

Proof of Lemma 1. To derive systems (7) and (8), we use the method of moments—we construct closed linear stochastic differential systems describing the evolution of the state up to the next observation jump and average them.

If $\{\zeta_\ell^{\eta,k}\}_{\ell\in\mathbb{Z}_+,\, k=\overline{1,K}}$ are the jump instants of the components of the counting observations η, and $\{\zeta_\ell^{\xi}\}_{\ell\in\mathbb{Z}_+}$ are the jump instants of the perfect observations ξ, the instant ζ_{j+1} following ζ_j is determined using an obvious recurrence

$$\zeta_{j+1} = \min_{\zeta_\ell^{\eta,k}>\zeta_j,\, \zeta_{\ell'}^{\xi}>\zeta_j} (\zeta_\ell^{\eta,k}, \zeta_{\ell'}^{\xi}).$$

On the interval $[\zeta_j, +\infty)$ we study the process

$$U_t \triangleq \mathbf{I}_{[\zeta_j,\zeta_{j+1})}(t) = \underbrace{\mathbf{I}_{[\zeta_j,\zeta_{\ell'}^{\xi})}(t)}_{\triangleq V_t} \prod_{k=1}^{K} \underbrace{\mathbf{I}_{[\zeta_j,\zeta_\ell^{\eta,k})}(t)}_{\triangleq W_t^k}.$$

By construction, on any interval $[\zeta_j, t)$ processes V_t and W_t^k experience no more than one jump, and the relations hold

$$\mathrm{diag}(\Xi(\overline{\xi}_j))\theta_t \equiv \theta_{\zeta_j} \text{ for } \forall\, t \in [\zeta_j, \zeta_{\ell'}^{\xi}), \qquad \mathrm{diag}(\Xi(\overline{\xi}_j))\theta_{\zeta_{\ell'}^{\xi}} = 0.$$

By the Doleans formula [23] the processes V_t and W_t^k can be represented as solutions of the equations

$$V_t = \mathbf{I}_{[\zeta_j,+\infty)}(t) \left(1 + \int_{\zeta_j}^{t} V_{s-}\Xi(\overline{\xi}_j) d\theta_s\right), \tag{A.1}$$

$$W_t^k = \mathbf{I}_{[\zeta_j,+\infty)}(t) \left(1 - \int_{\zeta_j}^{t} W_{s-}^k d\eta_s^k\right). \tag{A.2}$$

Indeed, the process $\int_{\zeta_j}^t \Xi(\overline{\xi}_j)d\theta_s$ is a purely discontinuous semimartingale, and the solution of equation (A.1) by the Doleans formula has the form

$$V_t = \mathbf{I}_{[\zeta_j,+\infty)}(t)\exp\left[\Xi(\overline{\xi}_j)(\theta_t-\theta_{\zeta_j})\right]\prod_{s:\,\zeta_j<s\leqslant t}\left(1+\Xi(\overline{\xi}_j)\Delta\theta_s\right). \tag{A.3}$$

If the process θ did not have any jumps prior to the instant t, then $V_t = V_{\zeta_j} = 1$. If at time $s > \zeta_j$ the first jump of θ occurred that did not lead to any jump of observations ξ, i.e., $\Delta\xi_s = 0$, then

$$\Xi(\overline{\xi}_j)(\theta_s-\theta_{s-}) = \Xi(\overline{\xi}_j)(\theta_s-\theta_{\zeta_j}) = 0,$$

and according to (A.3) $V_s = 1$. The process V will preserve the same value during subsequent jumps of θ that do not lead to jumps of observations ξ. If at time $s > \zeta_j$ the first jump of θ occurred that led to a jump of observations ξ, i.e., $\xi_s \neq \xi_{s-} = \overline{\xi}_j$ and $s = \min\limits_{\zeta^{\xi}_{\ell'}>\zeta_j} \zeta^{\xi}_{\ell'}$, then

$$\Xi(\overline{\xi}_j)(\theta_s-\theta_{s-}) = \Xi(\overline{\xi}_j)\theta_s - \Xi(\overline{\xi}_j)\theta_{s-} = 0-1 = -1,$$

and according to (A.3) $V_s = 0$. The process V_t will further preserve the same value. Thus, we showed that the solution of equation (A.1)—process (A.3)—coincides with the process $\mathbf{I}_{[\zeta_j,\zeta^{\xi}_{\ell'})}(t)$ on the ray $[\zeta_j,+\infty)$. We can similarly prove that the processes $W^k_t = \mathbf{I}_{[\zeta_j,\zeta^{\eta,k}_{\ell})}(t)$ can be represented as the solution to Eq. (A.2).

Further, from (1)–(3) it follows that V_t and W^k_t can be expanded as follows:

$$V_t = \mathbf{I}_{[\zeta_j,+\infty)}(t)\left(1+\int_{\zeta_j}^{t}\Xi(\overline{\xi}_j)A^\top \underbrace{\theta_s V_s}_{\triangleq v_s} ds + M^1_t\right), \tag{A.4}$$

$$W^k_t = \mathbf{I}_{[\zeta_j,+\infty)}(t)\left(1-\int_{\zeta_j}^{t} g^k \underbrace{\theta_s W^k_s}_{\triangleq w^k_s} ds + M^{2,k}_t\right), \tag{A.5}$$

where $\mathbf{I}_{[\zeta_j,+\infty)}(t)M^1_t$ and $\mathbf{I}_{[\zeta_j,+\infty)}(t)M^{2,k}_t$ are some martingales. Note that (A.4) and (A.5) can be interpreted as linear stochastic differential equations with martingales in their right-hand sides. Nevertheless these equations are not closed: the right-hand side of the equation for V_t contains the process v_t, and the right-hand side of W^k_t includes w^k_t. From (A.4) and (A.5) we obtain a closed system of linear stochastic differential equations for the vector process $u_t \triangleq \theta_t U_t$.

By Ito's rule and condition (B) the process U_t admits the expansion

$$\begin{aligned}U_t &= \mathbf{I}_{[\zeta_j,+\infty)}(t)\left[1+\int_{\zeta_j}^{t}\left(dV_s\prod_{k=1}^{K}W^k_{s-} + V_{s-}\sum_{k=1}^{K}\prod_{i:\,i\neq k}W^i_{s-}dW^k_s\right)\right]\\ &= \mathbf{I}_{[\zeta_j,+\infty)}(t)\left[1+\int_{\zeta_j}^{t}\left(\Xi(\overline{\xi}_j)A^\top - \sum_{k=1}^{K}g^k\right)u_s ds + M^3_t\right],\end{aligned}$$

where $\mathbf{I}_{[\zeta_j,+\infty)}(t)M^3_t$ is some martingale. From the definition of processes θ and U it follows that

$$\sum_{\zeta:\,\zeta_j<\zeta\leqslant t}\Delta\theta_\zeta\Delta U_\zeta = -\mathbf{I}_{[\zeta_j,+\infty)}(t)\int_{\zeta_j}^{t}\left[\theta_{s-}dV_s\prod_{k=1}^{K}W^k_{s-} - (I-\operatorname{diag}(\Xi(\overline{\xi}_j)))d\theta_s V_{s-}\prod_{k=1}^{K}W^k_{s-}\right],$$

therefore

$$
\begin{aligned}
u_t &= \mathbf{I}_{[\zeta_j,+\infty)}(t)\left[\theta_{\zeta_j} + \int\limits_{\zeta_j}^{t} (d\theta_s U_{s-} + \theta_{s-} dU_s) + \sum_{\zeta:\ \zeta_j<\zeta\leqslant t} \Delta\theta_\zeta \Delta U_\zeta\right] \\
&= \mathbf{I}_{[\zeta_j,+\infty)}(t)\left[\theta_{\zeta_j} + \int\limits_{\zeta_j}^{t} \left(\operatorname{diag}(\Xi(\overline{\xi}_j))A^\top - \sum_{k=1}^{K} \operatorname{diag}(g^k)\right) u_s ds + M_t^4\right],
\end{aligned} \tag{A.6}
$$

where $\mathbf{I}_{[\zeta_j,+\infty)}(t)M_t^4$ is some martingale. Calculating CME of both parts of (A.6) with respect to $\mathfrak{O}_j$ and using the fact that

$$
\mathsf{E}\left\{\mathbf{I}_{[\zeta_j,+\infty)}(t)M_t^4|\mathfrak{O}_j\right\} = \mathsf{E}\left\{\mathsf{E}\left\{\mathbf{I}_{[\zeta_j,+\infty)}(t)M_t^4|\mathcal{F}_{\zeta_j}\right\}|\mathfrak{O}_j\right\} = 0,
$$

we obtain a system of equations equivalent to (7):

$$
m_t = \widehat{\pi}_j + \int\limits_{\zeta_j}^{t} \left(\operatorname{diag}(\Xi(\overline{\xi}_j))A^\top - \sum_{k=1}^{K} \operatorname{diag}(g^k)\right) m_s ds.
$$

The fact that the function can be represented as a solution to (8) follows from the chain rule of function (6) and system (7).

Suppose $\mathcal{A} \in \mathfrak{O}_j$ is an arbitrary set and $\mathcal{A}' = \mathcal{A} \cap \{\omega :\ \zeta_{j+1} > t\}$. The CME properties lead to the following sequence of equalities being true

$$
\begin{aligned}
&\mathsf{E}\left\{\mathbf{I}_{[\zeta_j,+\infty)}(t)\left(\theta_t \mathbf{I}_{\mathcal{A}'}(\omega) - \mu_t \mathbf{I}_{\mathcal{A}'}(\omega)\right)\right\} \\
&= \mathsf{E}\left\{\theta_t \mathbf{I}_{[\zeta_j,\zeta_{j+1})}(t)\mathbf{I}_{\mathcal{A}}(\omega) - \mu_t \mathbf{I}_{[\zeta_j,\zeta_{j+1})}(t)\mathbf{I}_{\mathcal{A}}(\omega)\right\} \\
&= \mathsf{E}\left\{\mathsf{E}\left\{\theta_t \mathbf{I}_{[\zeta_j,\zeta_{j+1})}(t)\mathbf{I}_{\mathcal{A}}(\omega) - \mu_t \mathbf{I}_{[\zeta_j,\zeta_{j+1})}(t)\mathbf{I}_{\mathcal{A}}(\omega)\right\}|\mathfrak{O}_j\right\} \\
&= \mathsf{E}\left\{\left(\mathsf{E}\left\{\theta_t \mathbf{I}_{[\zeta_j,\zeta_{j+1})}(t)|\mathfrak{O}_j\right\} - \mu_t \mathsf{E}\left\{\mathbf{I}_{[\zeta_j,\zeta_{j+1})}(t)|\mathfrak{O}_j\right\}\right)\mathbf{I}_{\mathcal{A}}(\omega)\right\} \\
&= \mathsf{E}\left\{(m_t - \mathbf{1}m_t\mu_t)\mathbf{I}_{\mathcal{A}}(\omega)\right\} = 0,
\end{aligned}
$$

as well as equality (5). Lemma 1 is proved.

Proof of Lemma 2. The sequence $\{(\zeta_j, \overline{\theta}_j, \overline{\xi}_j, \overline{\eta}_j)\}_{j\in\mathbb{Z}_+}$ is Markov. We construct the elements of its transition kernel.

The processes $\theta_t(\eta_t^k - \overline{\eta}_j^k)\mathbf{I}_{[\zeta_j,+\infty)}(t)$ can be expanded as

$$
\theta_t(\eta_t^k - \overline{\eta}_j^k)\mathbf{I}_{[\zeta_j,+\infty)}(t) = \mathbf{I}_{[\zeta_j,+\infty)}(t)\left[\int\limits_{\zeta_j}^{t}\left(A^\top\theta_s(\eta_s^k - \overline{\eta}_j^k) + \operatorname{diag}(g^k)\theta_s\right)ds + M_t^5\right],
$$

where $\mathbf{I}_{[\zeta_j,+\infty)}(t)M_t^5$ is some martingale. On the other hand,

$$
\theta_t(\eta_t^k - \overline{\eta}_j^k)\mathbf{I}_{[\zeta_j,+\infty)}(t) = \theta_t \underbrace{(\eta_t^k - \overline{\eta}_j^k)\mathbf{I}_{[\zeta_j,\zeta_{j+1})}(t)}_{=0} + \theta_t(\eta_t^k - \overline{\eta}_j^k)\mathbf{I}_{[\zeta_{j+1},+\infty)}(t).
$$

From the last two equalities it follows that

$$\begin{aligned}
&\theta_{t\wedge\zeta_{j+1}}(\eta^k_{t\wedge\zeta_{j+1}}-\overline{\eta}^k_j)\mathbf{I}_{[\zeta_j,+\infty)}(t\wedge\zeta_{j+1})\\
&=\theta_{t\wedge\zeta_{j+1}}(\eta^k_{t\wedge\zeta_{j+1}}-\overline{\eta}^k_j)\mathbf{I}_{[\zeta_{j+1},+\infty)}(t\wedge\zeta_{j+1})\\
&=\overline{\theta}_{j+1}(\overline{\eta}^k_{j+1}-\overline{\eta}^k_j)\mathbf{I}_{[\zeta_{j+1},+\infty)}(t)\\
&=\mathbf{I}_{[\zeta_j,+\infty)}(t)\left[\int\limits_{\zeta_j}^{t}\left(A^\top\underbrace{u_s(\eta^k_s-\overline{\eta}^k_j)}_{=0}+\mathrm{diag}(g^k)u_s\right)ds+M^5_{t\wedge\zeta_{j+1}}\right].
\end{aligned}$$

Calculating the CME with respect to $\mathfrak{O}_j$ of the left and right parts of the last equality and using the optional stopping theorem of the right-continuous martingale, we obtain that

$$\begin{aligned}
&\mathsf{E}\left\{\overline{\theta}_{j+1}(\overline{\eta}^k_{j+1}-\overline{\eta}^k_j)\mathbf{I}_{[\zeta_{j+1},+\infty)}(t)|\mathfrak{O}_j\right\}\\
&=\mathsf{E}\left\{\overline{\theta}_{j+1}\mathbf{I}_{\{1\}}(\overline{\eta}^k_{j+1}-\overline{\eta}^k_j)\mathbf{I}_{[\zeta_{j+1},+\infty)}(t)|\mathfrak{O}_j\right\}\\
&=\mathbf{I}_{[\zeta_j,+\infty)}(t)\int\limits_{\zeta_j}^{t}\mathrm{diag}(g^k)m_sds=\mathbf{I}_{[\zeta_j,+\infty)}(t)\int\limits_{\zeta_j}^{t}\mathrm{diag}(g^k)\mu_s(\mathbf{1}m_s)ds.
\end{aligned}$$

The considered transition corresponds to a jump of the component η^k, i.e. $\zeta_{j+1}=\zeta^{\eta,k}_\ell$. Now consider the case when the transition is generated by a jump of observations ξ, i.e. when $\overline{\xi}_{j+1}\neq\overline{\xi}_j$ and $\zeta_{j+1}=\zeta^{\xi}_\ell$. Let $c\in\mathcal{C}$ (one of the possible values of observation ξ) be some column of matrix C. Note that

$$\mathrm{diag}(c)\left(I-\mathrm{diag}(\Xi(\overline{\xi}_j))\right)\overline{\theta}_{j+1}=\begin{cases}0, \text{ if } \overline{\xi}_{j+1}=\overline{\xi}_j,\\ \overline{\theta}_{j+1}, \text{ if } \overline{\xi}_{j+1}\neq\overline{\xi}_j.\end{cases}$$

The process $\mathrm{diag}(c)\left(I-\mathrm{diag}(\Xi(\overline{\xi}_j))\right)\theta_t\mathbf{I}_{[\zeta_j,+\infty)}(t)$ can be represented as

$$\begin{aligned}
&\mathrm{diag}(c)\left(I-\mathrm{diag}(\Xi(\overline{\xi}_j))\right)\theta_t\mathbf{I}_{[\zeta_j,+\infty)}(t)\\
&=\mathrm{diag}(c)\left(I-\mathrm{diag}(\Xi(\overline{\xi}_j))\right)\left[\int\limits_{\zeta_j}^{t}A^\top\theta_sds+M^6_t\right]\mathbf{I}_{[\zeta_j,+\infty)}(t),
\end{aligned}$$

where $\mathbf{I}_{[\zeta_j,+\infty)}(t)M^6_t$ is some martingale. On the other hand,

$$\begin{aligned}
&\mathrm{diag}(c)\left(I-\mathrm{diag}(\Xi(\overline{\xi}_j))\right)\theta_t\mathbf{I}_{[\zeta_j,+\infty)}(t)\\
&=\underbrace{\mathrm{diag}(c)\left(I-\mathrm{diag}(\Xi(\overline{\xi}_j))\right)\theta_t\mathbf{I}_{[\zeta_j,\zeta_{j+1})}(t)}_{=0}\\
&+\mathrm{diag}(c)\left(I-\mathrm{diag}(\Xi(\overline{\xi}_j))\right)\theta_t\mathbf{I}_{[\zeta_{j+1},+\infty)}(t).
\end{aligned}$$

From the last two equalities it follows that

$$\begin{aligned}
&\mathrm{diag}(c)\left(I-\mathrm{diag}(\Xi(\overline{\xi}_j))\right)\theta_{t\wedge\zeta_{j+1}}\mathbf{I}_{[\zeta_j,+\infty)}(t\wedge\zeta_{j+1})\\
&=\mathrm{diag}(c)\left(I-\mathrm{diag}(\Xi(\overline{\xi}_j))\right)\theta_{t\wedge\zeta_{j+1}}\mathbf{I}_{[\zeta_{j+1},+\infty)}(t\wedge\zeta_{j+1})
\end{aligned}$$

$$= \operatorname{diag}(c)\left(I - \operatorname{diag}(\Xi(\overline{\xi}_j))\right)\overline{\theta}_{j+1}\mathbf{I}_{[\zeta_{j+1},+\infty)}(t)$$

$$= \operatorname{diag}(c)\left(I - \operatorname{diag}(\Xi(\overline{\xi}_j))\right)\left[\int\limits_{\zeta_j}^{t} A^\top u_s ds + M^6_{t\wedge\zeta_{j+1}}\right]\mathbf{I}_{[\zeta_j,+\infty)}(t).$$

Again calculating the CME with respect to $\mathfrak{O}_j$ of the left and right parts of the equality and using the optional stopping theorem of the martingale, we obtain

$$\begin{aligned}
&\mathsf{E}\left\{\operatorname{diag}(c)\left(I - \operatorname{diag}(\Xi(\overline{\xi}_j))\right)\overline{\theta}_{j+1}\mathbf{I}_{[\zeta_{j+1},+\infty)}(t)|\mathfrak{O}_j\right\}\\
&= \mathsf{E}\left\{\overline{\theta}_{j+1}\mathbf{I}_{\{c\}}(\overline{\xi}_{j+1})\left(1 - \mathbf{I}_{\{\overline{\xi}_j\}}(\overline{\xi}_{j+1})\right)\mathbf{I}_{[\zeta_{j+1},+\infty)}(t)|\mathfrak{O}_j\right\}\\
&= \mathbf{I}_{[\zeta_j,+\infty)}(t)\int\limits_{\zeta_j}^{t}\operatorname{diag}(c)\left(I - \operatorname{diag}(\Xi(\overline{\xi}_j))\right)A^\top m_s ds\\
&= \mathbf{I}_{[\zeta_j,+\infty)}(t)\int\limits_{\zeta_j}^{t}\operatorname{diag}(c)\left(I - \operatorname{diag}(\Xi(\overline{\xi}_j))\right)A^\top \mu_s(\mathbf{1}m_s)ds.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\mathsf{P}\left\{\overline{\theta}_{j+1} = e_i,\ \overline{\xi}_{j+1} = c,\ \overline{\xi}_{j+1} \neq \overline{\xi}_j,\ \zeta_{j+1} \in [t, t+dt)|\mathfrak{O}_j\right\}\\
&= e_i^\top \operatorname{diag}(c)\left(I - \operatorname{diag}(\Xi(\overline{\xi}_j))\right)A^\top \mu_t(\mathbf{1}m_t)dt
\end{aligned} \tag{A.7}$$

and

$$\begin{aligned}
&\mathsf{P}\left\{\overline{\theta}_{j+1} = e_i,\ \overline{\xi}_{j+1} = \overline{\xi}_j,\ \overline{\eta}^k_{j+1} - \overline{\eta}^k_j = 1,\ \zeta_{j+1} \in [t, t+dt)|\mathfrak{O}_j\right\}\\
&= e_i^\top \operatorname{diag}(g^k)\mu_t(\mathbf{1}m_t)dt.
\end{aligned} \tag{A.8}$$

Further, we use a technique standard for deriving the equations of optimal state filtering of Markov observation systems with discrete time [24, 25]. Let (α, β, γ) be a block random vector, $P(\mathcal{A}, \mathcal{B}|\gamma)$ be the conditional distribution of the pair (α, β) with respect to γ, i.e.

$$\mathsf{P}\{\alpha \in \mathcal{A}, \beta \in \mathcal{B}|\gamma\} = P(\mathcal{A}, \mathcal{B}|\gamma) \qquad \mathsf{P} - \text{a.s.}$$

Let there also exist a measure $\chi(a, b|\gamma)$ such that $P \ll \chi$ and $\rho(a, b|\gamma) = \frac{dP}{d\chi}(a, b|\gamma)$ be the corresponding Radon-Nikodym derivative. Then the CME $\mathsf{E}\{\alpha|\beta,\gamma\}$ can be computed using the following variant of Bayes formula:

$$\mathsf{E}\{\alpha|\beta,\gamma\} = \left(\int \rho(a', \beta|\gamma)d\chi(a', \beta|\gamma)\right)^{-1}\int a\rho(a, \beta|\gamma)d\chi(a, \beta|\gamma). \tag{A.9}$$

Formula (9) is a special case of (A.9) obtained by substituting (A.7) and (A.8) into it. Lemma 2 is proved.

Proof of Theorem 1. By direct substitution, we can check that the estimate $\widehat{\theta}_t$, "glued" from solutions of systems (8) with jumps described by (9) and initial condition (11), is a solution of (12). Therefore, to prove the theorem it is sufficient to check the truth of equality (10).

The observable process (ξ_t, η_t) represents a multivariate point process (MPP) with state space $\mathbf{B} \triangleq \mathcal{C} \times \mathbb{Z}^K_+$, which can be represented in the equivalent form of stochastic measure ϕ [18], defined on the measurable space $([0, T] \times \mathbf{B}, \mathcal{B}([0, T]) \times 2^{\mathbf{B}})$:

$$\phi(\omega, dt, dy_1, dy_2) = \sum_{j\in\mathbb{Z}_+}\delta_{(\zeta_j(\omega),\overline{\xi}_j(\omega),\overline{\eta}_j(\omega))}(dt, dy_1, dy_2).$$

In [18] it was proved that the natural flow of σ-algebras generated by observations coincides with the one generated by the stochastic measure, i.e.

$$\sigma\left\{\phi([a,b)\times\{c\}\times\{\mathbf{z}\}):\ [a,b)\in\mathcal{B}([0,T]),\ c\in\mathcal{C},\ \mathbf{z}\in\mathbb{Z}_+^K\right\}\equiv\mathcal{O}_t,\quad t\in[0,T].$$

The base of the σ-algebra $\mathcal{B}([0,T])\times 2^{\mathbf{B}}$ consists of sets of the form $[a,b)\times\{c\}\times\{\mathbf{z}\}$, so by virtue of the theorem on monotone classes [23] to prove the truth of equality (10) it is sufficient to check the validity of equality

$$\mathsf{E}\left\{\left(\sum_{j\geqslant 0}\mathbf{I}_{[\zeta_j,\zeta_{j+1})}(t)\mu_t-\theta_t\right)\phi([a,b)\times\{c\}\times\{\mathbf{z}\})\right\}\equiv 0$$

for all sets $[a,b)\times\{c\}\times\{\mathbf{z}\}$ of the base.

From the properties of CME and (4)–(6) follows the sequence of equalities

$$\begin{aligned}
&\mathsf{E}\left\{\left(\sum_{j\geqslant 0}\mathbf{I}_{[\zeta_j,\zeta_{j+1})}(t)\mu_t-\theta_t\right)\phi([a,b)\times\{c\}\times\{\mathbf{z}\})\right\}\\
&=\mathsf{E}\left\{\sum_{j\geqslant 0}\mathbf{I}_{[\zeta_j,\zeta_{j+1})}(t)\left(\mu_t-\theta_t\right)\mathbf{I}_{[a,b)}(t)\sum_{\ell\geqslant 0}\mathbf{I}_{[\zeta_\ell,\zeta_{\ell+1})}(t)\mathbf{I}_{\{c\}}(\overline{\xi}_\ell)\mathbf{I}_{\{\mathbf{z}\}}(\overline{\eta}_\ell)\right\}\\
&=\mathbf{I}_{[a,b)}(t)\sum_{j\geqslant 0}\mathsf{E}\left\{\mathbf{I}_{[\zeta_j,\zeta_{j+1})}(t)\left(\mu_t-\theta_t\right)\mathbf{I}_{\{c\}}(\overline{\xi}_j)\mathbf{I}_{\{\mathbf{z}\}}(\overline{\eta}_j)\right\}\\
&=\mathbf{I}_{[a,b)}(t)\sum_{j\geqslant 0}\mathsf{E}\left\{\mathsf{E}\left\{\mathbf{I}_{[\zeta_j,\zeta_{j+1})}(t)\left(\mu_t-\theta_t\right)\mathbf{I}_{\{c\}}(\overline{\xi}_j)\mathbf{I}_{\{\mathbf{z}\}}(\overline{\eta}_j)|\mathfrak{O}_j\right\}\right\}\\
&=\mathbf{I}_{[a,b)}(t)\sum_{j\geqslant 0}\mathsf{E}\left\{\left(\underbrace{\mathbf{I}_{[\zeta_j,+\infty)}(t)\mathsf{E}\left\{\mathbf{I}_{[\zeta_j,\zeta_{j+1})}(t)|\mathfrak{O}_j\right\}}_{=\mathbf{I}_{[\zeta_j,+\infty)}(t)\mathbf{1}m_t}\mu_t\right.\right.\\
&\qquad\left.\left.-\underbrace{\mathbf{I}_{[\zeta_j,+\infty)}(t)\mathsf{E}\left\{\theta_t\mathbf{I}_{[\zeta_j,\zeta_{j+1})}(t)|\mathfrak{O}_j\right\}}_{=\mathbf{I}_{[\zeta_j,+\infty)}(t)m_t}\right)\mathbf{I}_{\{c\}}(\overline{\xi}_j)\mathbf{I}_{\{\mathbf{z}\}}(\overline{\eta}_j)\right\}=0.
\end{aligned}$$

Theorem 1 is proved.

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